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A NOTE ON RAMSEY NUMBERS FOR FANS

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Abstract

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 as a subgraph or the complement of G contains G_2 as a subgraph. A fan F_l is l triangles sharing exactly one vertex. In this note, it is shown that $R(F_n, F_m) = 4n + 1$ for $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$.

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1. Introduction

In this note we deal with finite simple graphs only. Let $G = (V(G), E(G))$ be a graph. For $S \subseteq V(G)$, we use $N_S(v)$ to denote the set of the neighbours of a vertex v that are contained in S, $N_S[v] = N_S(v) \cup \{v\}$ and $d_S(v) = |N_S(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. The maximum and minimum degrees of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Denote by $G[S]$ and $G - S$ the subgraphs induced by S and $V(G) - S$, respectively. For two vertex-disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes their disjoint union and $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2. We use mG to denote the union of m vertex-disjoint copies of G. A complete graph of order m is denoted by K_m. A star S_n is $K_1 + (n - 1)K_1$ and a fan F_n is $K_1 + nK_2$.

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 as a subgraph or $\overline{G}$ contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. Chvátal and Harary [2] constructed a general lower bound which often yields the exact values of $R(G_1, G_2)$. That is, $R(G_1, G_2) \geq (|V(G_1)| - 1)(\chi(G_2) - 1) + 1$, where G_1 is a connected graph and $\chi(G_2)$ is the chromatic number of G_2. Burr [1] generalised this lower bound by using another parameter $s(G_2)$, called the chromatic surplus of G_2, which is defined as the minimum number of vertices in some colour class under all proper vertex colourings of G_2 by $\chi(G_2)$ colours.

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THEOREM 1.1 [1]. *$R(G_1, G_2) \geq (|V(G_1)| - 1)(\chi(G_2) - 1) + s(G_2)$ for any connected graph G_1 with $|V(G_1)| \geq s(G_2)$.*

Burr defined G_1 to be G_2-good if the equality holds in Theorem 1.1. Based on this definition, one may ask, for a given graph G, which graphs F are G-good? This generated many questions in Ramsey theory and results were established for some special graphs G such as a tree, a cycle, a complete graph and so on. When G is a fan, Li and Rousseau showed that F_n is F_1-good for $n \geq 2$ and obtained lower and upper bounds for $R(F_n, F_m)$ in terms of n and m.

THEOREM 1.2 [4]. *$R(F_n, F_1) = 4n + 1$ for $n \geq 2$; and $4n + 1 \leq R(F_n, F_m) \leq 4n + 4m - 2$.*

Recently, Lin and Li proved that F_n is F_2-good for $n \geq 2$ and improved the upper bound for $R(F_n, F_m)$ in Theorem 1.2.

THEOREM 1.3 [5]. *$R(F_n, F_2) = 4n + 1$ for $n \geq 2$; and $R(F_n, F_m) \leq 4n + 2m$ for $n \geq m \geq 2$.*

Theorems 1.2 and 1.3 say that any F_n with $n \geq 2$ is both F_1-good and F_2-good. For a given $m \geq 3$, can we decide when F_n is F_m-good? Lin *et al.* established an approximate result by using the Erdős–Simonovits theorem.

THEOREM 1.4 [6]. *$R(F_n, F_m) = 4n + 1$ for sufficiently large n.*

It is not difficult to see that F_n is not always F_m-good for $n \geq m \geq 2$. In fact, we can prove that $R(F_n, F_m) \geq 4n + 2$ for $m \leq n < m(m - 1)/2$. Since $m(m - 1)/2 > m$, we must have $m \geq 4$ here. There exist positive integers p, q such that $2n + 1 = pm + q$ and $1 \leq q \leq m$. Let $H = pS_m \cup S_q$ if $q \neq 1$, and $H = (p - 1)S_m \cup S_{m-1} \cup S_2$ if $q = 1$. Since $n < m(m - 1)/2$, $2n + 1 \leq m(m - 1)$ and $p \leq m - 2$. It is easy to check that H is a graph of order $2n + 1$ with $\delta(H) \geq 1$, and that H contains neither S_{m+1} nor mK_2. Let $H' = K_{2n} \cup \overline{H}$. Then H' contains no F_n and $\overline{H'}$ contains no F_m. Thus, if $m \leq n < m(m - 1)/2$, then $R(F_n, F_m) \geq 4n + 2$.

In this note, our main goal is to determine a range of n with respect to m such that F_n is F_m-good for a given $m \geq 3$. Our main result is as follows.

THEOREM 1.5. *$R(F_n, F_m) = 4n + 1$ for $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$.*

REMARK 1.6. Since F_n is not F_m-good for $m \leq n < m(m - 1)/2$, we wonder whether F_n is F_m-good for $n \geq m(m - 1)/2$. If this is true, then we can see that the range $n \geq m(m - 1)/2$ is best possible.

2. Proof of Theorem 1.5

In order to prove Theorem 1.5, we need the following two lemmas.

LEMMA 2.1 [5]. *$R(F_t, sK_2) = \max\{s, t\} + s + t$.*

LEMMA 2.2 [3]. *A bipartite graph $G = (X, Y)$ has a matching which covers every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$, where $N(S) = \bigcup_{v \in S} N_Y(v)$.*

Proof of Theorem 1.5. The lower bound $R(F_n, F_m) \geq 4n+1$ is implied by the fact that $2K_{2n}$ contains no F_n and its complement contains no triangle and hence no F_m. It remains to prove that $R(F_n, F_m) \leq 4n+1$ for $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$.

Let G be a graph of order $4n+1$ with $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$, and suppose to the contrary that G does not contain an F_n and $\overline{G}$ does not contain an F_m. If $\Delta(G) \geq 2n+m$, let x be a vertex with $d(x) = \Delta(G)$ and $H = G[N(x)]$. By Lemma 2.1, either H contains nK_2, which, together with x, forms an F_n, or $\overline{H}$ contains an F_m, which is also a contradiction. Thus, we have $\Delta(G) \leq 2n+m-1$ and $\delta(\overline{G}) \geq 2n-m+1$.

Claim 1. For any vertex v of $V(G)$, $G - N_G[v]$ contains a subgraph H_v which satisfies one of the following conditions:

(1) $H_v = K_{2n-2m+2}$;
(2) $\overline{H_v} = K_3 \cup (2n-2m)K_1$;
(3) H_v is a graph of order $2n-m-l+1$ and at most $3m-2l-3$ vertices in $\overline{H_v}$ are of positive degree, where $0 \leq l \leq m-3$.

Moreover, there exists $X_v \subseteq V(H_v)$ such that $G[X_v] = K_{2n-3m+3}$ and $d_{X_v}(u) \geq 2n-3m+2$ for any $u \in V(H_v)$.

Proof. Since $\delta(\overline{G}) \geq 2n-m+1$, we have $|V(G) - N_G[v]| \geq 2n-m+1$. Let H_1 be an induced subgraph of $G - N_G[v]$ on $2n-m+1$ vertices and $M = \{x_1y_1, \ldots, x_ty_t\}$ a maximum matching of $\overline{H_1}$ and $H_2 = H_1 - V(M)$. We deduce that $t \leq m-1$, since otherwise M together with v forms an F_m in $\overline{G}$, which is a contradiction. Since M is a maximum matching in $\overline{H_1}$, $H_2 = K_{2n-m+1-2t}$. By the maximality of M, we can see that if $|N_{\overline{G}}(x_i) \cap V(H_2)| \geq 2$, then $|N_{\overline{G}}(y_i) \cap V(H_2)| = 0$ and *vice versa*. Assume without loss of generality that $x_1, x_2, \ldots, x_s$ are all the vertices of $V(M)$ such that $|N_{\overline{G}}(x_i) \cap V(H_2)| \geq 2$, where $s \leq t$. If $y_py_q \in E(\overline{G})$ for $1 \leq p < q \leq s$, then, since $|N_{\overline{G}}(x_p) \cap V(H_2)| \geq 2$ and $|N_{\overline{G}}(x_q) \cap V(H_2)| \geq 2$, we can find an M-augmenting path in $\overline{H_1}$, which contradicts the maximality of M. Thus, $y_py_q \in E(G)$ for all $1 \leq p < q \leq s$.

Set $H_3 = H_1 - \{x_1, x_2, \ldots, x_s\}$. We first show that H_3 contains an H_v, as required. By the assumption, $|N_{\overline{G}}(y_i) \cap V(H_2)| = 0$ for all $1 \leq i \leq s$. Noting that $y_py_q \in E(G)$ for all $1 \leq p < q \leq s$, we can see that $G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-m+1+s-2t}$.

If $s = m-1$, then $t = m-1$ and so $H_3 = G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-2m+2}$. Let $H_v = H_3$; then H_v is the subgraph, as required.

If $s = m-2$ and $t = m-2$, then $H_3 = G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-2m+3}$ and hence H_3 contains a subgraph $H_v = K_{2n-2m+2}$. If $s = m-2$ and $t = m-1$, then $G[V(H_2) \cup \{y_1, y_2, \ldots, y_s\}] = K_{2n-2m+1}$. If $V(H_2) \subseteq N_G(x_{m-1})$ or $V(H_2) \subseteq N_G(y_{m-1})$, then clearly H_3 contains an $H_v = K_{2n-2m+2}$. If not, then, by the maximality of M, we have $N_{\overline{G}}(x_{m-1}) \cap V(H_2) = N_{\overline{G}}(y_{m-1}) \cap V(H_2)$ and $|N_{\overline{G}}(x_{m-1}) \cap V(H_2)| = |N_{\overline{G}}(y_{m-1}) \cap V(H_2)| = 1$, which implies that $H_3 = K_{2n-2m+3} - \{x_{m-1}y_{m-1}, x_{m-1}u, y_{m-1}u\}$ for some $u \in V(H_2)$. Taking $H_v = H_3$, H_v is the subgraph, as required.

If $s \leq m-3$, we let $l = s$ and $H_v = H_3$. Obviously, $|H_v| = 2n-m-l+1$. By the assumption, $|N_{\overline{G}}(x_i) \cap V(H_2)| \leq 1$ and $|N_{\overline{G}}(y_i) \cap V(H_2)| \leq 1$ for $s+1 \leq i \leq t$. By the maximality of M, we have $|(N_{\overline{G}}(x_i) \cup N_{\overline{G}}(y_i)) \cap V(H_2)| \leq 1$ for $s+1 \leq i \leq t$.

Thus, H_v contains at most $l + 3(t - l) \leq 3m - 2l - 3$ vertices of positive degree in $\overline{H_v}$, where $0 \leq l \leq m - 3$.

Since $|V(H_2)| = 2n - m + 1 - 2t \geq 2n - 3m + 3$, we may let $X_v \subseteq V(H_2)$ with $|X_v| = 2n - 3m + 3$. Because H_2 is a complete graph, we have $G[X_v] = K_{2n-3m+3}$. Noting that each vertex of $V(H_v) - V(H_2)$ has at most one nonadjacent vertex in $V(H_2)$, we have $d_{X_v}(u) \geq 2n - 3m + 2$ for any $u \in V(H_v)$. □

Let $v \in V(G)$ be given. By Claim 1, there exist H_v and X_v attached to v. Since $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$, we have $2n - 2m \geq 1$ and $2n - m - l + 1 - (3m - 2l - 3) \geq 1$; it follows that $V(H_v)$ contains a vertex u such that $V(H_v) \subseteq N_G[u]$. By Claim 1, there exist H_u and X_u attached to u. Noting that $V(H_v) \subseteq N_G[u]$ and $V(H_u) \subseteq V(G) - N_G[u]$, we have $V(H_v) \cap V(H_u) = \emptyset$.

Set $V_1 = \{w \mid |X_w \cap X_u| \geq 2n - 7m + 6 \text{ and } X_w \cap X_v = \emptyset\}$ and $V_2 = \{w \mid |X_w \cap X_v| \geq 2n - 7m + 6 \text{ and } X_w \cap X_u = \emptyset\}$.

Claim 2. (V_1, V_2) is a partition of $V(G)$ with $V(H_v) \subseteq V_1$ and $V(H_u) \subseteq V_2$.

Proof. For any vertex w of $V(G)$, if $X_w \cap X_u = X_w \cap X_v = \emptyset$, then $4n + 1 \geq |X_u| + |X_v| + |X_w| \geq 3(2n - 3m + 3)$ and hence $n \leq 9m/2 - 4$, which is a contradiction. Thus, either $X_w \cap X_u \neq \emptyset$ or $X_w \cap X_v \neq \emptyset$. If $X_w \cap X_u \neq \emptyset$, then, since both $G[X_w]$ and $G[X_u]$ are complete graphs, we have $d(z) \geq |X_w| + |X_u| - |X_w \cap X_u| - 1$ for any vertex z in $X_w \cap X_u$. Because $d(z) \leq \Delta(G) \leq 2n + m - 1$, we obtain $|X_w \cap X_u| \geq |X_w| + |X_u| - 2n - m = 2n - 7m + 6$. Similarly, if $X_w \cap X_v \neq \emptyset$, then $|X_w \cap X_v| \geq 2n - 7m + 6$. If both $X_w \cap X_u \neq \emptyset$ and $X_w \cap X_v \neq \emptyset$, then $|X_w| \geq |X_w \cap X_u| + |X_w \cap X_v| \geq 2(2n - 7m + 6)$ and hence $n \leq (11m - 9)/2$, which contradicts $n \geq (11m - 8)/2$. Therefore, for any vertex w of $V(G)$, either $w \in V_1$ or $w \in V_2$, but not in both, that is, (V_1, V_2) is a partition of $V(G)$.

By Claim 1, for any $w \in V(H_v)$, w is nonadjacent to at most one vertex of X_v, and $X_w \subseteq V(G) - N_G[w]$; hence $|X_w \cap X_v| \leq 1$. Thus, $w \in V_1$ and $V(H_v) \subseteq V_1$. By symmetry, $V(H_u) \subseteq V_2$. □

Claim 3. For any two vertices $w_1, w_2 \in V_i$, $i = 1, 2$, we have $|X_{w_1} \cap X_{w_2}| \geq 4m - 2$.

Proof. By symmetry, it is sufficient to assume that $w_1, w_2 \in V_1$. Since $|X_{w_j} \cap X_u| \geq 2n - 7m + 6$ for $j = 1, 2$, we see that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1} \cap X_u| + |X_{w_2} \cap X_u| - |X_u| \geq 1$. Since both $G[X_{w_1}]$ and $G[X_{w_2}]$ are complete graphs, we have $d(z) \geq |X_{w_1}| + |X_{w_2}| - |X_{w_1} \cap X_{w_2}| - 1$ for any vertex z in $X_{w_1} \cap X_{w_2}$. Noting that $\Delta(G) \leq 2n + m - 1$ and $n \geq 11m/2 - 4$, we have $|X_{w_1} \cap X_{w_2}| \geq 4m - 2$. □

Assume that $|V_1| \geq |V_2|$. By Claim 2, $|V_1| \geq \lceil (4n + 1)/2 \rceil \geq 2n + 1$. For any vertex z of V_1, if $d_{V_1}(z) \geq m$ in $\overline{G}$, we choose m nonadjacent vertices of z from V_1, denoted by $z_1, \ldots, z_m$. By Claim 3, for $1 \leq i \leq m$, z_i and z have at least $4m - 2$ common nonadjacent vertices, and then z_i has at least $3m - 1$ nonadjacent vertices in $X_z - \{z_1, \ldots, z_m\}$. Thus, we may find a matching of m edges in $\overline{G}[N_{\overline{G}}(z)]$ by Lemma 2.2, which, together with z, forms an F_m in $\overline{G}$, which is a contradiction. Therefore, for any vertex z of V_1, we have $d_{V_1}(z) \leq m - 1$ in $\overline{G}$. Moreover, we may assume that $m \geq 2$, otherwise $G[V_1]$ is a complete graph which contains F_n, which is a contradiction. Since $V(H_v) \subseteq V_1$

and $|H_v| \leq 2n - 2m + 3$ by Claim 1, we let $V_1' \subseteq V_1$ be such that $V(H_v) \subseteq V_1'$ and $|V_1'| = 2n + 1$.

Now we prove that there exists some $z_0 \in V_1'$ such that $d_{V_1'}(z_0) = 2n$. By Claim 1, $H_v = K_{2n-2m+2}$; or $\overline{H_v} = K_3 \cup (2n - 2m)K_1$; or H_v is a graph of order $2n - m - l + 1$ and at most $3m - 2l - 3$ vertices in $\overline{H_v}$ are of positive degree, where $0 \leq l \leq m - 3$. Since each vertex of $V_1' - V(H_v)$ is of degree at most $m - 1$ in $\overline{G}[V_1']$, then at most $q = \max\{(2m - 1)m, (2m - 2)m + 3, (m + l)m + (3m - 2l - 3)\}$ vertices are of positive degree in $\overline{G}[V_1']$. Because $n \geq \max\{m^2 - m/2, 11m/2 - 4\}$, $m \geq 2$ and $l \leq m - 3$, it is easy to check that $q \leq 2n$. Thus, there is a vertex $z_0 \in V_1'$ such that $d_{V_1'}(z_0) = 2n$. Since $G[X_v - \{z_0\}]$ is a complete graph of order at least $2n - 3m + 2$, and every vertex of $V_1' - X_v$ has at least $2n - 3m + 2 - (m - 1) \geq n$ adjacent vertices in X_v, we can always find a perfect matching in $G[V_1' - \{z_0\}]$, which, together with z_0, forms an F_n, which is a contradiction. This completes the proof. □

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