A New Approach to Irreducibility in Multisectoral Models with Joint Production

ABSTRACT We present a new approach to irreducibility in multisectoral models, based on an axiom on the vectors of total outputs and inputs. We derive new results in joint production models, including an interpretation of von Neumann's famous condition A + B > 0, where A and B are respectively matrices of input and output coefficients. Also, an application of the axiom to the commodity technology model is presented.

1. Introduction

A number of authors have recently discovered the applicability of a theorem of Mangasarian (1971), which can be interpreted as a condition on input and output vectors, in multisectoral models with joint production. We mention Punzo (1980), who applies it to the Turnpike theorem, Semmler (1984), who applies it to labour economics, and Fujimoto & Krause (1988), who apply it to distribution problems. In this paper, we apply Mangasarian's results primarily to the issue of irreducibility.

Multisectoral models with joint production can generally be described by two coefficient matrices: an input matrix A and an output matrix B. In Section 2, we formulate the above-mentioned condition on the vectors of total outputs and inputs which will consequently be used as an additional axiom in the model. This axiom can be seen as a generalization of a fundamental property of single-product models, namely the non-negativity of the coefficient matrix.

The mathematical result of Mangasarian (1971) that we referred to above then tells us that, if the axiom holds, there exists a third non-negative matrix which we will call M that connects A and B through A=MB. This matrix M will be the fundamental tool for analysis in this paper. In Section 3 we use it to derive a condition that guarantees irreducibility of the general multisectoral system (A, B).

In sections 4 and 5 we apply it to two particular types of model, namely the von Neumann model (von Neumann, 1945--46) and the commodity technology model, a generalized input-output model (United Nations, 1968). In the former we establish a link between the famous von Neumann condition(n1) A + B > 0 and the concept of irreducibility, a link that so far has not been available.

2. Multisectoral Models

Typical of a multisectoral model with homogeneous production is that it can be represented by a non-negative coefficient matrix. Examples of these models are those of Sraffa (1960) and the Leontief input-output model (Leontief, 1951). The non-negativity of the coefficients is fundamental in these models. In Sraffa's model it is used to prove the existence of the Standard Commodity; in Leontief's model it is used for proving the existence and non-negativity of the Leontief inverse.

If we denote the coefficient matrix as A (where A is an m x m matrix, with m the number of commodities), then non-negativity of A is equivalent to

$x > 0 \quad Ax > 0$

provided that each row of A contains at least one non-zero element (if there is a zero row the corresponding commodity is irrelevant and can therefore be deleted). This fundamental property has the following interpretation: if we consider x to be a vector of changes in output levels, then a rise in the production of all commodities implies a rise in the use of all commodities. Note that this property implies nothing about the (ir)reducibility of the matrix A.

A multisectoral model that explicitly incorporates joint production should distinguish between commodities and activities. We shall distinguish m commodities and n activities. Such a model then typically consists of two matrices. A is the input matrix (of dimensions m x n), an element a_{ij} of which represents the use of commodity i in activity j if this activity operates at unit level. B is the output matrix (also sized m x n), and b_{ij} represents the production of commodity i in activity j when operating at unit level. Examples of such models are the Sraffa model with joint production (see Sraffa, 1960), where the problem becomes the existence of the Standard Commodity, the yon Neumann model (von Neumann, 1945--46), which studies the growth rate of the economy (see

Section 4), and the commodity technology model (United Nations, 1968), a generalized input-output model (see Section 5).

Now in these models problems arise because of the increased mathematical difficulties. In our view this is primarily due to the fact that the non-negativity property of the standard single-product models does not yet have an analogue in the extended models. We shall therefore introduce a condition that generalizes the nonnegativity property for joint production models. It is presented here as an axiom and we retain it throughout this paper to investigate the consequences of imposing the axiom on our models.

Axiom: x: Bx > 0 Ax > 0

The vector x may contain negative elements and it should therefore be interpreted as a vector of changes in activity levels. The economic interpretation of the axiom is the same as the interpretation of the fundamental property of single-product models described above: an increase in the production of all commodities implies that the use of all commodities should be increased. Again, nothing about irreducibility is implied yet.

The following proposition, which is due to Mangasarian (1971), tells us that the axiom is equivalent to the existence of a third non-negative matrix.

Proposition 1. For arbitrary matrices A, *(This character cannot be converted in ASCII text) the following equivalence holds, assuming that {x: Bx>0} ≠ Ø:

(x: Bx>0 Ax>0)*(This character cannot be converted in ASCII text) M≥0, i: $M_{i.}$≥0: A = MB

Proof. See Mangasarian (1971), theorem 3.1, part (vii).

Thus, a third matrix is involved. This matrix M can be seen as the generalization of the coefficient matrix of the single-product models. It is non-negative and has no zero rows, i.e. all commodities are used in some process.(n2) In Section 3 we shall give an application for general multisectoral models concerning the irreducibility of these models.

3. Irreducibility

The system (A, B) is called reducible (see Gale, 1960) if there exist permutation matrices P(m x m) and Q(n n) such that

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(1 ≤ k < m, 1 ≤ l < n), while each row of B_{11} contains at least one positive element. If such a permutation does not exist the system is called irreducible. A reducible system has a subsystem that can produce 'on its own'. Sraffa's notion of basic and non-basic products is related to the concept of reducibility. If m = n, k = l and B = I, the above definition reduces to the standard definition of irreducibility of a square matrix.

We now present our main result about the property of (ir)reducibility and the properties of M.

Proposition 2. Let the axiom be fulfilled, such that there exists a matrix M ≥ 0 with A = MB. Let this M be irreducible. Then the system (A, B) is irreducible.

Proof. Suppose to the contrary that (A, B) is reducible. Then there exist permutation matrices P and Q such that

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(1 ≤ l < n, 1 ≤ k < m) and $B_{11,\, i.}$ ≥ 0 (i). Now take

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From A = MB it follows that Â = PAQ = PMBQ = PMP'PBQ = MB and thus Âx = MBx, which can be written as

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Because $B_{11}x_1>0$ this implies $M_{21} = 0$. This contradicts the assumption of the irreducibility of M. Thus (A, B) must be irreducible.

If M is irreducible, it is impossible to find a system (A, B) such that the model become reducible. Thus reducibility of M is a prerequisite for the reducibility of (A, B). However, it is not sufficient. A simple counterexample is

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The following proposition gives a sufficient condition for (A, B) to be reducible when M is reducible.

Proposition 3. Let the axiom be fulfilled, such that there exists a matrix $M \geq 0$ with $A = MB$. Let this $M \geq 0$ be reducible, such that

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with P a suitable permutation matrix. Then the system (A, B) is reducible if there exists a vector $z \geq 0$ such that

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Proof.

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Thus, for this z we can produce a subset of goods without the use of goods from outside the subset. The system then must be reducible.

These proofs do not make use of the structure of any model in particular, except that it is represented by an input matrix and an output matrix. Therefore, the results hold for all models that can thus be described. The reducibility concept is in particular of great importance in the yon Neumann model. We discuss this in Section 4.

4. The von Neumann Condition

In the von Neumann model we have, besides A and B, a vector x which is the activity level vector and p' which gives the prices of the commodities. The real side of the von Neumann model can be written as

max α

such that $Bx \geq \alpha Ax$, $x \geq 0$

with α the growth factor of the economy. The dual price side is

min β

such that $p'B \leq \beta p'A$, $p' \geq 0$

with β the profit rate of the economy.

One of the central questions in the literature concerning the yon Neumann model is the equality of the growth rate α and the profit rate β. Two conditions ensuring this equality coexist, without any relation between them. yon Neumann imposed his famous condition $A + B > 0$ to guarantee that α would equal β. This approach is unsatisfactory from an economic point of view, since yon Neumann's condition is clearly not plausible. Gale (1960) and Kemeny et al. (1956) gave relaxations of von Neumann's condition. They impose the following, easily interpretable, conditions:

$A_{.j} \geq 0$ for all j, $B_{i.} \geq 0$ for all i

This guarantees that optimal values of α and β exist, and $\alpha \geq \beta$. If the system is irreducible we have $\alpha = \beta$.

However, the yon Neumann condition does not imply irreducibility. Robinson (1973) gave the following counterexample:

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Clearly, this system is reducible but satisfies the yon Neumann condition. Therefore, there is so far no relation between yon Neumann's condition and the condition of irreducibility of Gale et al. We shall show now that if our axiom is fulfilled we are able to establish a link.

Proposition 4. Suppose the von Neumann model (A, B) satisfies the axiom. Then if A + B > 0, the system (A, B) must be irreducible.

Proof. Since (A, B) satisfies Bx > 0 Ax > 0 there exists a matrix M ≥ 0, with $M_{i.}$ ≥ 0 for all i, and A = MB. Now, suppose to the contrary that (A, B) is reducible. Choose P, Q and x as in the proof of Proposition 2. From A + B > 0 we have Â + B > 0 and hence B_{21} > 0. Then

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which is clearly not a positive vector. But from A = MB we have again Â = MB with M = PMP'. From. M_i≥0 and Proposition 1, we have that Bx > 0 implies Â > 0. A contradiction thus exists, and (A, B) must be irreducible.

So we have established a link between the yon Neumann condition and irreducibility. However, we do not need the yon Neumann condition to prove irreducibility in the context of the axiom we use, as was shown in the previous section. The results of Section 3 can be viewed as an alternative relaxation of von Neumann's condition: if the axiom is accepted, the irreducibility of M is sufficient for the equality of the rates.

Another application will now be given, which deals with an upper bound to the growth rate of the economy. We give a relation between α and, μ_M, the PerronFrobenius eigenvalue of M. To save space, we present it without proof.(n3)

Proposition 5. Let the axiom be fulfilled. Let Bx ≥ αAx, with α > 0, x ≥ 0. Then α≤ 1/μ_M with μ_M ≠ 0 the Perron-Frobenius eigenvalue of M.

5. The Commodity Technology Model

The above theorems are very theoretical in nature; there exist only a small number of practical applications of yon Neumann's model (see, for example, Stolwijk, 1987). The implications of Mangasarian's theorem, however, may also be of use in the more practical world of national accounting and input-output analysis.

United Nations (1968) (see Stone (1984) for a survey) introduced the make-use framework, a system of two tables, one for the production of commodities by industries and one for the (intermediate) consumption of commodities by industries. U is the use matrix: element % describes the input of commodity i in the production of industry j. V' is the make matrix: element v_{ij} describes the production of commodity i by industry j. Furthermore, we have the vector e of final demands for commodities, the vector y of primary inputs into the industries, the vector q of total production of commodities and the vector g of total production of industries.

From the make matrix we can derive an output (coefficient) matrix B, an element b_{ij} of which represents the output share of commodity i in the production of industry j:

$B = V'(g)^{-1}$ g = diag(g)

From the use matrix we obtain similarly an input matrix A, an element a_{ij} of which gives the input share of commodity i in the production of industry j:

$A = U(g)^{-1}$

The commodity technology model assumes that each commodity has its own characteristic input structure. This can be given effect by combining the two (empirical) matrices A and B to derive an input-output matrix M as $M = AB^{-1}$ (here it should be assumed that A and B are square and that B is non-singular). Each column of M gives the characteristic input structure of the corresponding commodity. The matrix M, connecting A and B, which we introduced for general multisectoral models, therefore has a well-known meaning in input-output modelling. Now, imposing our axiom on this model yields a generalization of the fundamental property of the classical Leontief model, as described above. It yields, through Proposition 1, a non-negative matrix M (with Mi. ≥ 0 for all 0 such that A = MB. It thus appears that our axiom give additional grounds for the commodity technology approach. Any negative element of M found in empirical work is in contradiction with the axiom. This suggests another way of interpreting these negatives: they should not be regarded as rejecting the commodity technology assumption, but as indicators of flaws in the make and use tables. For a more detailed discussion on these issues see Steenge (1990).

6. Conclusions

In this paper we introduced an axiom for general multisectoral models with joint production. It is argued that it is necessary to impose such an axiom in order to extend results of our simple models with homogeneous production. Our major focus was on irreducibility, a subject of importance in most multisectoral models. We have given a new sufficient condition for irreducibility.

Furthermore, we have given a context for the yon Neumann model in which a clear connection exists between von Neumann's condition and irreducibility, and we have shown that the axiom gives an upper bound to the growth rate of the economy.

Another application of the axiom is for the commodity technology model. Here it gives a guarantee for the non-negativity of the coefficients. Thus the applications we have given illustrate the fact that the axiom gives new directions to solutions to long existing problems, and therefore deserves more attention than it has received in the literature so far.

Notes

(n1.) The following notational conventions are used.

A ≥ 0: all elements of A are non-negative A ≥ 0: all elements of A are non-negative, and some element is positive A > 0: all elements of A are positive

Similar conventions are used for vectors.

(n2.) Punzo (1980) formulates Mangasarian's theorem in terms of the polyhedral cones defined by the matrices A and B. This is just the geometrical way of presenting the economic condition which we present as an axiom. We prefer our algebraic way of putting it, because of the clearer economic understanding.

(n3.) The proof can be obtained from the authors upon request.

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