ON THE COMPLETE INSTABILITY OF EMPIRICALLY IMPLEMENTED DYNAMIC LEONTIEF MODELS

ABSTRACT

On theoretical grounds, real world implementations of forward-looking dynamic Leontief systems were expected to be stable. Empirical work, however, showed the opposite to be true: all investigated systems proved to be unstable. In fact, an extreme form of instability (`complete instability') appeared to be the rule. In contrast to this, backward-looking models and dynamic inverse versions appeared to be exceptionally stable. For this stability-instability switch a number of arguments have been put forward, none of which was convincing. Dual (in)stability theorems only seemed to complicate matters even more. In this paper we offer an explanation. We show that in the balanced growth case--under certain conditions--the spectrum of eigenvalues of matrix D equivalent to $(I - A)^{-1}B$, where A stands for the matrix of intermediate input coefficients and B for the capital matrix, will closely approximate the spectrum of a positive matrix of rank one. From this property the observed instability properties are easily derived. We argue that the employed approximations are not unrealistic in view of the data available up to now.

1. Introduction

Ever since Leontief (1953) discovered the instability of his closed forward-looking model

(1.1) $x(t) = Ax(t) + B[x(t + 1) - x(t)]$,

where A is the matrix of intermediate input coefficients (here including replacements and endogenized households consumption), B the capital coefficients matrix and x(t) total production during period t, much of the literature on the dynamic Leontief model has been concerned with stability problems. Stability would require that

(1.2) $1 + 1/Mu_1 > | 1 + 1/Mu_i |$ $i = 2, ..., n$

where the Mu_i are the eigenvalues (in descending order of magnitude, Mu_1 being the Frobenius eigenvalue) of $(I - A)^{-1}B$. (We suppose the capital matrix to have full rank). If (1.2) is not valid, the model is unstable. If it is stable, its output proportions will converge to those of the von Neumann ray.

Later research more than confirmed Leontief's findings: in fact, the literature shows that all implemented forward lag versions appeared to be not only unstable, but most of them were, in Tsukui's (1968) terms, "completely unstable", because it appeared that

(1.3) $1 + 1/Mu_1 < | 1 + 1/Mu_i |$ $i = 2, ..., n$

(The references contain a selected list of available studies on the issue.) All this implied, via the so-called Solow-Jorgenson dual instability result, that the corresponding price model

(1.4) $p(t + 1)[B + I - A] = (1 + r)p(t)B$

with p(t) the price vector at period t, was invariably stable.

A wide variety of explanations has been given, such as an incorrect incorporation of stock-flow characteristics, mathematical overdeterminacy, and incorrect time parameters, none of which has been convincing. In practice, the problem was circumvented by developing alternative structures such as linear or non-linear programming versions and the well-known Dynamic Inverse (Leontief, 1970). Yet other versions, such as the backward-lag model

(1.5) $x(t) = Ax(t) + B[x(t) - x(t - 1)]$

were also available (Wurtele, 1959). All these newer versions appeared to have stability properties often radically different from the forward-lag variant; Stability in model (1.5), for example, requires

(1.6) $(1- Mu_1)^{-1} > | (1-1/Mu_i)^{-1} |$ $i = 2, ..., n$

So far it has proved impossible to provide any convincing explanations for these differences in economic terms.

Because of the importance of the problem, especially in judging the various alternatives that have been presented up to now, we shall again focus on the stability issue in this paper. We shall restrict ourselves to the Brody (1970) variant of the model, although extensions to other variants now would seem within reach. We shall present a main theorem (proposition 3). Our conclusion will be that the stability properties of the model's many variants can be explained in terms of the properties of only one particular matrix, i.e. matrix D equivalent to $(I - A)^{-1}B$. Under plausible conditions, this matrix is shown to closely approximate a positive matrix of rank one. This fact then immediately explains the observed differences in stability properties. The model's eigenvalues being central in this paper, we shall use elements of spectral decomposition theory, a technique that, to our knowledge, has not been used before in this context.

2. The Brody model

In the Brody (1970) version of the model, $x(t + 1) = (1 + [1/Mu_1])x(t)$, with $1/Mu_1$ the economy's growth rate. Substituting in (1.1) and dropping the time index, we find

(2.1) $x = (A + [1/Mu_1]B)x$

or

(2.2) $x = Cx$

with

(2.3) C equivalent to $A + [1/Mu_1]B$.

The corresponding price equation reads

(2.4) $e = e(A + [1/Mu_1]B) = eC$.

(Henceforth we shall assume that units are such that e = (1, 1,..., 1) is the equilibrium price vector.) Rewriting (2.1) gives

$x = 1/Mu_1(I - A)^{-1}Bx = [1/Mu_1]Dx$.

We shall focus on D. Generally D can be written as the following sum:(n1)

(2.5) [Multiple line equation(s) cannot be represented in ASCII text]

with the Mu_i the n eigenvalues (again arranged in descending order) and the D_i the corresponding constituent matrices of D. Our paper aims at showing that under appropriate conditions D closely approximates matrix Mu_1D_1.

We shall employ special matrices, the columns of which represent overall system characteristics (see also section 6 for an observation on this type of matrices). Let us denote the standardized output vector (the right-hand eigenvector of C) as x (so that ex = 1). Then, using the fact that the right- and left-hand Frobenius eigenvectors of D are proportional to, respectively, x and eB, we have

(2.6) $D_1 = [1/(eBx)][x(eB)]$

The columns of D_1 thus are proportional to x; eB clearly is the vector of column sums of B. Being the outer product of two vectors, D_1 clearly has rank one. As we shall see, it will be useful to analyze D_1 in terms of matrices involving the column sums of A and B. To this end, therefore, let us introduce the following rank one matrices:

(2.7) $A = x(eA)$

and

(2.8) $B = x(eB)$

The columns of A are proportional to x, while its column sums are equal to those of A. The same is true concerning B and B. Total output x now can be written in terms of A and B:

$x = (A + [1/Mu_1]B)x$

Because (I - A) is nonsingular, we may rewrite to

$x = [1/Mu_1](I - A)^{-1}Bx.$

Writing D equivalent to $(I - A)^{-1}B$ and eAx equivalent to Alpha for short, and with ex = 1 as further standardization, we have

Proposition 1: $D = Mu_1 D_1$

Proof: $D = (I - A)^{-1}B = (I - A)^{-1}[x(eB)]$

$= (I + A + A^2 + \ldots)[x(eB)]$

$= (I + [x(eA)] + [x(eA)]^2 + \ldots)[(eB)]$

$= [x(eB)] + [eA)][x)eB)] + x(eA)]^2[x(eB)] + \ldots$

$= [x(eB)] + Alpha[x(eB)] + Alpha^2[x(eB)] + \ldots$

$= (1 + Alpha + Alpha^2 + \ldots)[x(eB)] = [1/(1- Alpha)][x(eB)].$

From (2.4) we have Alpha + $[1/Mu_1]eBx = eCx = ex\ 1$, so 1/(1-Alpha) = $Mu_1/(eBx)$. This gives

$D = [Mu_1/(eBx)][x(eB)] = Mu_1 D_1.$

Now we look for a convenient expression containing both D, D, A and B. To this end, let us introduce `rest' or 'remainder' matrices X and Y defined as

(2.9) X equivalent to $(I - A)^{-1} - (I - A)^{-1}$,

and

(2.10) Y equivalent to B - B.

Then

$D = (I - A)^{-1}B$

$= [(I - A)^{-1} + X][B + Y]$

$= (I - A)^{-1}B + (I - A)^{-1}Y + XB + XY$

With $(I - A)^{-1}B = Mu_1 D_1$, and taking the last two terms of the right-hand side together, we have that D may be written as the sum of the rank one matrix $Mu_1 D_1$ and two terms involving X and Y:

(2.11) $D = Mu_1 D_1 + [(I - A)^{-1}Y + XB].$

Because apparently

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we shall concentrate on the eigenvalues of the second term on the right-hand side of (2.11). Before presenting the main theorem, we shall first discuss a number of numerical aspects of the system matrices. Although some of these have received attention before in the literature, they have not--as far as we know--been placed in the context of the stability problem.

3. Quantitative Aspects of Matrices A and B

As we know from previous research in this area, in periods of growth, outlays on net investment will range from close to zero to a maximum of, say, 8-10% of sectoral outlays in periods of exceptionally rapid growth. This implies that outlays on intermediate inputs (as defined in this model, i.e. including household consumption) will be in the range of 90% or (much) more of aggregate outlays. If, in line with celebrated theories of economic growth, we would assume a tendency to profit rate equalization, this would immediately provide us with a quantitative estimate of the dominant eigenvalue of A: It would have to be put in the range from 0.9 to (much) higher values, which seems in accordance with empirically observed magnitudes. On the other hand, from $e(A+[1/Mu_1]B) = e$, we then have that the Frobenius eigenvalue of $[1/Mu_1]B$ should be at its very highest in the neighborhood of 0.1, with high probability of a much lower value. So eigenvalues of matrices A and $[1/Mu_1]B$ may be expected to differ by a factor 10 to 20 or even more.(n2)

Regarding matrix Y equivalent to (B - B), it will be interesting to take a look at the diagonal elements $(b_{ij} - b_{ij})$ and at the summation

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Concerning B, from above we have that in a situation of equilibrium, column sums of the capital matrix may be expected to be in the neighborhood of unity or somewhat less. Typically there are only a few sectors accounting for most of the outlays on capital. Therefore, the number of cases where outlays on a particular type of investment good account for more than, say, one-quarter to one-third of sector j's total outlays on capital goods cannot be large; necessarily, the number of sectors occupying such a large part of $Sigma_i b_{ij}$ must be limited to two or three. This also means that outlays on capital products generated by most of the sectors must be significantly smaller.

Columns of matrix B, on the other hand, reflect the overall production proportions in the system, i.e. current and capital outlays lumped together. Column sums of matrices B and B being the same, elements b_{ij} corresponding to typical non-capital providing sectors may be expected to be larger than the corresponding elements of b_{ij}, and elements of b_{ij} corresponding to typical investment goods producing sectors may be expected to be smaller than the corresponding elements of b_{ij}. Furthermore, magnitudes of the b_{ij} will also depend on n, the number of sectors distinguished. If n equals, say, 10 or 15, as is often the case, then an element b_{ij} accounting for more than some 10% of the column total already must be considered quite large. For larger n, this figure must be put even lower.

This means that absolute differences between diagonal elements b_{jj} and b_{jj} may be expected to be smaller than one-quarter to one-third of column totals. Absolute differences between other elements b_{ij} and b_{ij} in the jth column will also range from zero to, roughly, the same order of magnitude for cases of relatively large differences. However, for the reasons given above, the number of such relatively big differences must be limited. On the other hand, there are quite a few sectors whose outlays on intermediate inputs are significantly larger than the outlays on capital goods. Elements b_{ij} corresponding to such sectors will be larger than the corresponding b_{ij}. But again, absolute differences between coefficients will not be large, the number of (very) large elements of b_{ij} being limited by n. All in all, an estimate for

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relatively far above unity seems unlikely. But that means, using a result of Gerschgorin,(n3) that the eigenvalues of (B - B) are centered within a disk with center close to the origin and radius about unity.(n4) Naturally, in practice, much more information is available on the capital formation parameters, but the above will suffice for our purposes.

A further point that needs discussion concerns the relation between the elements of matrices A and $[1/Mu_1]B$. The closure of the model implies that besides the usual intermediate deliveries (including outlays for the maintenance of fixed capital), also endogenized final demand (with the sole exception of the net addition to productive fixed capital), i.e. private consumption, net exports, various excluded kinds of stocks, investment in housing and government demands, is included in the A matrix. This causes a massive `blow-up' of the matrix of intermediate input coefficients compared to the original one composed only of `pure' interindustry intermediate coefficients. Taking also in account our earlier observations regarding further quantitative aspects of matrices A and B, this implies that we may consider C in

(2.3) C equivalent to A + $[1/Mu_1]$B,

as a perturbation of A. That is, we may proceed by considering C as a perturbed matrix, $[1/Mu_1]$B as a first-order perturbation, and A as the unperturbed matrix. (See Deif, 1982, ch. 6, on general characteristics of this kind of problem. See also Saaty, 1980, ch. 7-7, for observations on orders of magnitude involved.) A being unperturbed, the unperturbed or zero-order eigenvalue problem is (i = 1, ..., n):

(3.1) Ax_{ai} = $Alpha_i x_{ai}$, $y_{ai}A$ = $Alpha_i y_{ai}$

where we again assume all eigenvectors to be bi-orthonormalized and the eigenvalues ordered according to size. The accompanying perturbed eigenvalue system is:

(3.2) Cx_{ci} = $Gamma_i c_{ci}$; y_{ci} = $Gamma_i y_{ci}$

where the $Gamma_i$ are the perturbed eigenvalues, and the x_{ci} and y_{ci} the corresponding perturbed right- and left-hand eigenvectors, bi-orthonormalization again being assumed. (Note that x_{c1} = x and y_{c1} = e.) Perturbation theory tells us that we have

(3.3) $Gamma_i$ = $Alpha_i$ + $Alpha^{*, \text{ sub } i}$; x_{ci} = x_{ai} + $x^{*, \text{ sub } ai}$; y_{ci} = y_{ai} + $y^{*, \text{ sub } ai}$,

where $Alpha^{*, \text{ sub } i}$, $x^{*, \text{ sub } ai}$ and $y^{*, \text{ sub } ai}$ are the first-order perturbations which, if required, can readily be obtained via established techniques in terms of the solutions to (3.1).(n5)

Consequences for the eigenvalue problem of A are straightforwardly obtained. By construction, the right-hand Frobenius eigenvector x_{a1} of A is a scalar multiple of x_{c1}, the right-hand Frobenius eigenvector of C. Thus x_{a1} is a perturbation of x_{a1} as well. Analogous arguments are valid regarding the left-hand eigenvector y_{a1} of A; its elements being the column sums of A, this vector is a perturbation of e, the left-hand eigenvector of C, and hence, of y_{a1}. In addition, because the column sums of A are equal to the column sums of A, we have that the dominant eigenvalue of A has the same lower and upper bounds as $Alpha_1$, the dominant eigenvalue of A. Denoting the variables associated with A by an overbar, we thus have

(3.4) $Alpha_1$ = $Alpha_1$ + $Alpha^{**, \text{ sub } 1}$; x_{a1} = x_{a1} = $x^{**, \text{ sub } a1}$; y_{a1} = y_{a1} + $y^{**, \text{ sub } a1}$

the x_{a1} and y_{a1} again bi-orthonormalized, the double asterisk denoting the perturbations.

It will be useful to devote a few words to the order of magnitude of the perturbation. It is known (see, e.g., Saaty, 1980, ch. 1) that a good approximation of the Frobenius eigenvector is obtained by dividing the elements of each column by the sum of that column, adding the elements in each resulting row, and finally dividing this sum by the order of the matrix. This fact enables us to estimate the order of magnitude of the elements of $x^{*, \text{ sub } a1}$. We know that column sums of matrix C are unity. We have also seen that capital coefficients exceeding unity already will be exceptionally large. This means that the elements of $[1/Mu_1]$B will be about 0.1 or smaller. If all entries in a certain row of B would be particularly large, say about unity, the corresponding row sum of $[1/Mu_1]$B would be about (0.1)n or less. The first-order perturbation of the eigenvector of A therefore would approximately be [(0.1)n]/n = 0.1. This implies that differences between the elements of x_{a1} and x would, maximally, be about 0.1.

Concerning the left-hand eigenvector a similar argument is available. (See earlier remarks on the level of $[1/Mu_1]$B). Here an additional tendency may be present. The perturbation approach yielded that eA approximately equal to $Alpha_1$e and x_{a1} approximately equal to x_{a1}. These approximations can be given additional support if we accept the classical position that a tendency for profit rate equalization is present in systems of the kind we are studying. Empirical work in any case seems to support this; dispersion among the elements of eA (and hence of eA) appears to be small.(n6) Observations about the size of annual new capital formation may lend additional, though probably less marked, support to the approximation of the right-hand eigenvector. The result, therefore, can be expected to be rather robust.

4. The Explanation of Instability

Matrix A can be written (see footnote 1) as the sum:

(4.1) $A = Alpha_1A_1 + Alpha_2A_2 + ... + Alpha_nA_n$

where the $Alpha_i$ are the eigenvalues of A (ordered according to decreasing value), and the A_i its constituent matrices. Below we shall first derive a result on the norms of the A_i matrices.(n7) Again denoting the right- and left-hand eigenvectors corresponding to $Alpha_i$ by x_{ai} and y_{ai}, we have:

Proposition 2: Norms of the constituent matrices Ai are of the same order, and near unity.

Proof: We know that the products v_iw_i of the unit length eigenvectors

w_i equivalent to $x_{ai}/|x_{ai}|$ and v_i equivalent to $y_{ai}/|y_{ai}|$

are of the same order of magnitude; see, e.g., Saaty (1980,pp. 193-195). To establish the order of magnitude of the products, let us consider the product v_1w_1, associated with the Frobenius eigenvalue of A. Both vectors being positive, v_1w_1 is seen to reach its maximum at unity if all elements of w_1 and v_1 are equal. Because elements of w_1 are not very small (sectors accounting for only a very small portion of production are not considered) and because of the tendency towards equalization of column totals, v_1w_1 cannot be far from unity. All v_iw_i being of the same order of magnitude, they must be of the order of v_1w_1, i.e. near unity (see further Sasty, 1980, on similar issues). Considering now the matrices

A_i equivalent to w_iv_i,

we see that $||A_i|| = 1$, because

$tr(A'_iA_i) = ||A_i||^2 = (v_iv'_i) = 1$.

Also, $A_i = [1/(v_iw_i)]A_i$. Thus

$||A_i||$ </= $| 1/(v_iw_i) |$ $||A_i||$ </= $| 1/(v_iw_i)|$.

It follows that also $||A_i||$ is near unity.

Let us now recall the expression for D we derived earlier:

(2.11) $D = Mu_i D + [(I - A)^{-1}Y + XB]$.

With symbols as before, we now have the following result regarding the second part of the right-hand side of (2.11):

Proposition 3: Let $x = (A + [1/Mu_1]B)x$ and $e = e(A + [1/Mu_1]B)$. Let further $[1/Mu_1]B$ be a perturbation of A. Then $(I - A)^{-1} Y + XB$ approximately equal to $H(B - B)$, with

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Proof: (a) Recalling that x has the proportions of the Frobenius eigenvector of C, we have

$(I - A)^{-1} = I + A + A^2 + ...$

$= I + x(eA) + x(eA)x(eA) + ...$

From the above we have that e, the left-hand Frobenius eigenvector of C, is a perturbation of y_{a1}, the left-hand (Frobenius) eigenvector of A, implying eA approximately equal to $Alpha_1e$. From ex = 1, we now obtain

$(I - A)^{-1}$ approximately equal to $I + Alpha_1xe + Alpha_{2, sub 1}xe + ...$

$= I + [Alpha_1/(1- Alpha_1)]xe$.

(b) Concerning the second term of D, we then have

$(I - A)^{-1}$ Y approximately equal to (I + [Alpha$_1$/(1-Alpha$_1$)]xe)Y. As eY = 0 by definition, we obtain $(I - A)^{-1}$Y approximately equal to Y.

(c) Concerning matrix X we have

$X = (I - A)^{-1} - (I - A)^{-1}$

approximately equal to (I + A + A^2 + ...) - (I + [Alpha$_1$ (1- Alpha$_1$)]xe)

= ([1/(1- Alpha$_1$]A_1 + [1/(1- Alpha$_2$)] A_2 + ... + [1/(1- Alpha$_n$)]A_n)

- (I + [Alpha$_1$/(1- Alpha$_1$)]xe)

= ([Alpha$_1$/(1- Alpha$_1$)]A_1 + [Alpha$_2$/(1-Alpha$_2$)]A_2 + ...+ [Alpha$_n$/(1- Alpha$_n$)] A_n)

x and e being perturbations of x_{a1} and y_{a1}, the right- and left-hand Frobenius eigenvectors of A, we have A_1 approximately equal to xe. Consequently [Alpha$_1$/ (1- Alpha$_1$)]A_1 -[Alpha$_1$/(1- Alpha$_1$)] xe approximately equal to 0. (d) Therefore,

XB approximately ([Alpha$_2$/(1- Alpha$_2$)]A_2 + ... + [Alpha$_n$/(1- Alpha$_n$)]A_n)B.

Continuing, from A_iB = A_i(xeB) approximately 0 (i = 2, ..., n), implying XB approximately equal to 0, we have XB approximately equal to X(B - B). Combining (a) to (d), we now have that

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Now let us see what we can say about parameter values in the above formula for D. We have already discussed Mu$_1$ and matrix (B - B). Realistic estimates for the Alpha$_1$ are also readily available. As we have seen, in the closed model, matrix A is obtained as the sum of the matrix of (interindustry) intermediate input coefficients and a matrix representing the endogenization of the household consumption bundle. Thus, A is the sum of the original matrix of interindustry coefficients and a positive matrix of rank one. As under the present-day classification schemes the consumption bundle easily accounts for one-half or more of the aggregate intermediate deliveries, also the bounds for the Frobenius eigenvalue of the present A matrix, compared with the earlier interindustry matrix, will increase by that order of magnitude. As we have seen earlier, our matrix A may be expected to have a Frobenius eigenvalue of 0.9 or more. In view of the construction of A, an acceptable first estimate for the subdominant eigenvalue, a2, then would be about one half of this value. Furthermore,

(4.2) [Multiple line equation(s) cannot be represented in ASCII text]

But

(4.3) A_1(B - B) approximately (x_{a1}e)(B - B) = 0,

so

(4.4) (A_2 + ... + A_n)(B - B) approximately equal to (B - B).

Thus premultiplication by (A_2 + ... + A_n) does not increase the elements of (B - B). In the second term of matrix H, the weights of the A_i are all less than unity. In view of the values of the Alpha$_i$, and the norms of the idempotent matrices A_i being near unity, we know that the columns of matrix (B - B) will not be amplified by premultiplication by this (second) term: the norm of the term

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will be of the same order of magnitude as the norm of (B - B).(n8)

In view of the maximum size of the subdominant eigenvalue Alpha$_2$, and the elements of (B - B) being (very) small as we had observed earlier, we have that the matrix forming the second term on the right-hand side of (2.11) must have very small eigenvalues. These eigenvalues being the remaining (n - 1) eigenvalues of D, we therefore conclude that D must have one big eigenvalue and (n - 1) very small ones, i.e. D approximately equal to $Mu_1 D_1$.

Finally, of course, we have to consider the case where growth is in the neighborhood of the balanced growth path and where the price system is near equilibrium. Ceteris paribus, further application of the continuity properties of eigenvectors and eigenvalues leads to the conclusion that D again will have a spectrum of one large positive eigenvalue and (n -1) very small ones. (Because the proof is nearly identical to the proof of Proposition 3, we shall leave the proof to the reader.) This of course means that again relative instability will characterize the forward looking system. Depending on the exact size of the (n - 1) small eigenvalues, complete instability may be the case. Before concluding this paper, however, let us take a further look at empirical work.

5. Empirical Results

By now a number of detailed studies on the closed dynamic Leontief model are available. In obtaining the relevant coefficient matrices, a variety of techniques has been used. For example, in constructing the capital matrix B, one may employ building permits and investment survey data, as well as customs data on machinery and equipment imports, in combination with detailed information regarding final destinations. Another method to obtain estimates of the matrix is based on observed patterns of capital use in recent investment projects. To these methods we may add the method of turnover times developed by Brody (1970). Regarding the matrix of input coefficients A, the literature describes a great number of techniques to estimate the required coefficients. Special methods to endogenize final demands and primary inputs, are discussed in Tsukui (1968) and Meyer & Schumann (1977).

By whichever methods the estimates of the coefficient matrices A and B have been obtained, provided they offer a reasonably accurate picture of the entire economy, they can be employed to provide a first, limited test for our framework. Necessary, in addition, is that the total output vectors for the two consecutive years on which the estimates have been based, are approximately similar. That is, the period of estimation should have been one of relatively smooth growth. In that case, as mentioned at the end of the previous section, we have x(t + 1) approximately equal to $(1+[1/Mu_1])s(t)$, and the Brody model may be applied. The studies we have investigated all seemed to satisfy this requirement.

Not all investigations based on dynamic Leontief models were sufficiently detailed to suit our purposes. However, in a number of cases, either the complete matrices A and B had been published, or they could be reconstructed by us. In a few cases, the complete spectrum of eigenvalues has been published. Table 1 contains the spectra of eigenvalues of matrices D of the models presented in Leontief (1953), Tsukui (1968) and Meyer & Schumann (1977). As can be seen, the spectra closely approximate the spectrum of a positive matrix of rank one. Table 2 gives the spectra of eigenvalues of matrices A and B of Tsukui's detailed model. This table has been included to illustrate that these matrices cannot be satisfactorily approximated by a matrix of rank one. As we have seen already, in the footnotes further calculations have been given.(n9)

We have only discussed the Brody equilibrium form of the model. If this particular kind of equilibrium is not present, possibly because the economy is far from equilibrium (or simply because inaccurate data have been used), results may be expected to be less clear-cut. In such cases, perturbations will be larger and one may expect, for instance, the subdominant eigenvalue of D to be rather large, say 2 or 3. This, however, would not affect the overall conclusions of this study: The system would still be unstable, though complete instability may not be the case.

6. Additional Remarks and Conclusion

A substantial part of the literature on the dynamic Leontief model has been devoted to discussing stability conditions for the various versions of the model; stability in the forward-lag version, e.g. required (1.3) to be satisfied. Results on what to actually expect if a particular version was implemented were not available, however.

In this paper we have shown that under economically easily interpretable conditions (such as conditions on the yearly net investment outlays or the size of the household consumption basket), matrix D equivalent to $(I - A)^{-1}B$ can be written as the sum of a positive rank one matrix, having Mu_1 as its (only) nonzero eigenvalue, and a function of another matrix, (B - B), with--on average--very much smaller elements. As the remaining (n - 1) eigenvalues of D are the eigenvalues of this second matrix, our approach thus enabled us to obtain a quantitative estimate of the complete set of eigenvalues of D. As our conditions seemed to be satisfied during the periods for which a number of models were implemented, this immediately explained why all empirically implemented forward-lag models were found to be unstable. It also immediately explained why backward-lag accelerator models and even Leontief's Dynamic Inverse (although that model is an open one), with their radically different stability zones, are normally stable (Kigyossy-Schmidt, 1981). The rank of the capital matrix was found to be irrelevant here.

The paper also illustrates the need for theoretical reinforcements. For example, the incorporation of certain 'stylized facts' about numerical magnitudes, much like the use of a priori information in simultaneous equation econometrics, should be given priority. For instance, we should put more effort in 'explaining' the 'normal' values of the eigenvalues of the intermediate and capital coefficient matrices. As part of this, we also should put more effort in knowing if a certain estimated coefficient (especially in the capital matrix) is `normal', 'large' or simply 'too large' to be convincing. Also more results should be available on topics such as the behavior of the Frobenius eigenvalue of sums or other functions of matrices. We have already seen that specific elements of capital theory, such as trends at equalizing profit rates, can be easily incorporated. Such inclusions, naturally, might provide alternatives to using perturbation analysis.

Furthermore, it may be worthwhile to try to explain the appearance of the rank one matrices we have employed (such as $D = Mu_1 D_1$) in terms of (functions of) 'infinite order' input coefficients, as described in Steenge (1986) (although in a quite different context). Let T stand for the coefficients matrix of a static closed Leontief model. If T is primitive (which usually is the case), matrices T^2, T^3, etc. give us the indirect inputs of second, third, etc. order. Continuing,

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then gives us 'infinite order' inputs where all indirect effects have been accounted for. $T^{Infinity}$, of course, has rank one. Now, the elements of $(I - A)^{-1}B$ are a kind of integrated coefficients themselves. Thus, if some link between D and $T^{Infinity}$ could be established, a further theoretical foundation might well be within reach. Clearly, in this area we virtually are still at the very beginning.

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Notes

(n1) *In spectral decomposition form, a matrix M is written as*

$M = Mu_1 M_1 + Mu_2 M_2 + \dots + Mu_n M_n$

where the Mu_i are the eigenvalues of M and the constituent matrices M_i the outer products of the bi-orthonormalized eigenvectors of M. Properties of the M_i include $Sigma_i M_i = I$, $M^{2, \text{ sub } i} = M_i$, $M_i M_j = 0$ for i not equal to j. For simplicity, we assume here that all roots are simple.

(n2) The highest and lowest column (row) sums are bounds for the Probenius eigenvalue of a nonnegative matrix. As an illustration: Frobenius eigenvalues of matrices A in, respectively, the Tsukui (1968) and Meyer & Schumann (1977) studies, are 0.919 and 0.926. Eigenvalues of the capital matrix B as calculated from the same studies are, respectively, 0.313 and 0.481.

(n3) Every eigenvalue of a matrix A lies in at least one of the circular disks with centres a_{ii} end radius $Sigma_i$, j not equal to i] | a_{ij}|.

(n4) Again as an illustration, we provide the exercise for Tsukui's first sector (agriculture and foods). Sectoral outlays on intermediate inputs (i.e. $Sigma_i a_{ij}$) amount to 0.928. We have, in three digits:

b x_1 = (0.089 0.002 0.002 0.002 0.006 0.002 0.055 0.028 0.002 0.401 0.003 0.026), and

b x_1 = (0.103 0.055 0.031 0.032 0.026 0.067 0.025 0.014 0.023 0.064 0.026 0.152).

Column sums are 0.618 in both cases. We easily find $b_{11} - b_{11} = - 0.014$, and $Sigma_{i, \text{ i not equal to } 1}$ | $b_{i1} - b_{i1}$ = 0.748.

(n5) It is well known that the Frobenius eigenvector is very sensitive to perturbations in A if $Alpha_1$ is close to any of the other eigenvalues (e.g. Saaty, 1980, p. 194). In our case, this will hardly be a problem because $Alpha_1$ is well separated from the other eigenvalues (See below).

(n6) For instance, from Tsukui (1968) and van Schaik (1975) we may calculate, respectively, for column sums:

eA = (0.928, 0.884, 0.934, 0.929, 0.905, 0.940, 0.952, 0.961, 0.784, 0.952, 0.943, 0.917), and

eA = (0.991, 0.986, 0.922, 0.960, 0.953, 0.992, 0.759, 0.998, 0.823, 0.942).

(n7) We employ the least upper bound (lub) of A, i.e.

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That is, lub,(A) measures the largest amount by which any vector is amplified by matrix multiplication. We have

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Lub(A) thus can be calculated as the square root of the largest eigenvalue of A'A.

(n8) This can be seen as follows. Let $Theta_1$ be the angle between vectors y_{a1} and x_{a1} as defined in section 4. From Proposition 2 we have that

$v_1 w_1 = y_{a1} x_{a1} / |y_{a1}||x_{a1}| = \cos(Theta_1) = 1 - Epsilon_1$,

with $Epsilon_1 > 0$ small. This means that $Theta_1$ itself will be relatively small. Therefore, because y_{a1} is perpendicular to all other right-hand eigenvectors of A, the cosine of the angle between x_{a1} and these (n - 1) other right-hand eigenvectors also will be relatively small, implying that this angle is relatively large. Given that the norms of the A_1 matrices are approximately unity, this means that

$|(B - B)| = |(A_1 + A_2 + \ldots + A_{n-1} + A_n)(B - B)|$

approximately equal to $|(A_1 + A_2 + \ldots + A_{n-1} + Theta_n A_n)(B - B)|$,

approximately equal to $|(A_2 + \ldots + A_{n-1} + Theta_n A_n)(B - B)|$,

etc., where $|Theta_n|$ is (much) smaller than unity. For weights attached to the other A' matrices, a similar argument applies.

(n9) A survey of spectra of eigenvalues, calculated from major studies available up to now, can be obtained from the author upon request.

Table 1. Eigenvalues of matrices (I - A)-1B[a]

Country	Real part	Imaginary part
USA	8.333	--
	0.679	--
	0.278	--
	0.190	--
	0.149	--
	0.105	0.024
	0.105	-0.024
	0.076	--
	0.034	--
	-0.005	--
Japan	7.788	--
	0.198	--
	-0.064	0.082
	-0.064	-0.082
	0.074	0.027
	0.074	-0.027
	0.045	--
	-0.033	--

	0.028	--
	0.018	--
	0.003	0.003
	0.003	-0.003
FRG	15.873	--
	0.489	--
	0.300	--
	-0.116	--
	0.081	--
	0.069	0.025
	0.069	-0.025
	0.069	--
	0.013	0.062
	0.013	-0.062
	0.052	--
	-0.024	--

[a] Calculated from Leontief (1953), Tsukui (1968) and Meyer & Schumann (1977), respectively.

Table 2. Characteristic roots of matrices A and B for Japan

Real part	Imaginary part	Modulus
Matrix A		
0.919	--	0.919
0.456	--	0.456
0.391	--	0.391
0.378	--	0.378
0.335	--	0.335
0.209	--	0.209
0.158	0.053	0.167
0.158	-0.053	0.167
0.163	--	0.163
0.057	--	0.057
0.042	--	0.042
0.007	--	0.007
Matrix B		
0.353	--	0.353
-0.053	0.098	0.111
-0.053	-0.098	0.111
0.096	--	0.096
-0.039	--	0.039
0.042	0.025	0.049
0.042	-0.025	0.049
0.002	0.003	0.004
0.002	-0.003	0.004
0.024	0.002	0.024
0.024	-0.002	0.024
0.015	--	0.015

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