Not every 2-tough graph is Hamiltonian

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The first two authors dedicate this paper to the memory of their dear friend and coauthor Henk Tau Veldman, who died October 12, 1998.

Abstract

We present \((\frac{2}{4} - \epsilon)\)-tough graphs without a Hamilton path for arbitrary \(\epsilon > 0\), thereby refuting a well-known conjecture due to Chvátal. We also present \((\frac{3}{4} - \epsilon)\)-tough chordal graphs without a Hamilton path for any \(\epsilon > 0\). © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

We use Bondy and Murty’s book [5] for terminology and notation not defined here, and consider finite simple graphs only.

A graph \(G\) is Hamiltonian if it contains a Hamilton cycle (a cycle containing every vertex of \(G\)); \(G\) is traceable if \(G\) contains a Hamilton path (a path containing every vertex of \(G\)); \(G\) is Hamiltonian-connected if for every pair of distinct vertices \(x\) and \(y\) of \(G\) there is a Hamilton path with endvertices \(x\) and \(y\).

The number of components of a graph \(G\) is denoted by \(\omega(G)\). The graph \(G\) is \(t\)-tough \((t \in \mathbb{R}, t \geq 0)\) if \(|S| \geq t \cdot \omega(G - S)\) for every subset \(S\) of \(V(G)\) with \(\omega(G - S) > 1\). The toughness of \(G\), denoted by \(\tau(G)\), is the maximum value of \(t\) for which \(G\) is \(t\)-tough.

The concept of toughness of a graph was introduced by Chvátal [7]. Clearly, 1-toughness is a necessary condition for hamiltonicity, but it is not sufficient. In [7] the following conjecture is stated.

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Conjecture 1 (Chvátal [7]). There exists $t_0$ such that every $t_0$-tough graph is Hamiltonian.

The stronger conjecture that every $t$-tough graph with $t > \frac{1}{2}$ is Hamiltonian, also occurring in [7], was first disproved by Thomassen (see [4]). Enomoto et al. [8] showed that every $2$-tough graph contains a 2-factor (a 2-regular spanning subgraph), while for arbitrary $\epsilon > 0$ there exist $(2 - \epsilon)$-tough graphs without a 2-factor, and hence without a Hamilton cycle. Therefore the following conjecture, usually attributed to Chvátal, appeared to be both reasonable and best possible.

Conjecture 2. Every 2-tough graph is Hamiltonian.

In [1] a construction of a nontraceable graph from non-Hamiltonian-connected building blocks was used to show that Conjecture 2 is equivalent to several other statements, some (seemingly) weaker, some (seemingly) stronger than Conjecture 2. This construction was inspired by examples of graphs of high toughness without 2-factors occurring in [3]. In the next section, we use the same construction to obtain $(\frac{2}{3} - \epsilon)$-tough nontraceable graphs for arbitrary $\epsilon > 0$, thereby refuting Conjecture 2. Conjecture 1 remains open.

2. Counterexamples to Conjecture 2

For a given graph $H$ and two vertices $x$ and $y$ of $H$ we define the graph $G(H, x, y, \ell, m)$ ($\ell, m \in \mathbb{N}$) as follows. Take $m$ disjoint copies $H_1, \ldots, H_m$ of $H$, with $x_i, y_i$ the vertices in $H_i$ corresponding to the vertices $x$ and $y$ in $H$ ($i = 1, \ldots, m$). Let $F_m$ be the graph obtained from $H_1 \cup \cdots \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$. Let $T = K_{\ell}$ and let $G(H, x, y, \ell, m)$ be the join $T \cup F_m$ of $T$ and $F_m$.

The proof of the following theorem occurs almost literally in [1]. For convenience we repeat it here.

Theorem 3. Let $H$ be a graph and $x, y$ two vertices of $H$ which are not connected by a Hamilton path of $H$. If $m \geq 2\ell + 3$, then $G(H, x, y, \ell, m)$ is nontraceable.

Proof. Suppose $G(H, x, y, \ell, m)$ contains a Hamilton path $P$. The intersection of $P$ and $F_m$ consists of a collection $\mathcal{P}$ of at most $\ell + 1$ disjoint paths, together containing all vertices in $F_m$. Since $m \geq 2(\ell + 1) + 1$, there is a subgraph $H_{t_0}$ in $F_m$ such that no endvertex of a path of $\mathcal{P}$ lies in $H_{t_0}$. Hence the intersection of $P$ and $H_{t_0}$ is a path with endvertices $x_0$ and $y_0$ that contains all vertices of $H_{t_0}$. This contradicts the fact that $H_{t_0}$ is a copy of the graph $H$ without a Hamilton path between $x$ and $y$. \qed

Consider the graph $L$ of Fig. 1.
Theorem 4. For $\ell \geq 2$ and $m \geq 1$,
\[ \tau(G(L, u, v, \ell, m)) = \frac{\ell + 4m}{2m + 1} \]

Proof. Let $G = G(L, u, v, \ell, m)$ for some $\ell \geq 2$ and $m \geq 1$, and choose $S \subseteq V(G)$ such that $\omega(G - S) > 1$ and $\tau(G) = |S|/\omega(G - S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let $\omega_i$ be the number of components of $L_i - S_i$ that contain neither $u_i$ nor $v_i$ ($i = 1, \ldots, m$). Then

\[ \tau(G) = \frac{\ell + \sum_{i=1}^{m} s_i}{c + \sum_{i=1}^{m} \omega_i} \geq \frac{\ell + \sum_{i=1}^{m} s_i}{1 + \sum_{i=1}^{m} \omega_i}, \]

where

\[ c = \begin{cases} 0 & \text{if } u_i, v_i \in S_i \text{ for all } i \in \{1, \ldots, m\}, \\ 1 & \text{otherwise}. \end{cases} \]

We now show that

\[ s_i \geq 2\omega_i \quad (i = 1, \ldots, m). \]

First note that $\omega_i \leq 2$, since $L - \{u, v\}$ has independence number 2. Clearly $s_i \geq 2\omega_i$ if $\omega_i = 0$ or $\omega_i = 1$. By exhaustion it is readily checked that if $s_i \leq 3$, then $\omega_i \leq 1$. In other words, $s_i \geq 2\omega_i$ if $\omega_i = 2$.

It follows that

\[ \tau(G) \geq \frac{\ell + 2 \sum_{i=1}^{m} \omega_i}{1 + \sum_{i=1}^{m} \omega_i}. \]

Since $\ell \geq 2$, this lower bound for $\tau(G)$ is a nonincreasing function of $\sum_{i=1}^{m} \omega_i$, and is hence minimized if $\omega_i = 2$ for all $i \in \{1, \ldots, m\}$. Thus

\[ \tau(G) \geq \frac{\ell + 4m}{2m + 1}. \]

Set $U = V(T) \cup U_1 \cup \cdots \cup U_m$, where $U_i$ is the set of vertices of $L_i$ having degree 4 in $L_i$ ($i = 1, \ldots, m$). The proof is completed by observing that

\[ \tau(G) \leq \frac{|U|}{\omega(G - U)} = \frac{\ell + 4m}{2m + 1}. \quad \square \]

Corollary 5. For every $\varepsilon > 0$ there exists a $(\frac{\ell}{4} - \varepsilon)$-tough nontraceable graph.
Proof. Clearly the graph $L$ has no Hamilton path with endvertices $u$ and $v$. Hence by Theorem 3 the graph $G(L, u, v, \ell, 2\ell + 3)$ is nontraceable for every $\ell$. By Theorem 4 it has toughness $(9\ell + 12)/(4\ell + 7)$ for $\ell \geq 2$. The result follows.

Remark 1. It is easily seen that Theorem 3 remains valid if “$m \geq 2\ell + 3$” and “nontraceable” are replaced by “$m \geq 2\ell + 1$” and “non-Hamiltonian”, respectively. Thus the graph $G(L, u, v, 2, 5)$ is a non-Hamiltonian graph, which by Theorem 4 has toughness 2. This graph is sketched in Fig. 2. It follows that a smallest counterexample to Conjecture 2 has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most 58 $(|V(G(L, u, v, 2, 7))|)$ vertices.

Remark 2. A graph $G$ is neighborhood-connected if the neighborhood of each vertex of $G$ induces a connected subgraph of $G$. In [7] Chvátal also states the following weaker version of Conjecture 2: every 2-tough neighborhood-connected graph is Hamiltonian. Since all counterexamples to Conjecture 2 described above are neighborhood-connected, this weaker conjecture is also false.

Remark 3. Most of the ingredients used in the above counterexamples to Conjecture 2 were already present in [1]. It only remained to observe that using the specific graph $L$ as our “building block” produced a graph with toughness at least 2.

3. Chordal graphs

A graph $G$ is chordal if it contains no induced cycles of length at least 4. Chvátal [7] obtained $(\frac{3}{2} - \varepsilon)$-tough graphs without a 2-factor for arbitrary $\varepsilon > 0$. These examples are all chordal. Recently it was shown in [2] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [9] raised the question whether every $\frac{3}{2}$-tough chordal graph is Hamiltonian. Using Theorem 3 we now show that this conjecture, too, is false. A key observation in this context is that the graphs $G(H, x, y, \ell, m)$ are chordal whenever $H$ is chordal, as is easily shown.
Consider the graph $M$ of Fig. 3. The graph $M$ is chordal and has no Hamilton path with endvertices $p$ and $q$. Hence by Theorem 3 the chordal graph $G(M, p, q, \ell, m)$ is nontraceable whenever $m \geq 2\ell + 3$. By arguments as used in the proof of Theorem 4 its toughness is $(\ell + 3m)/(2m + 1)$ if $\ell \geq 2$. Hence for $\ell \geq 2$ the graph $G(M, p, q, \ell, 2\ell + 3)$ is a chordal nontraceable graph with toughness $(7\ell + 9)/(4\ell + 7)$. We have thus obtained the following result.

**Theorem 6.** For every $\varepsilon > 0$ there exists a $(7/4-\varepsilon)$-tough chordal nontraceable graph.

On the other hand Chen et al. [6] recently proved that every 18-tough chordal graph is Hamiltonian, which means that Conjecture 1 is true when restricted to chordal graphs.

**References**