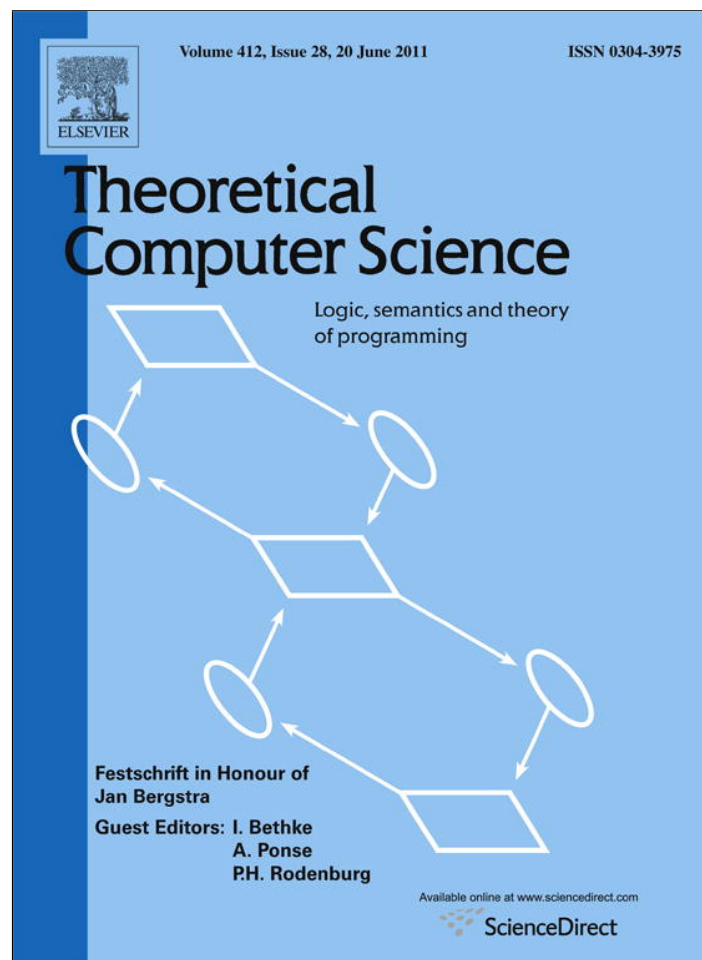


Provided for non-commercial research and education use.  
Not for reproduction, distribution or commercial use.



This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

<http://www.elsevier.com/copyright>



Contents lists available at ScienceDirect

## Theoretical Computer Science

journal homepage: [www.elsevier.com/locate/tcs](http://www.elsevier.com/locate/tcs)

## A calculus for four-valued sequential logic

Jan A. Bergstra<sup>a,1</sup>, Jaco van de Pol<sup>b,\*,1</sup><sup>a</sup> University of Amsterdam, Programming Research Group, Kruislaan 403, 1098 SJ Amsterdam, The Netherlands<sup>b</sup> University of Twente, Formal Methods and Tools, P.O. Box 217, 7500 AE Enschede, The Netherlands

## ARTICLE INFO

## Keywords:

Four-valued sequential logic  
Complete axiomatisation  
Finite basis

## ABSTRACT

We present a complete axiomatisation for four-valued sequential logic. It consists of nine axioms, from which all valid laws can be derived by equational reasoning. These nine axioms are independent of each other.

© 2011 Elsevier B.V. All rights reserved.

## 1. Introduction

This paper presents a complete algebraic axiomatisation for four-valued sequential logic. ‘Sequential’ means that arguments are evaluated from left to right, until an answer can be obtained. This models the ‘short-cutting’ semantics of connectives in programming languages.

*Related work.* Three-valued sequential logic is due to McCarthy [10]. In [4], four truth values are introduced: *true*, *false*, *mistake*, and *divergent*. The purpose of introducing these four values is to distinguish between computation errors, both in specification and programming. Several four-valued logics arise by restricting the set of connectives. In the nomenclature of [4], four-valued sequential logic is characterised as  $\Sigma_4(\neg, \wedge, \vee)$ . An axiomatisation of this system has not been published before.

In [5], it is examined whether four-valued sequential logic can serve as a basis for data type specifications. That application motivates and justifies the meta-mathematical study of four-valued logics. Our complete axiomatisation can also itself be viewed as an  $\omega$ -complete data type specification, see [8], which is quite different in nature from a Gentzen-style deduction system as in [2]. We refer to [4] for an introduction to three-valued and four-valued logic, and also for further references.

In [7], a complete axiomatisation is given for McCarthy’s system. Completeness is obtained by characterising all algebras satisfying the axioms. The completeness proof for the axiomatisation of the four-valued system that we give is quite different. Our proof yields a systematic method to prove each valid formula from the axioms.

In [12], a complete proof system for full four-valued logic is provided, consisting of 25 equations. It is based on the functionally complete set of connectives  $\vee$ ,  $\wedge$  and  $\downarrow$ , with  $\neg$  as a complex defined operation. Our result concerns a simpler axiomatisation for a simpler logic, and consequently has a simpler completeness proof as well. It is unlikely that our completeness result follows from [12].

Belnap [3] introduced a four-valued logic with a different purpose; we discuss it briefly in Section 4. A functionally complete version has been studied, with eight sublogics [2, Theorem 14], all including a non-monotonic form of implication. Our sequential four-valued logic has only monotonic operators and is strictly less expressive than all eight, according to the conditions of that theorem.

\* Corresponding author.

E-mail addresses: [J.A.Bergstra@uva.nl](mailto:J.A.Bergstra@uva.nl) (J.A. Bergstra), [J.C.vandePol@ewi.utwente.nl](mailto:J.C.vandePol@ewi.utwente.nl) (J. van de Pol).URL: <http://fmt.cs.utwente.nl/~vdpol> (J. van de Pol).<sup>1</sup> The main part of this research was carried out when both authors were affiliated to the Department of Philosophy, Utrecht University.

**Table 1**  
Parallel conjunction, parallel disjunction, and definedness.

$\wedge$	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>	$\vee$	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>	$\downarrow$	
<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>f</b>
<b>t</b>	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>	<b>t</b>	<b>m</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>
<b>f</b>	<b>m</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>	<b>f</b>	<b>t</b>
<b>d</b>	<b>m</b>	<b>d</b>	<b>f</b>	<b>d</b>	<b>d</b>	<b>m</b>	<b>t</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>f</b>

**Table 2**  
The connectives of  $\Sigma_4(\neg, \delta\wedge, \delta\vee)$ .

$\neg$		$\delta\wedge$	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>	$\delta\vee$	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>
<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>	<b>m</b>
<b>t</b>	<b>f</b>	<b>t</b>	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>
<b>f</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>m</b>	<b>t</b>	<b>f</b>	<b>d</b>
<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>	<b>d</b>

*Contribution.* In this paper, we provide a complete finite equational axiomatisation of a sequential logic with four values, solving a question posed in [4]. Moreover, we show that its nine axioms are independent, so in some sense our axiomatisation is minimal. This finally solves an open question posed in [6]. There, an additional axiom of left-distributivity was provided, but its independence could not be established, nor its derivability from the other axioms. The latter fact has now been established by the use of an automated theorem prover. This finally finishes off the result from 1996, and justifies (at least according to the second author) publication in this Festschrift.

## 2. Four-valued sequential logic

Following [4], we extend the usual truth values **t** and **f** (for *true* and *false*) with two other constants **d** and **m**, modelling a *diverging* computation and an error situation (i.e., a *mistake* has been made and there is no point in going on). The difference between **d** and **m** can be illustrated by the following equations for conjunction (see Table 1 for the complete definition).

$$\mathbf{f} \wedge \mathbf{d} = \mathbf{d} \wedge \mathbf{f} = \mathbf{f} \quad \text{but} \quad \mathbf{f} \wedge \mathbf{m} = \mathbf{m} \wedge \mathbf{f} = \mathbf{m}.$$

The first equation shows that a divergent computation may be circumvented, because the final result will be **f** in any case. In the second case, a mistake has been made, and this has to be reported. Conjunction behaves strictly w.r.t. **m**, but non-strictly w.r.t. **d**.

As the first equation reveals, conjunction needs a parallel computation. If one of the arguments can be evaluated to false, a diverging computation in the other argument must be avoided. The definedness operator ( $\downarrow$ ) is not computable at all (in a plausible model of computation). By definition,  $\downarrow X = \mathbf{f}$  when  $X = \mathbf{d}$  or  $X = \mathbf{m}$  and  $\downarrow X = \mathbf{t}$  otherwise. Intuitively, we cannot know whether an ongoing computation of  $X$  will diverge, or result in an answer eventually.

It is also possible to study sequential connectives. Here, the computation starts on the left and terminates as soon as an answer can be given. We write  $\delta\wedge$  for the *left sequential conjunction* (also known as ‘conditional and’). See Table 2 for the definition. Here, a mistake is not always reported. Some typical equations are

$$\mathbf{f} \delta\wedge \mathbf{d} = \mathbf{f} \quad \mathbf{d} \delta\wedge \mathbf{f} = \mathbf{d} \quad \mathbf{f} \delta\wedge \mathbf{m} = \mathbf{f} \quad \mathbf{m} \delta\wedge \mathbf{f} = \mathbf{m}.$$

The system that extends the truth values  $\{\mathbf{t}, \mathbf{f}, \mathbf{m}, \mathbf{d}\}$  with the connectives  $o_1, \dots, o_n$  is denoted by  $\Sigma_4(o_1, \dots, o_n)$ . In [4], it is proved that the system  $\Sigma_4(\neg, \wedge, \downarrow, \delta\wedge)$  is truth-functionally complete. This means that every four-valued function can be expressed in terms of the truth values, combined with negation, conjunction, definedness, and the left sequential conjunction.

In this paper, we focus on  $\Sigma_4(\neg, \delta\wedge, \delta\vee)$ , the system of strongly sequential truth functions. Although  $\delta\vee$  can be defined from  $\delta\wedge$  and  $\neg$  in the usual way, we incorporate it in the language. Table 2 contains the truth tables for the sequential connectives.

In this system, several classical principles are lacking. We mention commutativity of  $\delta\wedge$  and right-distributivity. In Table 3 we list nine laws that hold in  $\Sigma_4(\neg, \delta\wedge, \delta\vee)$ . These laws are self-explanatory, except the last, which expresses a valid variant of right-distributivity. We claim that all other valid laws can be derived from these nine. So, Table 3 gives a complete axiomatisation of  $\Sigma_4(\neg, \delta\wedge, \delta\vee)$ . This claim is proved in Section 3. In Section 4, we show that each of the laws (1)–(9) is independent of the other laws.

**Table 3**  
Nine axioms of  $\Sigma_4(\neg, \delta, \vee)$ .

(1)	$\neg \mathbf{d} = \mathbf{d}$
(2)	$\neg \mathbf{m} = \mathbf{m}$
(3)	$\neg \mathbf{t} = \mathbf{f}$
(4)	$\neg \neg X = X$
(5)	$\mathbf{t} \delta X = X$
(6)	$\mathbf{f} \delta X = \mathbf{f}$
(7)	$X \vee Y = \neg(\neg X \delta \neg Y)$
(8)	$(X \delta Y) \delta Z = X \delta (Y \delta Z)$
(9)	$(X \vee Y) \delta Z = (\neg X \delta (Y \delta Z)) \vee (X \delta Z)$

**Table 4**  
Ten more laws of  $\Sigma_4(\neg, \delta, \vee)$ .

		Derived from:
(10)	$X \delta \mathbf{t} = X$	(3,5,5d,6d,9d)
(11)	$X \delta Y = (\neg X \vee Y) \delta X$	(9d,10d)
(12)	$X \delta \mathbf{f} = \neg X \delta X$	(10d,11)
(13)	$X \delta \neg X = \neg X \delta X$	(4,5d,6,9,9d,11,12)
(14)	$X \delta X = X$	(4,10,11,12d,13d)
(15)	$X \delta Y = X \delta (Y \delta X)$	(8,11,14)
(16)	$X \delta (Y \vee Z) = (X \delta Y) \vee (X \delta Z)$	(4,8,8d,9,11,11d,15)
(17)	$X \delta (Y \delta Z) = (X \delta Y) \delta (X \delta Z)$	(8,15)
(18)	$\mathbf{d} \delta X = \mathbf{d}$	(1,6,8,12,13,14)
(19)	$\mathbf{m} \delta X = \mathbf{m}$	(2,6,8,12,13,14)

### 3. A complete axiomatisation of $\Sigma_4(\neg, \delta, \vee)$

We write  $P, Q$  for arbitrary open terms over  $\Sigma_4(\neg, \delta, \vee)$ .  $X$  and  $Y$  are arbitrary variables, while  $b$  and  $c$  range over the four constants. By  $P[X := c]$ , we will denote the term  $P$ , with all occurrences of  $X$  replaced by  $c$ . We write  $\models P = Q$  if  $P = Q$  holds in  $\Sigma_4(\neg, \delta, \vee)$ . With  $\vdash P = Q$ , we denote that  $P = Q$  is derivable with equational logic, using laws (1)–(9) of Table 3.

**Proposition 1.** For all  $P$  and  $Q$ , if  $\vdash P = Q$  then  $\models P = Q$ .

**Proof.** Laws (1)–(9) can be checked straightforwardly. This gives the required result.  $\square$

The *dual* of a term is obtained by interchanging all occurrences of  $\mathbf{t}$  with  $\mathbf{f}$ , and  $\delta$  with  $\vee$ .

**Lemma 2.** For all  $P$  and  $Q$ , if  $\vdash P = Q$  then also  $\vdash P^{dual} = Q^{dual}$ .

**Proof.** By (4), it suffices to prove  $\neg \neg P^{dual} = \neg \neg Q^{dual}$ . Using (1)–(4) and (7), the second  $\neg$  can be pushed inside step by step. Eventually, an instance of  $\neg P = \neg Q$  is obtained (with all variables  $X$  replaced by  $\neg X$ ), which can be derived since by assumption  $\vdash P = Q$ .  $\square$

In what follows, we will denote the use of the dual of a derived law by the postfix  $d$ . For example, (5d) is the dual of (5):  $\mathbf{f} \vee X = X$ .

**Lemma 3.** Laws (10)–(19) from Table 4 are derivable from (1)–(9).

**Proof.** 10.  $X \stackrel{5d,5}{=} (\mathbf{t} \delta \mathbf{f}) \vee X \stackrel{9d}{=} (\neg \mathbf{t} \vee \mathbf{f} \vee X) \delta (\mathbf{t} \vee X) \stackrel{6d,3,5d}{=} X \delta \mathbf{t}$ .  
 11.  $X \delta Y \stackrel{10d}{=} (X \delta Y) \vee \mathbf{f} \stackrel{9d}{=} (\neg X \vee Y \vee \mathbf{f}) \delta (X \vee \mathbf{f}) \stackrel{10d}{=} (\neg X \vee Y) \delta X$ .  
 12.  $X \delta \mathbf{f} \stackrel{11}{=} (\neg X \vee \mathbf{f}) \delta X \stackrel{10d}{=} \neg X \delta X$ .  
 13. First, we prove the auxiliary identity (\*):  $(X \delta \mathbf{f}) \vee (X \delta \mathbf{f}) = X \delta \mathbf{f}$ .  
 $(X \delta \mathbf{f}) \vee (X \delta \mathbf{f})$   
 $\stackrel{4,6,12}{=} (\neg \neg X \delta (\mathbf{f} \delta X)) \vee (\neg X \delta X)$   
 $\stackrel{9}{=} (\neg X \vee \mathbf{f}) \delta X$   
 $\stackrel{11}{=} X \delta \mathbf{f}$

Next, we prove 13.

$$\begin{aligned}
 & X \triangleleft \neg X \\
 \stackrel{11}{=} & (\neg X \triangleleft \neg X) \triangleleft X \\
 \stackrel{9}{=} & (\neg \neg X \triangleleft (\neg X \triangleleft X)) \triangleleft (\neg X \triangleleft X) \\
 \stackrel{4,12}{=} & (X \triangleleft (X \triangleleft \mathbf{f})) \triangleleft (X \triangleleft \mathbf{f}) \\
 \stackrel{9}{=} & (\neg X \triangleleft ((X \triangleleft \mathbf{f}) \triangleleft (X \triangleleft \mathbf{f}))) \triangleleft (X \triangleleft (X \triangleleft \mathbf{f})) \\
 \stackrel{(*)}{=} & (\neg X \triangleleft (X \triangleleft \mathbf{f})) \triangleleft (X \triangleleft (X \triangleleft \mathbf{f})) \\
 \stackrel{5d}{=} & (\neg X \triangleleft (\mathbf{f} \triangleleft (X \triangleleft \mathbf{f}))) \triangleleft (X \triangleleft (X \triangleleft \mathbf{f})) \\
 \stackrel{9d}{=} & (X \triangleleft \mathbf{f}) \triangleleft (X \triangleleft \mathbf{f}) \\
 \stackrel{(*)}{=} & X \triangleleft \mathbf{f} \\
 \stackrel{12}{=} & \neg X \triangleleft X
 \end{aligned}$$

$$\begin{aligned}
 14. & X \triangleleft X \\
 \stackrel{11}{=} & (\neg X \triangleleft X) \triangleleft X \\
 \stackrel{13d}{=} & (X \triangleleft \neg X) \triangleleft X \\
 \stackrel{4,12d}{=} & (\neg X \triangleleft \mathbf{t}) \triangleleft X \\
 \stackrel{11}{=} & X \triangleleft \mathbf{t} \\
 \stackrel{10}{=} & X
 \end{aligned}$$

$$\begin{aligned}
 15. & X \triangleleft Y \\
 \stackrel{11}{=} & (\neg X \triangleleft Y) \triangleleft X \\
 \stackrel{14}{=} & (\neg X \triangleleft Y) \triangleleft (X \triangleleft X) \\
 \stackrel{8}{=} & ((\neg X \triangleleft Y) \triangleleft X) \triangleleft X \\
 \stackrel{11}{=} & (X \triangleleft Y) \triangleleft X \\
 \stackrel{8}{=} & X \triangleleft (Y \triangleleft X)
 \end{aligned}$$

$$\begin{aligned}
 16. & X \triangleleft (Y \triangleleft Z) \\
 \stackrel{11}{=} & (\neg X \triangleleft (Y \triangleleft Z)) \triangleleft X \\
 \stackrel{8d}{=} & ((\neg X \triangleleft Y) \triangleleft Z) \triangleleft X \\
 \stackrel{4,11d}{=} & (((X \triangleleft Y) \triangleleft \neg X) \triangleleft Z) \triangleleft X \\
 \stackrel{8d}{=} & ((X \triangleleft Y) \triangleleft (\neg X \triangleleft Z)) \triangleleft X \\
 \stackrel{9}{=} & (\neg(X \triangleleft Y) \triangleleft ((\neg X \triangleleft Z) \triangleleft X)) \triangleleft ((X \triangleleft Y) \triangleleft X) \\
 \stackrel{11}{=} & (\neg(X \triangleleft Y) \triangleleft (X \triangleleft Z)) \triangleleft ((X \triangleleft Y) \triangleleft X) \\
 \stackrel{8,15}{=} & (\neg(X \triangleleft Y) \triangleleft (X \triangleleft Z)) \triangleleft (X \triangleleft Y) \\
 \stackrel{11d}{=} & (X \triangleleft Y) \triangleleft (X \triangleleft Z)
 \end{aligned}$$

$$17. X \triangleleft (Y \triangleleft Z) \stackrel{8}{=} (X \triangleleft Y) \triangleleft Z \stackrel{15}{=} (X \triangleleft (Y \triangleleft X)) \triangleleft Z \stackrel{8}{=} (X \triangleleft Y) \triangleleft (X \triangleleft Z).$$

18. If  $Z = \neg Z$  then  $Z = Z \triangleleft \mathbf{f} (*)$ , for  $Z \stackrel{14}{=} Z \triangleleft Z = Z \triangleleft \neg Z \stackrel{13,12}{=} Z \triangleleft \mathbf{f}$ ; hence  $Z \triangleleft X \stackrel{(*)}{=} (Z \triangleleft \mathbf{f}) \triangleleft X \stackrel{8}{=} Z \triangleleft (\mathbf{f} \triangleleft X) \stackrel{6}{=} Z \triangleleft \mathbf{f} \stackrel{(*)}{=} Z$ . Now, using (1), the required result follows.

19. Similar to (18), but now using (2).  $\square$

**Lemma 4.** Every closed term is provably equal to  $\mathbf{t}$ ,  $\mathbf{f}$ ,  $\mathbf{d}$ , or  $\mathbf{m}$ .

**Proof.** This is proved by term induction. In the case of a negation, (1), (2), (3), and (3d) are used. In the case of conjunction we use (5), (6), (18), and (19). Disjunction is the dual of conjunction.  $\square$

**Lemma 5.** Every term  $P$  is either provably equal to a closed term, or it is provably equal to  $(X \triangleleft P_1) \triangleleft P_2$ , for some variable  $X$  and terms  $P_1$  and  $P_2$ . Moreover,  $X$  and the variables occurring in  $P_1$  and  $P_2$  also occur in  $P$ .

**Proof.** The lemma is proved by induction on  $P$ .

Case  $c$ : Constants are clearly closed.

Case  $X$ : By (10d) and (10),  $\vdash X = (X \triangleleft \mathbf{f}) \triangleleft \mathbf{t}$ .

Case  $\neg P$ : If  $P$  is provably closed, then  $\neg P$  is provably closed too. Otherwise, we obtain

$$\begin{aligned}
 & \neg P \\
 \stackrel{\text{IH}}{=} & \neg((X \overset{\circ}{\vee} P_1) \overset{\circ}{\wedge} P_2) \\
 \stackrel{9}{=} & \neg((\neg X \overset{\circ}{\wedge} (P_1 \overset{\circ}{\wedge} P_2)) \overset{\circ}{\vee} (X \overset{\circ}{\wedge} P_2)) \\
 \stackrel{7}{=} & \neg\neg(\neg(\neg X \overset{\circ}{\wedge} (P_1 \overset{\circ}{\wedge} P_2)) \overset{\circ}{\wedge} \neg(X \overset{\circ}{\wedge} P_2)) \\
 \stackrel{4}{=} & \neg(\neg X \overset{\circ}{\wedge} \neg\neg(P_1 \overset{\circ}{\wedge} P_2)) \overset{\circ}{\wedge} \neg(X \overset{\circ}{\wedge} P_2) \\
 \stackrel{7}{=} & (X \overset{\circ}{\vee} \neg(P_1 \overset{\circ}{\wedge} P_2)) \overset{\circ}{\wedge} \neg(X \overset{\circ}{\wedge} P_2),
 \end{aligned}$$

which is of the required format.

Case  $P \overset{\circ}{\wedge} Q$ : If  $P$  is provably closed, then, by Lemma 4, it is provably equal to **t**, **f**, **d**, or **m**. Then  $P \overset{\circ}{\wedge} Q$  is either provably closed, or provably equal to  $Q$ . In the latter case, the induction hypothesis for  $Q$  yields the required format.

If  $P$  is not provably closed, then, by the induction hypothesis for  $P$ , we obtain  $P_1$  and  $P_2$  such that  $\vdash P \overset{\circ}{\wedge} Q = ((X \overset{\circ}{\vee} P_1) \overset{\circ}{\wedge} P_2) \overset{\circ}{\wedge} Q$ . Using (8), this can be brought into the required form.

Case  $P \overset{\circ}{\vee} Q$ : The case that  $P$  is provably closed is similar to  $\overset{\circ}{\wedge}$ . Otherwise, we find  $P_1$  and  $P_2$  by the induction hypothesis for  $P$ , such that

$$\begin{aligned}
 & P \overset{\circ}{\vee} Q \\
 \stackrel{\text{IH}}{=} & ((X \overset{\circ}{\vee} P_1) \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} Q \\
 \stackrel{9}{=} & ((\neg X \overset{\circ}{\wedge} (P_1 \overset{\circ}{\wedge} P_2)) \overset{\circ}{\vee} (X \overset{\circ}{\wedge} P_2)) \overset{\circ}{\vee} Q \\
 \stackrel{8d}{=} & (\neg X \overset{\circ}{\wedge} (P_1 \overset{\circ}{\wedge} P_2)) \overset{\circ}{\vee} ((X \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} Q) \\
 \stackrel{9d}{=} & (\neg\neg X \overset{\circ}{\vee} ((P_1 \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} ((X \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} Q))) \overset{\circ}{\wedge} (\neg X \overset{\circ}{\vee} ((X \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} Q)) \\
 \stackrel{4}{=} & (X \overset{\circ}{\vee} ((P_1 \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} ((X \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} Q))) \overset{\circ}{\wedge} (\neg X \overset{\circ}{\vee} ((X \overset{\circ}{\wedge} P_2) \overset{\circ}{\vee} Q)),
 \end{aligned}$$

which is of the required form.  $\square$

**Lemma 6.** For any term  $P$  and variable  $X$ , we have

- (a)  $\vdash X \overset{\circ}{\wedge} P = X \overset{\circ}{\wedge} P[X := \mathbf{t}]$
- (b)  $\vdash \neg X \overset{\circ}{\wedge} P = \neg X \overset{\circ}{\wedge} P[X := \mathbf{f}]$ .

**Proof.** Without loss of generality, we assume that  $P$  is built from constants, variables, and negated variables, using the connectives  $\overset{\circ}{\wedge}$  and  $\overset{\circ}{\vee}$ . Using (1)–(4) and (7) (including their duals), we can write each term in such a form. The lemma is then proved with induction on  $P$ .

Cases  $c$ ,  $Y$ , or  $\neg Y$ , with  $c$  a constant and  $Y \neq X$ , are trivial.

Case  $X$ : use (10, 14) for (a); use (13, 4, 12) for (b).

Case  $\neg X$ : use (13, 12, 3) for (a); use (14, 10, 3d) to obtain (b).

Case  $P \overset{\circ}{\wedge} Q$ : Use (17) and the induction hypotheses for  $P$  and  $Q$ .

Case  $P \overset{\circ}{\vee} Q$ : Use (16) and the induction hypotheses for  $P$  and  $Q$ .  $\square$

**Theorem 7.** Axioms (1)–(9) form a complete axiomatisation for  $\Sigma_4(\neg, \overset{\circ}{\wedge}, \overset{\circ}{\vee})$ .

**Proof.** Assume that  $\vDash P = Q$ . We prove that  $\vdash P = Q$  by induction on the number of different variables occurring in this equation. By Lemma 5, we have that either

- (a)  $P$  is provably closed; or
- (b)  $\vdash P = (X \overset{\circ}{\vee} P_1) \overset{\circ}{\wedge} P_2$ .

Similarly, we obtain that either

- (c)  $Q$  is provably closed; or
- (d)  $\vdash Q = (Y \overset{\circ}{\vee} Q_1) \overset{\circ}{\wedge} Q_2$ .

By Lemma 4, each provably closed term is provably equal to **t**, **f**, **d**, or **m**. We first prove that cases (a) and (d) cannot both occur, for assume both (a) and (d). Then  $\vdash P = c$ , where  $c$  is one of the constants. By soundness,  $\vDash P = c$  and  $\vDash Q = (Y \overset{\circ}{\vee} Q_1) \overset{\circ}{\wedge} Q_2$ ; hence, also,  $\vDash c = (Y \overset{\circ}{\vee} Q_1) \overset{\circ}{\wedge} Q_2$ . Now, taking  $Y = \mathbf{d}$  and  $Y = \mathbf{m}$ , respectively, we get  $\vDash \mathbf{d} = \mathbf{m}$ , quod non. Similarly, (b) and (c) cannot both occur. Two cases remain.

- (a) and (c) hold (this includes the base of the induction). In this case,  $\vdash P = b$  and  $\vdash Q = c$ , for some constants  $b$  and  $c$ . Using soundness and the assumption that  $\vDash P = Q$ , we obtain  $b \equiv c$ , so  $\vdash P = Q$ .

- (b) and (d) hold. In this case,  $X \equiv Y$ , for otherwise we could substitute  $\mathbf{d}$  for  $X$  and  $\mathbf{m}$  for  $Y$ , implying (via 18, 18d, 19, 19d and soundness) that  $\models \mathbf{m} = \mathbf{d}$ .

Define  $P'_1 \equiv (P_1 \circlearrowleft P_2)[X := \mathbf{f}]$ ,  $P'_2 \equiv (P_2[X := \mathbf{t}])$ . Then, using (9), we have  $\vdash P = (\neg X \circlearrowleft P_1 \circlearrowleft P_2) \circlearrowleft (X \circlearrowleft P_2)$ . By Lemma 6, we have  $\vdash P = (\neg X \circlearrowleft P'_1) \circlearrowleft (X \circlearrowleft P'_2)$ . In a similar way, we can find  $Q'_1$  and  $Q'_2$  that do not contain  $X$  such that  $\vdash Q = (\neg X \circlearrowleft Q'_1) \circlearrowleft (X \circlearrowleft Q'_2)$ .

Using  $\models P = Q$  and soundness, and taking  $X = \mathbf{t}$ , we find that  $\models P'_2 = Q'_2$ . Taking  $X = \mathbf{f}$ , we find that  $\models P'_1 = Q'_1$ . Now, by the induction hypothesis,  $\vdash P'_1 = Q'_1$  and  $\vdash P'_2 = Q'_2$ . By equational logic, we find that  $\vdash P = Q$ .  $\square$

#### 4. Concluding remarks

*Extension.* The existence of at least two error values is needed in the proof of Theorem 7 to make sure that  $X \equiv Y$ . If there is only one error value (i.e., McCarthy's logic [10]) then the following law becomes valid; note that the leftmost variable changes:

$$((X \circlearrowleft Y) \circlearrowleft (Y \circlearrowleft X)) = ((Y \circlearrowleft X) \circlearrowleft (X \circlearrowleft Y)). \quad (1)$$

Our proof easily generalises to more than two error values. For a new error value, add an axiom  $e = \neg e$ . As in Lemma 3.18, we can then prove  $e \circlearrowleft X = e$ ; via (7) we obtain  $e \circlearrowleft X = e$ . With these equations, Lemma 4 can be extended to the new situation. Then the proof of Theorem 7 remains valid.

*Independence of axioms.* Below we list the arguments that each of the laws (1)–(9) is independent of the other laws. In order to prove that some law is independent, we use semantic (1, 2, 7, 8, 9) and syntactic (3, 4, 5, 6) arguments. The semantic argument provides a model for the remaining laws that refutes this one. The syntactic argument shows that the remaining laws obey some invariant, which prevents the derivation of this law.

1. Take as a model the restriction of  $\neg, \circlearrowleft$ , and  $\circlearrowright$  to the carrier set  $\{\mathbf{t}, \mathbf{f}, \mathbf{m}\}$  and interpret  $\mathbf{d}$  by  $\mathbf{t}$ . Then law (1) is false, but laws (2)–(9) hold.
2. Similar to 1.
3. If a term contains  $\mathbf{f}$ , then after application of a rule from (1, 2, 4–9), it still contains  $\mathbf{f}$ . So law (3) is not derivable from those laws.
4. Without law (4), a term in which no symbols from  $\{\neg, \circlearrowleft, \mathbf{f}, \mathbf{m}, \mathbf{d}\}$  occur cannot be proved equal to a term that contains some of these symbols.
5. With the laws other than (5), terms without constants cannot be proved equal to a term containing a constant.
6. Without law (6), a closed term cannot be proved equal to an open term.
7. In the following model, laws (1)–(6) and (8)–(9) hold, but (7) fails. The carrier set is  $\{\mathbf{t}, \mathbf{f}, \mathbf{m}, \mathbf{d}\}$ . Interpret  $\neg$  as usual negation,  $\circlearrowleft$  as the constant function  $\mathbf{f}$ , and interpret  $x \circlearrowleft y$  as  $y$  whenever  $x = \mathbf{t}$  and  $\mathbf{f}$  otherwise.
8. Consider the model  $\Sigma_4(\neg, \circlearrowleft, \circlearrowright)$ , modified only by  $\mathbf{d} \circlearrowleft \mathbf{m} = \mathbf{d} \circlearrowright \mathbf{m} = \mathbf{m}$ . In this model, (1)–(7) and (9) hold, but (8) does not hold:  
 $(\mathbf{d} \circlearrowleft \mathbf{f}) \circlearrowleft \mathbf{m} = \mathbf{m} \neq \mathbf{d} = \mathbf{d} \circlearrowleft (\mathbf{f} \circlearrowleft \mathbf{m})$ .
9. In the following model, (1)–(8) hold but (9) fails. Take  $\{\mathbf{t}, \mathbf{f}, \mathbf{d}\}$  as the carrier set, with  $\mathbf{m} = \mathbf{d}$ . Interpret  $\circlearrowleft$  and  $\circlearrowright$  as the restriction of  $\wedge$  and  $\vee$ , respectively (Table 1). This model is known as Kleene's three-valued logic [9].

*Use of tools.* In [6], the proofs of laws (13) and (16) were much simpler, based on the axiom of left-distributivity (here law 16). Actually, the current proofs were found with help of the automated theorem prover Prover9,<sup>2</sup> a successor of Otter [11], and its companion Mace4 for finding finite counterexample models. Although this prover finds proofs by paramodulation within seconds, it still took several days to manually transform them to straight proofs in equational logic, and to introduce suitable intermediate lemmas.

*Alternatives.* The reader might wonder why in law (9) the disjuncts seem to be reordered. Indeed, the alternative rule (9b) from Table 5 is valid as well. However, replacing rule (9) by (9b) does not yield a complete axiomatisation. Apparently, some form of commutativity is needed as well. We claim that replacing (9) by (9b+11) or even by (9b+13) yields a complete axiomatisation. This is readily checked by deriving law (9) from them. Eventually, the question which minimal set of axioms to choose as a basis is a matter of taste.

*Belnap's four-valued logic.* Belnap [3] introduced four-valued logic, with the purpose of reasoning about incomplete (*none*) and inconsistent (*both*) information from different sources. Linking  $\mathbf{d}$  to *none* and  $\mathbf{m}$  to *both*, our values are based on the same information lattice as his:  $\mathbf{d} < \mathbf{f}, \mathbf{t} < \mathbf{m}$ . Belnap's conjunction and disjunction are the meet and join in a single logical lattice  $\mathbf{f} < \mathbf{d}, \mathbf{m} < \mathbf{t}$ .

<sup>2</sup> See <http://www.cs.unm.edu/~mccune/prover9/>.

**Table 5**

Alternative axiom for right-distributivity.

$$(9b) \quad (X \overset{\circ}{\vee} Y) \overset{\circ}{\wedge} Z = (X \overset{\circ}{\wedge} Z) \overset{\circ}{\vee} (\neg X \overset{\circ}{\wedge} (Y \overset{\circ}{\wedge} Z))$$

Our purpose is to study errors in specification and programming. We distinguish errors which may be avoided (e.g., divergence **d**) and errors that should be reported (e.g., mistakes **m**). Our four-valued logic is not based on a bilattice. As we want to be strict in **m**, our parallel conjunction can be viewed as the meet in the logical lattice  $\mathbf{m} < \mathbf{f} < \mathbf{d} < \mathbf{t}$ , while our parallel disjunction corresponds to the join in the different lattice  $\mathbf{f} < \mathbf{d} < \mathbf{t} < \mathbf{m}$ .

In conclusion, we study *how we might think about computing* rather than *how computers should think*.

## References

- [1] A.R. Anderson, N.D. Belnap, J.M. Dunn (Eds.), A useful Four-Valued Logic: How Computers should think, in: Entailment II, The Logic of Relevance and Necessity, vol. II, Princeton University Press, 1992, pp. 506–541 (Chapter XII, Section 81).
- [2] Ofer Arieli, Arnon Avron, The value of the four values, *Journal of Artificial Intelligence* 102 (1) (1998) 97–141.
- [3] Nuel D. Belnap, How a computer should think, in: G. Ryle (Ed.), *Contemporary Aspects of Philosophy*, Oriel Press, 1977, pp. 30–55. cf. [1].
- [4] Jan A. Bergstra, Inge Bethke, Piet H. Rodenburg, A propositional logic with 4 values: true, false, divergent and meaningless, *Journal of Applied Non-Classical Logics* 5 (2) (1995) 199–217.
- [5] Jan A. Bergstra, Alex Sellink, *Sequential data algebra primitives*, Utrecht University, Dept. of Philosophy, 1996.
- [6] Jan A. Bergstra, Jaco van de Pol, A calculus for sequential logic with 4 values. Technical Report Logic Group Preprint Series Nr. 160, Utrecht University, Dept. of Philosophy, 1996.
- [7] F. Guzman, C.S. Squier, The algebra of conditional logic, *Algebra Universalis* 27 (1990) 88–110.
- [8] Jan Heering, Partial evaluation and  $\omega$ -completeness of algebraic specifications, *Theoretical Computer Science* 43 (1986) 149–167.
- [9] S.C. Kleene, On a notation for ordinal numbers, *Journal of Symbolic Logic* 3 (1938) 150–155.
- [10] John McCarthy, A basis for a mathematical theory of computation, in: P. Braffort, D. Hirshberg (Eds.), *Computer Programming and Formal Systems*, North-Holland, 1963, pp. 33–70.
- [11] William McCune, Larry Wos, Otter — the CADE-13 competition incarnations, *Journal of Automated Reasoning* 18 (2) (1997) 211–220.
- [12] Piet H. Rodenburg, A complete system of four-valued logic, *Journal of Applied Non-Classical Logics* 11 (3–4) (2001) 367–389.