

# Generalized semi-infinite programming: A tutorial

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## Abstract

This tutorial presents an introduction to generalized semi-infinite programming (GSIP) which in recent years became a vivid field of active research in mathematical programming. A GSIP problem is characterized by an infinite number of inequality constraints, and the corresponding index set depends additionally on the decision variables. There exist a wide range of applications which give rise to GSIP models; some of them are discussed in the present paper. Furthermore, geometric and topological properties of the feasible set and, in particular, the difference to the standard semi-infinite case are analyzed. By using first-order approximations of the feasible set corresponding constraint qualifications are developed. Then, necessary and sufficient first- and second-order optimality conditions are presented where directional differentiability properties of the optimal value function of the so-called lower level problem are used. Finally, an overview of numerical methods is given.

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## 1. Introduction

This article describes theory, applications and methods for the so-called *generalized semi-infinite optimization problems*. These problems have the form

$$\text{GSIP: } \min f(x) \quad \text{s.t. } x \in M$$

with

$$M = \{x \in \mathbb{R}^n \mid g(x, y) \leq 0 \text{ for all } y \in Y(x)\}$$

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and

$$Y(x) = \{y \in \mathbb{R}^m \mid v_\ell(x, y) \leq 0, \ell \in L\}.$$

All defining functions  $f, g, v_\ell, \ell \in L = \{1, \dots, s\}$ , are assumed to be real-valued and at least continuous in their respective domains. Moreover, we assume that the set-valued mapping  $Y : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is locally bounded, that is, for each  $\bar{x} \in \mathbb{R}^n$  there exists a neighborhood  $U$  of  $\bar{x}$  such that  $\bigcup_{x \in U} Y(x)$  is bounded in  $\mathbb{R}^m$ .

As opposed to a standard semi-infinite optimization problem (SIP), the possibly infinite index set  $Y(x)$  of the semi-infinite inequality constraint is allowed to vary with  $x$  in a GSIP. For surveys and detailed studies about *standard* semi-infinite optimization we refer to [13,18,23,24,31,34,45,68,70].

In applications (cf. Section 2) often finitely many semi-infinite constraints  $g_i(x, y) \leq 0, y \in Y_i(x), i \in I$ , describe the feasible set  $M$  of GSIP, along with finitely many equality constraints in the definitions of  $M$  and  $Y(x)$ . In order to avoid technicalities in this tutorial article we focus on the basic case of a single semi-infinite constraint and refer the interested reader to [85] for more general formulations.

First systematic studies of GSIP in [36,52] gave the impression that GSIP is merely a slight generalization of standard SIP. However, in [47] first indications appeared that GSIP is an *essentially harder* problem than SIP. In particular it turned out that the feasible set of GSIP can possess topological structures that are neither known from finite nor from standard semi-infinite optimization. From a geometrical point of view it was also clear that these phenomena are stable under data perturbations. These observations inspired a number of authors to have a closer look at the topological structure of  $M$  [75,81–83,91,99], at optimality conditions [46,47,73,74,76,84,88,103], and at solution methods [8,30,51,61,89,92,93,98] for GSIP.

This tutorial is structured as follows. After pointing out some important applications of GSIP in Section 2, we explain the geometry of the feasible set  $M$ , including appropriate constraint qualifications, in Section 3. Based on these results first- and second-order optimality conditions are presented in Sections 4 and 5. Section 6 reviews numerical methods for GSIP before the tutorial closes with some final remarks in Section 7.

## 2. Applications

From the numerous real-life applications of generalized semi-infinite programming this section explains three important classes in some detail: Chebyshev approximation, design centering, and robust optimization. Examples for further applications are the optimal layout of an assembly line [50,98], time minimal control [51,55,98], and disjunctive optimization [85]. Many examples for *standard* semi-infinite optimization problems can be found in [34] and the references cited therein. We also remark that *semi-definite programming* [96,102] can be interpreted as a special case of standard semi-infinite programming. This approach is elaborated in [17,97].

### 2.1. Chebyshev and reverse Chebyshev approximation

In applications one often seeks to approximate a given continuous function  $F$  on a nonempty and compact set  $Z \subset \mathbb{R}^M$  by a simpler function  $a(p, \cdot)$  which can be chosen from a parameterized family of continuous functions  $\{a(p, \cdot) \mid p \in P\}$  with some parameter set  $P \subset \mathbb{R}^N$ . Depending on the application, different norms may be used to measure the deviation between  $F$  and  $a(p, \cdot)$  on  $Z$ . For *computational* reasons one often uses the Euclidean norm since this gives rise to an optimization problem with a smooth objective function.

However, in many applications it is not sufficient to minimize some averaged deviation, but one actually needs to minimize the *maximal* deviation, that is, the Chebyshev norm is used instead of the Euclidean norm.

This leads to the nondifferentiable problem of *Chebyshev approximation* (cf., e.g., [14,22])

$$\text{CA: } \min_{p \in P} \|F(\cdot) - a(p, \cdot)\|_{\infty, Z} = \min_{p \in P} \max_{z \in Z} |F(z) - a(p, z)|.$$

The epigraph reformulation of CA yields the equivalent problem

$$\min_{(p, q) \in P \times \mathbb{R}} q \quad \text{s.t. } \|F(\cdot) - a(p, \cdot)\|_{\infty, Z} \leq q,$$

which can be rewritten as

$$\text{SIP}_{\text{CA}}: \quad \min_{(p,q) \in P \times \mathbb{R}} q \quad \text{s.t.} \quad F(z) - a(p, z) \leq q \quad \text{for all } z \in Z, \\ -F(z) + a(p, z) \leq q \quad \text{for all } z \in Z,$$

that is, as a standard semi-infinite optimization problem. The main advantage of this reformulation of CA is that  $\text{SIP}_{\text{CA}}$  is a *smooth* optimization problem if all defining functions are smooth, whereas CA is intrinsically nonsmooth. The price to pay for smoothness is, of course, the presence of infinitely many inequality constraints. Solution methods for this specially structured SIP can be found for example in [37].

In engineering applications a modification of CA, termed *reverse Chebyshev approximation*, has received interest as it can be used to model for example the approximation of a thermo-couple characteristic or for the construction of low pass filters in digital filtering theory [38,51]. In this framework, let  $F$  be a real-valued continuous function on a nonempty and compact set  $Z(q) \subset \mathbb{R}^M$  which depends on a parameter  $q \in Q$ . Given an approximating family of functions  $a(p, \cdot)$  and a desired precision  $e(p, q)$ , the aim is to find parameter vectors  $p$  and  $q$  such that the domain  $Z(q)$  is as large as possible without exceeding the approximation error  $e(p, q)$ . This yields the problem

$$\text{RCA}: \quad \max_{(p,q) \in P \times Q} \text{Vol}(Z(q)) \quad \text{s.t.} \quad \|F - a(p, \cdot)\|_{\infty, Z(q)} \leq e(p, q),$$

where  $\text{Vol}(Z(q))$  denotes the  $M$ -dimensional volume of  $Z(q)$ . Again, this intrinsically nonsmooth optimization problem can be reformulated with semi-infinite constraints.

However, as opposed to the situation in standard Chebyshev approximation, we now obtain a *generalized* semi-infinite optimization problem:

$$\text{GSIP}_{\text{RCA}}: \quad \max_{(p,q) \in P \times Q} \text{Vol}(Z(q)) \\ \text{s.t.} \quad F(z) - a(p, z) \leq e(p, q) \quad \text{for all } z \in Z(q), \\ -F(z) + a(p, z) \leq e(p, q) \quad \text{for all } z \in Z(q).$$

Numerical approaches to this problem class for small dimensions are presented in [38,51].

## 2.2. Design centering

A design centering problem consists in maximizing some measure, for example the volume, of a parameterized body  $B(x)$  while it is inscribed into a container set  $C$ :

$$\text{DC}: \quad \max_{x \in \mathbb{R}^n} \text{Vol}(B(x)) \quad \text{s.t.} \quad B(x) \subset C.$$

In applications the set  $C$  often has a complicated structure, while  $B(x)$  possesses a simpler geometry (cf. Fig. 1).

If the container  $C$  is described by functional constraints,

$$C = \{y \in \mathbb{R}^m \mid c(y) \leq 0\},$$

an equivalent formulation of the design centering problem as a GSIP is

$$\text{GSIP}_{\text{DC}}: \quad \max_{x \in \mathbb{R}^n} \text{Vol}(B(x)) \quad \text{s.t.} \quad c(y) \leq 0 \quad \text{for all } y \in B(x).$$

Note that the semi-infinite constraint function  $c$  does not depend on  $x$  in this case, but only its index set does. In applications, if the set  $C$  or the function  $c$ , respectively, are not too complicated, parts of the decision variable  $x$  like translations and rotations may also be modeled to affect  $C$ , so that the inclusion constraint in DC becomes  $B(x) \subset C(x)$ . In the latter case also the function  $c$  in  $\text{GSIP}_{\text{DC}}$  depends on  $x$  and  $y$ .

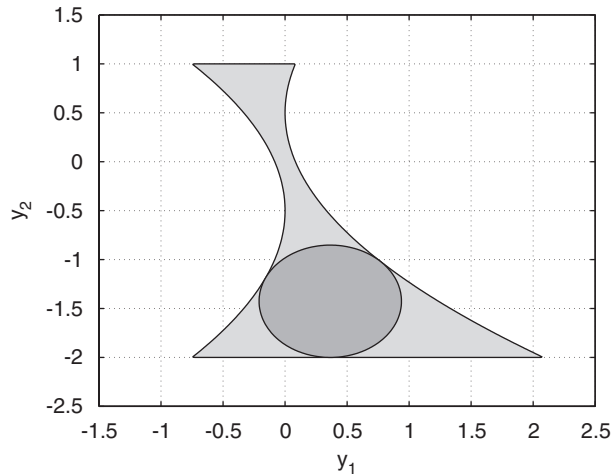


Fig. 1. A disk  $B(x^*)$  with maximal area in a container  $C$ .

Design centering problems have been studied extensively, see for example [27,41,65,67,87] and the references therein. They are also related to the so-called containment problem from [63].

Applications for design centering arise in different circumstances. It is used for example to determine lower bounds for the volume of a complicated container set by inscribing ellipsoids in the so-called *maneuverability problem of a robot* from [26]. This problem gave rise to one of the first formulations of a generalized semi-infinite optimization problem in [35].

If  $B(x)$  is a norm ball, design centering can also be used to find “innermost” points of  $C$ . As described in [41] at such a point a company should produce a good, if its uncertain quality parameters are to be contained in a set of feasible parameters.

A third major application of design centering is the cutting stock problem. The problem of cutting a gem of maximal volume with prescribed shape features from a raw gem is treated in [65] and, with the numerical method from [90], in the recent thesis [101].

### 2.3. Robust optimization

Robustness questions arise when an optimization problem is subject to uncertain data. If one wishes to treat uncertainty in an optimization problem without using stochastic information, one possible approach is to solve the problem for some nominal choice of parameters, and then to study the influence of parameter perturbations on the solution. Apparently this stability and sensitivity investigation is an *a posteriori* approach, and reasonable criteria for the special choice of the nominal parameters are needed.

In contrast to this, one can use the *a priori* approach of robust optimization which has attracted a lot of attention in recent years. In fact, robust counterparts of finite optimization problems constitute a very important application of semi-infinite programming since they arise naturally in a large number of real-life situations.

If an inequality constraint function  $G(x, p)$  depends on some uncertain parameter vector  $p$  from a so-called uncertainty set  $P \subset \mathbb{R}^m$ , then the “most cautious” or “pessimistic” way to deal with this constraint is to use its worst case reformulation

$$G(x, p) \leq 0 \quad \text{for all } p \in P,$$

which is clearly of semi-infinite type. If a point  $x$  is feasible for this semi-infinite constraint, then we have  $G(x, p) \leq 0$ , no matter what the actual parameter  $p \in P$  is. This approach is also known as the “principle of guaranteed results” (cf. [21]). When the uncertainty set  $P$  also depends on the decision variable  $x$ , we arrive at a generalized semi-infinite constraint. For example, uncertainties concerning small displacements of an aircraft may be modeled as being dependent on its speed. For an example from portfolio analysis see [90].

Similarly, if an objective function  $F(x, p)$  depends on the unknown parameter  $p \in P(x)$ , in the worst case one has to minimize the *maximal* objective value, that is, one considers the problem

$$\min_{x \in \mathbb{R}^n} \max_{p \in P(x)} F(x, p).$$

Such a minimax problem can be cast in a semi-infinite form by epigraph reformulation. Summarizing, the robust formulation of a finite problem

$$\text{FP}(p): \min_{x \in \mathbb{R}^n} F(x, p) \quad \text{s.t. } G_i(x, p) \leq 0, \quad i \in I,$$

with an unknown parameter  $p \in P(x)$  is given by

$$\begin{aligned} \text{GSIP}_{\text{RO}}: \quad & \min_{(x,z) \in \mathbb{R}^n \times \mathbb{R}} z \quad \text{s.t. } F(x, p) \leq z \quad \text{for all } p \in P(x), \\ & G_i(x, p) \leq 0 \quad \text{for all } p \in P(x), \quad i \in I. \end{aligned}$$

In [6] it is shown that under special structural assumptions the semi-infinite problem  $\text{GSIP}_{\text{RO}}$  can be reformulated as a semi-definite problem and then be solved with polynomial time algorithms [5]. The structural assumptions are essentially bilinearity of  $F$  and  $G_i$  as well as an ellipsoidal fixed uncertainty set  $P$ . Under similarly special assumptions a saddle point approach for robust programs is given in [95]. As a tailored solution method for robust optimization the so-called cascading algorithm is introduced in [54].

### 3. Geometry of the feasible set

This section focuses on the structure of the feasible set  $M$  of a generalized semi-infinite optimization problem, that is, the objective function  $f$  of GSIP will not play a role throughout Section 3.

#### 3.1. A projection formula

We define the sets

$$\begin{aligned} \mathcal{G} &= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid g(x, y) \leq 0\}, \\ \mathcal{Y} &= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid v_\ell(x, y) \leq 0, \ell \in L\}, \end{aligned}$$

for a set  $A \subset \mathbb{R}^N$  we denote by  $A^c$  the set complement of  $A$  in  $\mathbb{R}^N$ , and  $\pi$  stands for the orthogonal projection from  $\mathbb{R}^n \times \mathbb{R}^m$  to  $\mathbb{R}^n$ . The following formula provides basic geometrical insight into the topological features of feasible sets in generalized semi-infinite programming. It is proved straightforwardly by characterizing the elements of  $M^c$ .

**Lemma 3.1** (Stein [85]). *An alternative description of the feasible set is*

$$M = [\pi(\mathcal{Y} \cap \mathcal{G}^c)]^c. \tag{3.1}$$

Formula (3.1) reveals important features of  $M$  in the case of a variable index set mapping  $Y(x)$  by a simple geometric consideration. Assume for the moment that the function  $g$  is affine in  $(x, y)$  and that  $\mathcal{Y}$  is a polytope. Then  $\mathcal{Y} \cap \mathcal{G}^c$  is the intersection of the polytope  $\mathcal{Y}$  with the open halfspace  $\mathcal{G}^c$  and hence, unless  $g$  is a redundant constraint, it is a “polytope with a missing facet”. The feasible set  $M$  is the complement of the orthogonal projection to the  $x$ -space of this object. In Fig. 2, which illustrates this situation with  $x \in \mathbb{R}^2$  and  $y \in \mathbb{R}^1$ , it becomes clear geometrically that  $M$  can be the union of finitely many closed and open halfspaces. For an explicit description of  $M$  under linearity assumptions see [75,85].

These considerations show that two topological phenomena arise which are not known from standard semi-infinite optimization and which are, of course, not related to our temporary linearity assumptions:  $M$  is endowed with an inherent disjunctive structure, and  $M$  is not necessarily a closed set.

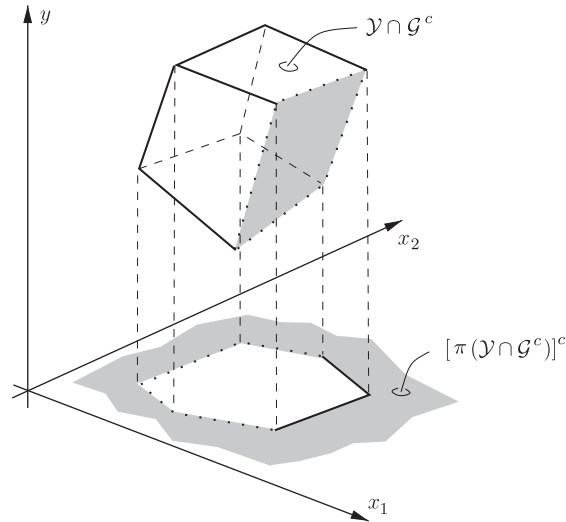


Fig. 2. Illustration of the projection formula.

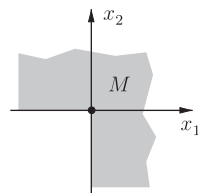


Fig. 3. A re-entrant corner point.

The following examples illustrate these features some more, with  $g$  and  $Y$  given in functional form.

**Example 3.2 (Re-entrant corner point).** For  $x \in \mathbb{R}^2$  consider the index set

$$Y(x) = \{y \in \mathbb{R} \mid y \geq x_1, y \geq x_2\}$$

and put  $g(x, y) = -y$ . Then we obtain

$$\begin{aligned} M &= \{x \in \mathbb{R}^2 \mid g(x, y) \leq 0 \text{ for all } y \in Y(x)\} \\ &= \{x \in \mathbb{R}^2 \mid y \geq 0 \text{ for all } y \in [\max(x_1, x_2), +\infty)\} \\ &= \{x \in \mathbb{R}^2 \mid \max(x_1, x_2) \geq 0\}. \end{aligned}$$

Fig. 3 illustrates that  $M$  is the union of two closed halfplanes. Note that  $M$  is nonconvex, although all defining functions are linear. More precisely,  $M$  exhibits a so-called *re-entrant corner point* at the origin.

**Example 3.3 (Local nonclosedness).** For  $x \in \mathbb{R}^2$  consider the index set

$$Y(x) = \{y \in \mathbb{R} \mid y \geq x_1, y \leq x_2\}$$

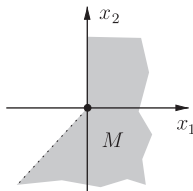


Fig. 4. Local nonclosedness.

and put again  $g(x, y) = -y$ . Now we have

$$\begin{aligned}
 M &= \{x \in \mathbb{R}^2 \mid g(x, y) \leq 0 \text{ for all } y \in Y(x)\} \\
 &= \{x \in \mathbb{R}^2 \mid y \geq 0 \text{ for all } y \in [x_1, x_2]\} \\
 &= \{x \in \mathbb{R}^2 \mid x_1 \leq x_2, y \geq 0 \text{ for all } y \in [x_1, x_2]\} \\
 &\quad \cup \{x \in \mathbb{R}^2 \mid x_1 > x_2, y \geq 0 \text{ for all } y \in \emptyset\} \\
 &= \{x \in \mathbb{R}^2 \mid x_1 \leq x_2, x_1 \geq 0\} \cup \{x \in \mathbb{R}^2 \mid x_1 > x_2\}.
 \end{aligned}$$

As depicted in Fig. 4,  $M$  is the union of an open with a closed halfplane, although all defining inequalities are nonstrict.

By a well-known theorem of Whitney [12] re-entrant corner points as in Example 3.2 can also occur in finite optimization, even with a single (degenerate) smooth inequality constraint function. There, however, the local disjunctive structure of the feasible set is destroyed under small perturbations of the defining function. In contrast to this, re-entrant corner points are *stable* in GSIP [82,85]. Even the local nonclosedness phenomenon, which does not have any analog in finite or standard semi-infinite programming, is a stable phenomenon in GSIP.

### 3.2. Topological properties of the feasible set

The key to a theoretical treatment of the topological features in the feasible set of GSIP lies in the bilevel structure of semi-infinite programming. In the following we sketch the main ideas of this approach. For stability properties of the feasible set in SIP we refer the reader to [47,48].

Under our assumptions it is easy to see that the semi-infinite constraint in GSIP is equivalent to

$$\varphi(x) := \max_{y \in Y(x)} g(x, y) \leq 0,$$

which means that the feasible set  $M$  of GSIP is the lower level set of some optimal value function:

$$M = \{x \in \mathbb{R}^n \mid \varphi(x) \leq 0\}. \tag{3.2}$$

The usual convention “ $\max_{\emptyset} = -\infty$ ” is consistent here, as an empty index set  $Y(x)$  corresponds, loosely speaking, to “the absence of restrictions” at  $x$  and, hence, to the feasibility of  $x$ .

The function  $\varphi$  is the optimal value function of the so-called *lower level problem*

$$Q(x): \quad \max_{y \in \mathbb{R}^m} g(x, y) \quad \text{s.t. } v_\ell(x, y) \leq 0, \quad \ell \in L. \tag{3.3}$$

In contrast to the upper level problem which consists in minimizing  $f$  over  $M$ , in the lower level problem  $x$  plays the role of an  $n$ -dimensional parameter, and  $y$  is the decision variable. The main computational problem in semi-infinite programming is that the lower level problem has to be solved to global optimality, even if only a stationary point of the upper level problem is sought.

The alternative description of the feasible set in (3.2) shows that the topological properties of  $M$  are determined by continuity properties of  $\varphi$ . In view of Example 3.3  $\varphi$  may be discontinuous, even if the underlying optimization problem

is defined by smooth functions. Properties of optimal value functions have been studied extensively in parametric optimization [2]. For a brief introduction to these results we refer to [85].

Recall that we assume the functions  $g$  and  $v_\ell, \ell \in L$ , to be continuous. In particular the set-valued mapping  $Y$  is closed. Since we also assume local boundedness of  $Y$ , the optimal value function  $\varphi$  can be shown to be at least upper semi-continuous. Thus points  $x \in \mathbb{R}^n$  with  $\varphi(x) < 0$  belong to the topological interior of  $M$ . However, upper semi-continuity of  $\varphi$  does not imply that the set of points with  $\varphi(x) \leq 0$  is closed, but for this we need lower semi-continuity.

A sufficient condition for lower semi-continuity of  $\varphi$  can be given in terms of another topological property of the index set mapping: the set-valued mapping  $Y$  is called *inner semi-continuous* at  $\bar{x}$  if and only if for all  $y \in Y(\bar{x})$  and all sequences  $x^v \rightarrow \bar{x}$  there are points  $y^v \in Y(x^v)$  with  $y^v \rightarrow y$ . Inner semi-continuous mappings are also called *lower semi-continuous* [7] or *open* [39]. In [39] it is shown that  $\varphi$  is lower semi-continuous at  $\bar{x}$  if  $Y$  is inner semi-continuous at  $\bar{x}$ . This implies the following result:

**Proposition 3.4.** *Let the index set mapping  $Y$  be inner semi-continuous on  $\mathbb{R}^n$ . Then  $M$  is a closed set.*

Obviously  $Y$  is not inner semi-continuous in Example 3.3. Simple examples show that inner semi-continuity of  $Y$  is not necessary for the closedness of  $M$  [85]. In applications  $Y$  is usually inner semi-continuous and the feasible set is thus closed.

If the functions  $v_\ell, \ell \in L$ , are continuously differentiable with respect to  $y$ , a sufficient condition for inner semi-continuity of  $Y$  at  $\bar{x}$  is the validity of the *Mangasarian–Fromovitz constraint qualification* (MFCQ) everywhere in  $Y(\bar{x})$ . MFCQ is said to hold at  $\bar{y}$  in  $Y(\bar{x})$  if the system  $D_y v_\ell(\bar{x}, \bar{y})\eta < 0, \ell \in L_0(\bar{x}, \bar{y})$ , has a solution  $\eta$ . Here  $D_y v_\ell$  stands for the row vector of partial derivatives of  $v_\ell$  with respect to  $y$ , and  $L_0(\bar{x}, \bar{y}) = \{\ell \in L | v_\ell(\bar{x}, \bar{y}) = 0\}$  is the lower level active index set. The stronger *linear independence constraint qualification* (LICQ) holds at  $\bar{y}$  in  $Y(\bar{x})$  if the gradients  $D_y v_\ell(\bar{x}, \bar{y}), \ell \in L_0(\bar{x}, \bar{y})$ , are linearly independent. Proposition 3.4 yields that  $M$  is closed if for all  $x \in \mathbb{R}^n$  MFCQ or even LICQ hold everywhere in  $Y(x)$ .

This sufficient condition for closedness can be weakened considerably. In fact, for investigations of the local structure of  $M$  or of local optimality conditions we are only interested in points from the boundary  $\partial M$  of  $M$ . In view of the upper semi-continuity of  $\varphi$  it suffices to consider the zeros of  $\varphi$ , that is, points  $x \in \mathbb{R}^n$  for which  $Q(x)$  has vanishing maximal value. We denote the corresponding globally maximal points of  $Q(x)$  by

$$Y_0(x) = \{y \in Y(x) | g(x, y) = 0\}.$$

The set  $Y_0(x)$  is also called the upper level active index set of GSIP. In [20] it is shown that under our assumptions  $\varphi$  is continuous at some  $\bar{x} \in \mathbb{R}^n$  if  $Y(\bar{x})$  is nonempty and MFCQ holds at *some* element of  $Y_0(\bar{x})$ . Hence  $M$  is closed if for all  $x \in \mathbb{R}^n$  the index set  $Y(x)$  is nonempty and MFCQ or even LICQ hold at some element of  $Y_0(x)$ .

### 3.3. The reduction ansatz

For theoretical as well as numerical purposes it is of crucial importance to keep track of the elements of  $Y_0(x)$  for varying  $x$ . An important piece of information is that these points solve the lower level problem, that is, for functions  $g$  and  $v_\ell, \ell \in L$ , which are continuously differentiable with respect to  $y$  they satisfy the first-order necessary optimality condition of Karush–Kuhn–Tucker (KKT): let

$$\mathcal{L}(x, y, \gamma) = g(x, y) - \gamma^T v(x, y)$$

denote the Lagrangian of  $Q(x)$  with multiplier vector  $\gamma \in \mathbb{R}^s$ . Then for  $\bar{x} \in M$  and each  $\bar{y} \in Y_0(\bar{x})$  such that MFCQ holds at  $\bar{y}$  in  $Q(\bar{x})$ , there exist multipliers  $\bar{\gamma} \geq 0$  with  $D_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\gamma}) = 0$  and  $\gamma_\ell \cdot v_\ell(\bar{x}, \bar{y}) = 0, \ell \in L$ . Note that the multiplier vector  $\bar{\gamma}$  is uniquely determined if instead of MFCQ the stronger LICQ holds at  $\bar{y}$ .

Keeping track of the elements of  $Y_0(x)$  can now be achieved for example by means of the implicit function theorem, if the functions  $g$  and  $v_\ell, \ell \in L$ , are  $C^2$  with respect to  $y$ . For  $\bar{x} \in M$  a local maximizer  $\bar{y}$  of  $Q(\bar{x})$  is called nondegenerate in the sense of Jongen/Jonker/Twilt [43], if LICQ, strict complementary slackness (SCS) and the second-order sufficiency condition (SOSC)  $D_{yy}^2 \mathcal{L}(\bar{x}, \bar{y}, \bar{\gamma})|_{T_{\bar{y}} Y(\bar{x})} < 0$  are satisfied. Here  $T_{\bar{y}} Y(\bar{x})$  is the tangent space to  $Y(\bar{x})$  at  $\bar{y}$ , and  $A < 0$  stands for the negative definiteness of a matrix  $A$ . SCS means  $\bar{\gamma}_\ell > 0$  for all  $\ell \in L_0(\bar{x}, \bar{y})$ . The *reduction ansatz* is said to hold at  $\bar{x} \in M$  if all global maximizers of  $Q(\bar{x})$  are nondegenerate. Since nondegenerate maximizers are isolated,



and  $Y(\bar{x})$  is a compact set, the set  $Y_0(\bar{x})$  can only contain finitely many points, say  $Y_0(\bar{x}) = \{\bar{y}^1, \dots, \bar{y}^p\}$  with  $p \in \mathbb{N}$ . By a result from [19] the local variation of these points with  $x$  can be described by the implicit function theorem.

In fact, for  $x$  locally around  $\bar{x}$  there exist continuously differentiable functions  $y^i(x)$ ,  $1 \leq i \leq p$ , with  $y^i(\bar{x}) = \bar{y}^i$  such that  $y^i(x)$  is the locally unique local maximizer of  $Q(x)$  around  $\bar{y}^i$ . Moreover, if  $\bar{\gamma}^i$  is the uniquely determined multiplier vector corresponding to  $\bar{y}^i$ , then there exists a continuously differentiable function  $\gamma^i(x)$  with  $\gamma^i(\bar{x}) = \bar{\gamma}^i$  such that  $\gamma^i(x)$  is the unique multiplier vector corresponding to  $y^i(x)$ ,  $1 \leq i \leq p$ . It turns out that the functions  $\varphi_i(x) := g(x, y^i(x))$  are even  $C^2$  in a neighborhood of  $\bar{x}$ . Their gradients are

$$D\varphi_i(\bar{x}) = D_x \mathcal{L}(\bar{x}, \bar{y}^i, \bar{\gamma}^i). \tag{3.4}$$

The reduction ansatz was originally formulated for standard semi-infinite problems in [32,100] under weaker regularity assumptions. It was transferred to generalized semi-infinite problems in [36]. A major consequence of the reduction ansatz is the so-called reduction lemma [36]: if the reduction ansatz holds at  $\bar{x}$ , then for all  $x$  from a neighborhood  $U$  of  $\bar{x}$  we have

$$\varphi(x) = \max_{1 \leq i \leq p} \varphi_i(x).$$

In view of (3.2) this means that  $M$  can locally be described by finitely many  $C^2$ -constraints, that is, GSIP locally looks like a smooth finite optimization problem:

$$M \cap U = \{x \in U \mid g(x, y^i(x)) \leq 0, \quad i = 1, \dots, p\}. \tag{3.5}$$

In particular, locally around  $\bar{x}$  the set  $M$  is closed and only in degenerate situations will it possess a re-entrant corner point at  $\bar{x}$ .

For *standard* semi-infinite problems the reduction ansatz is a natural assumption in the sense that for problems with defining functions in general position it holds at each local minimizer [80,104]. For GSIP this result can be transferred to local maximizers  $\bar{x}$  with  $|Y_0(\bar{x})| \geq n$  [82]. Moreover, in [89] it is shown that it holds in the “completely linear” case, that is, when the defining functions  $f, g$  and  $v_\ell, \ell \in L$ , of GSIP are affine on their respective domains. For GSIP without these special structures it is not yet known whether the reduction ansatz generically holds at all local minimizers. The reduction ansatz also serves as a basic regularity condition for numerical solution methods (see Section 6).

### 3.4. First-order properties of the feasible set

Since the reduction ansatz cannot be expected to hold generically *everywhere* in  $M$ , we study the first-order structure of  $M$  under considerably weaker assumptions in this section. In particular, we will explain re-entrant corner points in terms of the lower level KKT multipliers.

For the first-order approximation of  $M$  we define the *contingent cone*  $\Gamma^*(\bar{x}, M)$  to  $M$  at  $\bar{x}$  as follows:  $\bar{d} \in \Gamma^*(\bar{x}, M)$  if and only if there exist sequences of scalars  $(t^v)_{v \in \mathbb{N}}$  and of vectors  $(d^v)_{v \in \mathbb{N}}$  such that

$$t^v \searrow 0, \quad d^v \rightarrow \bar{d} \quad (v \rightarrow \infty) \quad \text{and} \quad \bar{x} + t^v d^v \in M \quad \text{for all } v \in \mathbb{N}.$$

Moreover, we define the *inner tangent cone*  $\Gamma(\bar{x}, M)$  to  $M$  at  $\bar{x}$  as:  $\bar{d} \in \Gamma(\bar{x}, M)$  if and only if there exist some  $\bar{t} > 0$  and a neighborhood  $D$  of  $\bar{d}$  such that

$$\bar{x} + td \in M \quad \text{for all } t \in (0, \bar{t}), \quad d \in D.$$

The contingent cone is a closed cone, not necessarily convex, containing first-order information about  $M$ . In Example 3.2 the set  $M$  coincides with  $\Gamma^*(0, M)$ . In view of (3.2) the contingent cone to  $M$  at  $\bar{x}$  should be related to a level set of a first-order approximation of  $\varphi$  at  $\bar{x}$ . Unfortunately the differentiability properties of  $\varphi$  can be very weak, so that we recall the definition of *lower and upper directional derivatives* of  $\varphi$  at  $\bar{x}$  in direction  $\bar{d}$  in the Hadamard sense [11]:

$$\varphi'_-(\bar{x}, \bar{d}) = \liminf_{t \searrow 0, d \rightarrow \bar{d}} \frac{\varphi(\bar{x} + td) - \varphi(\bar{x})}{t}$$

and

$$\phi'_+(\bar{x}, \bar{d}) = \limsup_{t \searrow 0, d \rightarrow \bar{d}} \frac{\phi(\bar{x} + td) - \phi(\bar{x})}{t}.$$

$\phi$  is called *directionally differentiable* at  $\bar{x}$  (in the Hadamard sense) if for each direction  $\bar{d} \neq 0$  we have  $\phi'_-(\bar{x}, \bar{d}) = \phi'_+(\bar{x}, \bar{d})$ . In this case, we put

$$\phi'(\bar{x}, \bar{d}) = \lim_{t \searrow 0, d \rightarrow \bar{d}} \frac{\phi(\bar{x} + td) - \phi(\bar{x})}{t}.$$

We define the *outer linearization cone* of  $M$  at  $\bar{x}$  as

$$L^*(\bar{x}, M) = \{d \in \mathbb{R}^n \mid \phi'_-(\bar{x}, d) \leq 0\}$$

and the *inner linearization cone* by

$$L(\bar{x}, M) = \{d \in \mathbb{R}^n \mid \phi'_+(\bar{x}, d) < 0\}.$$

**Lemma 3.5** (Laurent [58], Stein [86]). *For  $\bar{x} \in \partial M \cap M$  the following chain of inclusions holds:*

$$L(\bar{x}, M) \subset \Gamma(\bar{x}, M) \subset \Gamma^*(\bar{x}, M) \subset L^*(\bar{x}, M).$$

A good first-order description of  $M$  around  $\bar{x}$  by the contingent cone  $\Gamma^*(\bar{x}, M)$  can thus be obtained if the linearization cones  $L(\bar{x}, M)$  and  $L^*(\bar{x}, M)$  do not differ too much from each other.

For example, in standard semi-infinite programming the index set mapping  $Y(x) \equiv Y$  is constant, and the Theorem of Danskin [15] then says that  $\phi$  is directionally differentiable with

$$\phi'(\bar{x}, d) = \max_{y \in Y_0(\bar{x})} D_x g(\bar{x}, y)d$$

for all  $d \in \mathbb{R}^n$ . The linearization cones

$$L(\bar{x}, M) = \bigcap_{y \in Y_0(\bar{x})} \{d \in \mathbb{R}^n \mid D_x g(\bar{x}, y)d < 0\}$$

and

$$L^*(\bar{x}, M) = \bigcap_{y \in Y_0(\bar{x})} \{d \in \mathbb{R}^n \mid D_x g(\bar{x}, y)d \leq 0\}$$

thus differ only by the strictness of inequalities, and they do not possess a disjunctive structure.

If in GSIP the reduction ansatz (cf. Section 3.3) holds at  $\bar{x}$ , using (3.4) it is not hard to see that  $\phi$  is directionally differentiable with

$$\phi'(\bar{x}, d) = \max_{1 \leq i \leq p} D_x \mathcal{L}(\bar{x}, \bar{y}^i, \bar{\gamma}^i)d$$

for all  $d \in \mathbb{R}^n$ . The linearization cones

$$L(\bar{x}, M) = \bigcap_{i=1}^p \{d \in \mathbb{R}^n \mid D_x \mathcal{L}(\bar{x}, \bar{y}^i, \bar{\gamma}^i)d < 0\}$$

and

$$L^*(\bar{x}, M) = \bigcap_{i=1}^p \{d \in \mathbb{R}^n \mid D_x \mathcal{L}(\bar{x}, \bar{y}^i, \bar{\gamma}^i)d \leq 0\}$$

again differ only by the strictness of inequalities, and they do not possess a disjunctive structure.

Under weaker assumptions than the reduction ansatz the situation in GSIP becomes more involved, since  $\varphi$  does not even have to be directionally differentiable. The following estimates for the upper and lower directional derivatives from [20,59] are known to be tight: for  $\bar{x} \in \partial M \cap M$  such that MFCQ is satisfied at each  $y \in Y_0(\bar{x})$  we have for each  $d \in \mathbb{R}^n$

$$\begin{aligned} \sup_{y \in Y_0(\bar{x})} \min_{\gamma \in \text{KKT}(\bar{x}, y)} D_x \mathcal{L}(\bar{x}, y, \gamma) d &\leq \varphi'_-(\bar{x}, d) \\ &\leq \varphi'_+(\bar{x}, d) \leq \max_{y \in Y_0(\bar{x})} \max_{\gamma \in \text{KKT}(\bar{x}, y)} D_x \mathcal{L}(\bar{x}, y, \gamma) d. \end{aligned}$$

Here

$$\text{KKT}(x, y) = \{\gamma \in \mathbb{R}^s \mid \gamma \geq 0, D_y \mathcal{L}(x, y, \gamma) = 0, \gamma_\ell \cdot v_\ell(x, y) = 0, \ell \in L\}$$

denotes the set of KKT multipliers at  $y$  in  $Q(x)$ .

At least this gives us estimates for the linearization cones:

$$\begin{aligned} &\bigcap_{y \in Y_0(\bar{x})} \bigcap_{\gamma \in \text{KKT}(\bar{x}, y)} \{d \in \mathbb{R}^n \mid D_x \mathcal{L}(\bar{x}, y, \gamma) d < 0\} \\ &\subset L(\bar{x}, M) \subset \Gamma^*(\bar{x}, M) \subset L^*(\bar{x}, M) \\ &\subset \bigcap_{y \in Y_0(\bar{x})} \bigcup_{\gamma \in \text{KKT}(\bar{x}, y)} \{d \in \mathbb{R}^n \mid D_x \mathcal{L}(\bar{x}, y, \gamma) d \leq 0\}. \end{aligned}$$

In [84] an analogous result is given without the assumption of MFCQ in  $Y_0(\bar{x})$ . However, the estimate for the inner linearization cone is rather poor in many situations in which the problem data are endowed with a special structure. In fact, in Example 3.2 it coincides with the topological interior of the *first orthant* rather than with the topological interior of  $M$ . A critical point notion based on this estimate treats re-entrant corner points as candidates for local minimizers of GSIP, although many feasible directions of first-order descent for  $f$  may exist in the actual feasible set  $M$ .

In the case  $\Gamma^*(\bar{x}, M) = L^*(\bar{x}, M)$  (see also Section 3.5) we see that a disjunctive structure of  $\Gamma^*(\bar{x}, M)$  is intimately related to the nonuniqueness of the lower level KKT multipliers.

This becomes clearer if we assume that the lower level problems  $Q(x)$ ,  $x \in U$ , are convex for some neighborhood  $U$  of  $\bar{x}$ , and that  $Y(\bar{x})$  possesses a Slater point. Note that these assumptions are satisfied in Example 3.2. Due to results from [25,40,71] the multiplier set  $\text{KKT}(\bar{x})$  then does not depend on  $y \in Y_0(\bar{x})$  and  $\varphi$  is directionally differentiable at  $\bar{x}$  with

$$\varphi'(\bar{x}, d) = \min_{\gamma \in \text{KKT}(\bar{x})} \max_{y \in Y_0(\bar{x})} D_x \mathcal{L}(\bar{x}, y, \gamma) d$$

for all  $d \in \mathbb{R}^n$ . As a consequence we obtain

$$L(\bar{x}, M) = \bigcup_{\gamma \in \text{KKT}(\bar{x})} \bigcap_{y \in Y_0(\bar{x})} \{d \in \mathbb{R}^n \mid D_x \mathcal{L}(\bar{x}, y, \gamma) d < 0\}$$

and

$$L^*(\bar{x}, M) = \bigcup_{\gamma \in \text{KKT}(\bar{x})} \bigcap_{y \in Y_0(\bar{x})} \{d \in \mathbb{R}^n \mid D_x \mathcal{L}(\bar{x}, y, \gamma) d \leq 0\}.$$

Now both the inner and the outer linearization cone possess a disjunctive structure, and they only differ by the strictness of inequalities. Moreover it becomes obvious that the occurrence of stable re-entrant corner points in GSIP is caused by nonunique lower level KKT multipliers. Consequently a stable re-entrant corner point at  $\bar{x}$  can be avoided if LICQ holds at all  $y \in Y_0(\bar{x})$ . A characterization of unique multipliers which is weaker than LICQ can be found in [57]. For more details on lower level problems with a special structure see [73,76,85].

Together with the results of Section 3.2 we find that local nonclosedness of  $M$  around  $\bar{x}$  is related to the failure of MFCQ at all  $y \in Y_0(\bar{x})$ , and that a re-entrant corner point at  $\bar{x}$  is related to the failure of LICQ at some  $y \in Y_0(\bar{x})$ .

Thus we see that the major difference between standard and generalized semi-infinite programming is the possibility of violated constraint qualifications in the lower level problem. In fact, in [98] it is shown that up to a smooth coordinate transformation a GSIP is equivalent to a standard SIP if for all  $x \in \mathbb{R}^n$  LICQ holds everywhere in  $Y(x)$ . On the other hand, results from [82] show that generically LICQ cannot be expected to hold everywhere in the lower level problem. Therefore not only in degenerate cases does the feasible set of GSIP have richer structural features than the feasible set in standard SIP.

### 3.5. Constraint qualifications

Throughout this section let the functions  $f, g$ , and  $v_\ell, \ell \in L$ , be continuously differentiable. It is well known [3] that at a local minimizer  $\bar{x}$  of  $f$  on  $M$  the following primal first-order necessary optimality condition holds:

$$\{d \in \mathbb{R}^n \mid Df(\bar{x})d < 0\} \cap \Gamma^*(\bar{x}, M) = \emptyset. \tag{3.6}$$

To obtain a more explicit condition from (3.6) we need an explicit description of  $\Gamma^*(\bar{x}, M)$ . A good candidate would be the outer linearization cone  $L^*(\bar{x}, M)$  which contains the contingent cone by Lemma 3.5. The simple example

$$M = \{x \in \mathbb{R} \mid x^2 \leq 0\} \tag{3.7}$$

shows, however, that  $\Gamma^*(0, M)$  can be a proper subset of  $L^*(0, M)$ . In this case we cannot replace the contingent cone in (3.6) by the outer linearization cone.

On the other hand, in view of Lemma 3.5 it is always possible to replace the contingent cone in (3.6) by the inner linearization cone. However, the example in (3.7) shows that the resulting optimality condition may be trivially satisfied since  $L(0, M)$  can be void itself.

These observations give rise to the following definitions.

**Definition 3.6.** We say that the *extended Mangasarian–Fromovitz constraint qualification* (briefly: EMFCQ) holds at  $\bar{x} \in M$  if  $L(\bar{x}, M) \neq \emptyset$  and that the *Extended Abadie constraint qualification* (briefly: EACQ) holds at  $\bar{x} \in M$  if  $\Gamma^*(\bar{x}, M) = L^*(\bar{x}, M)$ .

Note that EMFCQ coincides with the Mangasarian–Fromovitz constraint qualification [64] for finite differentiable optimization problems. Furthermore it is obvious that EACQ coincides with the Abadie constraint qualification (ACQ, [1]) for finite differentiable optimization problems. For a survey of the multitude of other constraint qualifications in smooth finite optimization see [66].

In finite optimization MFCQ is stronger than ACQ. For GSIP this is not necessarily the case as an example in [86] shows. We have, however, the following result:

**Proposition 3.7** (Stein [84]). *Let  $\varphi$  be directionally differentiable at  $\bar{x} \in M$ , and let the directional derivative  $\varphi'(\bar{x}, \cdot)$  be subadditive with respect to the direction. Then EMFCQ implies EACQ at  $\bar{x}$ .*

Explicit formulations of EMFCQ under different structural assumptions on the lower level problem  $Q(\bar{x})$  can easily be obtained from the descriptions of  $L(\bar{x}, M)$  in Section 3.4.

In general, for  $\bar{x} \in M$  MFCQ need not hold at each  $\bar{y} \in Y_0(\bar{x})$ ; but,  $\bar{y} \in Y_0(\bar{x})$  as a solution of the lower level problem  $Q(\bar{x})$  satisfies the first-order necessary optimality condition of Fritz John [42]: there exist nonnegative multipliers  $\bar{\alpha} \in \mathbb{R}, \bar{\gamma} \in \mathbb{R}^s$ , with

$$\bar{\gamma}_\ell v_\ell(\bar{x}, \bar{y}) = 0, \quad \ell \in L$$

and

$$D_y \mathcal{L}^0(\bar{x}, \bar{y}, \bar{\alpha}, \bar{\gamma}) = 0, \quad \bar{\alpha} + \sum_{\ell \in L} \bar{\gamma}_\ell = 1, \tag{3.8}$$

where

$$\mathcal{L}^0(x, y, \alpha, \gamma) = \alpha g(x, y) - \gamma^T v(x, y).$$

In [47] it is shown, that the nonempty sets

$$F(\bar{x}, \bar{y}) = \{(\alpha, \gamma) \in \mathbb{R}^{1+s} | (\alpha, \gamma) \text{ satisfies (3.8)}\}$$

and

$$V(\bar{x}) = \bigcup_{y \in Y_0(\bar{x})} \{-D_x \mathcal{L}^0(\bar{x}, y, \alpha, \gamma) | (\alpha, \gamma) \in F(\bar{x}, y)\}$$

are compact. We use the approximations

$$H^*(\bar{x}, M) = \left\{ d \in \mathbb{R}^n \mid \max_{y \in Y_0(\bar{x})} \min_{(\alpha, \gamma) \in F(\bar{x}, y)} D_x \mathcal{L}^0(\bar{x}, y, \alpha, \gamma) d \leq 0 \right\}$$

and

$$H(\bar{x}, M) = \{d \in \mathbb{R}^n | w^T d > 0, w \in V(\bar{x})\},$$

of the outer and inner linearization cone of  $M$  at  $\bar{x}$ , respectively, and introduce the following corresponding constraint qualifications.

**Definition 3.8.** We say that EMFCQ\* holds at  $\bar{x} \in M$  if  $H(\bar{x}, M) \neq \emptyset$ . We say that EACQ\* holds at  $\bar{x} \in M$  if  $\Gamma^*(\bar{x}, M) = H^*(\bar{x}, M)$ .

In order to generalize the *Kuhn–Tucker constraint qualification KTCQ* (cf. [56,94]) to GSIP define the cone of attainable directions of  $M$  at  $\bar{x}$ :

$$A(\bar{x}, M) = \left\{ \zeta \in \mathbb{R}^n \setminus \{0\} \mid \begin{array}{l} \text{There exist some } \tau > 0 \text{ and a} \\ \text{continuously differentiable arc} \\ C : [0, \tau] \rightarrow \mathbb{R}^n \text{ such that } C(0) = \bar{x}, \\ C'(0) = \zeta, \text{ and } C(t) \in M, t \in [0, \tau] \end{array} \right\}$$

(where  $C'(0) = (DC_1(0), \dots, DC_n(0))$ ). In particular, it is  $\text{cl}A(\bar{x}, M) \subset \Gamma^*(\bar{x}, M)$  [29].

**Definition 3.9.** We say that the *extended Kuhn–Tucker constraint qualification* (briefly: EKTCQ) holds at  $\bar{x} \in M$  if  $H^*(\bar{x}, M) \subset \text{cl}(A(\bar{x}, M))$ .

EKTCQ is a stronger condition than EACQ\*. In finite optimization, MFCQ is stronger than KTCQ; however, an example in [29] shows that, in general, this is not the case for GSIP.

#### 4. First-order optimality conditions

Assume throughout this section that the functions  $f, g$  and  $v_\ell, \ell \in L$  are continuously differentiable. We present three approaches to first-order conditions for  $\bar{x} \in M$  to be a local minimizer of GSIP. For a more detailed study on first-order optimality conditions for GSIP we refer to [47,73,84,85,88].

##### 4.1. A Fritz–John and a KKT condition

For a set  $A \subset \mathbb{R}^n$  define its dual cone by  $A^0 = \{d \in \mathbb{R}^n | d^T x \geq 0, x \in A\}$ . Let  $\bar{x}$  be a local minimizer of GSIP. If EKTCQ holds at  $\bar{x}$ , then

$$V(\bar{x})^0 \subset H^*(\bar{x}, M) \underset{\text{by EKTCQ}}{\subset} \text{cl}(A(\bar{x}, M)) \subset \Gamma^*(\bar{x}, M) \underset{\text{by (3.6)}}{\subset} \{Df(\bar{x})\}^0.$$

If  $\text{co}(V(\bar{x}))$  is closed (co denotes the convex cone hull), then the latter condition  $V(\bar{x})^0 \subset \{Df(\bar{x})\}^0$  is equivalent to the KKT first-order optimality condition

$$Df(\bar{x}) \in \text{co}(V(\bar{x})). \tag{4.1}$$

Furthermore, without assuming any constraint qualification at  $\bar{x}$  we obtain the following Fritz–John first-order optimality condition.

**Theorem 4.1** (Jongen et al. [47], Guerra Vázquez and Rückmann [29]). *Let  $\bar{x} \in M$  be a local minimizer of GSIP with  $Y_0(\bar{x}) \neq \emptyset$  (if  $Y_0(\bar{x}) = \emptyset$ , a first-order condition reduces to  $Df(\bar{x}) = 0$ ). Then:*

- (i) *There exist  $y^j \in Y_0(\bar{x})$ ,  $(\alpha^j, \gamma^j) \in F(\bar{x}, y^j)$ ,  $j = 1, \dots, p$  and multipliers  $\mu_j \geq 0$ ,  $j = 0, \dots, p$ , not all of them being zero, such that*

$$\mu_0 Df(\bar{x}) + \sum_{j=1}^p \mu_j D_x \mathcal{L}^0(\bar{x}, y^j, \alpha^j, \gamma^j) = 0. \tag{4.2}$$

- (ii) *If EMFCQ\* holds at  $\bar{x} \in M$ , then we obtain (4.1).*
- (iii) *If EKTCQ or EACQ\* holds at  $\bar{x} \in M$  and  $\text{co}(V(\bar{x}))$  is closed, then we obtain (4.1).*

If EMFCQ\* holds at  $\bar{x} \in M$  or when  $Y_0(\bar{x})$  is finite, then  $\text{co}(V(\bar{x}))$  is always closed; however, if only EKTCQ or EACQ\* holds at  $\bar{x}$ , then, in general, one has to assume the closedness of  $\text{co}(V(\bar{x}))$  in order to satisfy (4.1). It is obvious that (4.2) describes a family of first-order optimality conditions which is parameterized by the choice of the multipliers  $(\alpha^j, \gamma^j) \in F(\bar{x}, y^j)$ ,  $j = 1, \dots, p$ . In [47] it is shown that there do not exist for each possible choice of  $(\alpha^j, \gamma^j) \in F(\bar{x}, y^j)$ ,  $j = 1, \dots, p$  corresponding multipliers  $\mu_j \geq 0$ ,  $j = 1, \dots, p$ , not all of them being zero, such that (4.2) can be satisfied. However, we have the following result.

**Proposition 4.2** (Rückmann and Shapiro [73]). *Assume that  $\bar{x}$  is a local minimizer of GSIP and that the optimal value function  $\varphi$  is directionally differentiable with*

$$\varphi'(\bar{x}, d) = \max_{y \in Y_0(\bar{x})} \inf_{\gamma \in \text{KKT}(\bar{x}, y)} D_x \mathcal{L}(\bar{x}, y, \gamma) d. \tag{4.3}$$

Furthermore, choose for each  $y \in Y_0(\bar{x})$  a vector of multipliers  $\gamma(y) \in \text{KKT}(\bar{x}, y)$  such that the set  $\{D_x \mathcal{L}(\bar{x}, y, \gamma(y)), y \in Y_0(\bar{x})\}$  is compact. Then, there exist  $y^j \in Y_0(\bar{x})$ ,  $j = 1, \dots, p$  and corresponding multipliers  $\mu_j \geq 0$ ,  $j = 0, \dots, p$ , not all of them being zero, satisfying (4.2) with

$$\mu_0 Df(\bar{x}) + \sum_{j=1}^p \mu_j D_x \mathcal{L}^0(\bar{x}, y^j, 1, \gamma(y^j)) = 0.$$

An example in [47] shows that a choice of  $\gamma(y)$ ,  $y \in Y_0(\bar{x})$  as described in the latter proposition is not always possible; however, in the important case that  $Y_0(\bar{x})$  is a finite set, such a choice always exists. In particular, if  $Y_0(\bar{x}) = \{\bar{y}\}$  is a singleton and there exist  $\gamma^1, \gamma^2 \in \text{KKT}(\bar{x}, \bar{y})$  such that  $D_x \mathcal{L}(\bar{x}, \bar{y}, \gamma^1)$  and  $D_x \mathcal{L}(\bar{x}, \bar{y}, \gamma^2)$  are linearly independent, then  $Df(\bar{x}) = 0$  [73].

#### 4.2. First-order conditions obtained from the linearized problem

In the remainder of this section let  $\bar{x} \in M$  with  $Y_0(\bar{x}) \neq \emptyset$  and assume that  $\varphi$  is directionally differentiable with (4.3). We consider the following linearization of GSIP [73]:

$$\text{Min } Df(\bar{x})d \quad \text{s.t. } d \in M^1 \tag{4.4}$$

with

$$M^1 = \{d \in \mathbb{R}^n \mid \varphi'(\bar{x}, d) \leq 0\}$$

$$= \left\{ d \in \mathbb{R}^n \mid \inf_{\gamma \in \text{KKT}(\bar{x}, y)} D_x \mathcal{L}(\bar{x}, y, \gamma) d \leq 0, y \in Y_0(\bar{x}) \right\}.$$

If the constraint qualification

$$\{d \in \mathbb{R}^n \mid \varphi'(\bar{x}, d) \leq 0\} = \text{cl}\{d \in \mathbb{R}^n \mid \varphi'(\bar{x}, d) < 0\}$$

holds, then a first-order necessary optimality condition for GSIP is that  $\bar{d} = 0$  is a local minimizer of (4.4). If one does not wish to use a constraint qualification, one can consider the auxiliary directionally differentiable function

$$\psi(x) := \max\{f(x) - f(\bar{x}), \varphi(x)\}. \tag{4.5}$$

If  $\bar{x}$  is a local minimizer of GSIP, then  $\bar{x}$  is also a local minimizer of the (unconstrained) function  $\psi$  with  $\psi(\bar{x}) = 0$  and, hence,  $\psi'(d) \geq 0$  for all  $d \in \mathbb{R}^n$ . Obviously, the latter condition means that either the optimal value of the problem

$$\inf Df(\bar{x})d \quad \text{s.t. } d \in M^2 \tag{4.6}$$

is zero with

$$M^2 = \{d \in \mathbb{R}^n \mid \varphi'(\bar{x}, d) < 0\}$$

$$= \left\{ d \in \mathbb{R}^n \mid \inf_{\gamma \in \text{KKT}(\bar{x}, y)} D_x \mathcal{L}(\bar{x}, y, \gamma) d < 0, y \in Y_0(\bar{x}) \right\}$$

or  $M^2 = \emptyset$ . Now, we assume for a moment that  $Y_0(\bar{x}) = \{\bar{y}\}$  is a singleton and that  $\text{KKT}(\bar{x}, \bar{y})$  is not a singleton. Then, the feasible set  $M^1$  of (4.4) is the union of halfspaces (cf. Example 3.2), that is, the feasible set of the linearized problem (4.4) is not convex! If, furthermore,  $\bar{d} = 0$  is a local minimizer of (4.4), then for each  $\gamma \in \text{KKT}(\bar{x}, \bar{y})$  it is also a local minimizer of the problem

$$\text{Min } Df(\bar{x})d \quad \text{s.t. } \{d \in \mathbb{R}^n \mid D_x \mathcal{L}(\bar{x}, \bar{y}, \gamma) d \leq 0\}$$

and the vectors  $Df(\bar{x})$  and  $D_x \mathcal{L}(\bar{x}, \bar{y}, \gamma)$  have to be linearly dependent.

### 4.3. First-order conditions based on quasidifferentiable functions

In the remainder of this section we assume that  $Y_0(\bar{x}) = \{y^1, \dots, y^p\}$  and that MFCQ holds at each  $y^j, j = 1, \dots, p$ . Following [73], we derive first-order optimality conditions for GSIP by using the calculus of quasidifferentiable functions based on the linearizations (4.4) and (4.6). Let

$$B^j = -\text{conv} \left( \bigcup_{\gamma \in \text{KKT}(\bar{x}, y^j)} D_x \mathcal{L}(\bar{x}, y^j, \gamma) \right), \quad j = 1, \dots, p$$

(conv denotes the convex hull) and

$$\delta(d, B^j) = \sup_{\xi \in B^j} \xi^T d, \quad j = 1, \dots, p$$

be the corresponding support function. Since MFCQ holds at  $y^j$ , the sets  $B^j, j = 1, \dots, p$  are compact. Then, (4.3) implies

$$\varphi'(\bar{x}, d) = \max\{-\delta(d, B^j), j = 1, \dots, p\}$$

and, by Demyanov and Vasilev [16], we have

$$\varphi'(\bar{x}, d) = \delta(d, C_1) - \delta(d, C_2),$$

where

$$C_1 = \text{conv} \left( \bigcup_{j=1}^p \left( \sum_{v \neq j} B^v \right) \right)$$

and  $C_2 = B^1 + B^2 + \dots + B^p$  are compact sets, too. We obtain the following necessary and sufficient first-order optimality conditions.

**Proposition 4.3** (Demyanov and Vasilev [16], Shapiro [77]). (i) If  $\bar{x}$  is a local minimizer of GSIP, then

$$C_2 \subset \text{conv}(C_1 \cup (Df(\bar{x}) + C_2)).$$

(ii) If

$$C_2 \subset \text{int}(\text{conv}(C_1 \cup (Df(\bar{x}) + C_2))), \tag{4.7}$$

then  $\bar{x}$  is a local minimizer of GSIP (int denotes the set of interior points).

Note that (4.7) means that  $\psi'(\bar{x}, d) > 0$  for all  $d \in \mathbb{R}^n$  (where  $\psi$  is defined as in (4.5)).

### 5. Second-order optimality conditions

Assume throughout this section that the functions  $f, g$ , and  $v_\ell, \ell \in L$ , are twice continuously differentiable and that  $\bar{x} \in M$  is the point under consideration with  $Y_0(\bar{x}) \neq \emptyset$ . In [36], second-order optimality conditions for GSIP have been presented under the assumption that the reduction ansatz holds at  $\bar{x}$ . Later, [74] derived second-order optimality conditions for GSIP without assuming the reduction ansatz; the results in this section are taken from [74]. We also refer to the related papers [53,79].

#### 5.1. Second-order optimality conditions for an unconstrained nonsmooth problem

According to Section 4.2, if  $\bar{x} \in M$  is a local minimizer of GSIP, then  $\bar{x}$  is also a local minimizer of the (unconstrained) function  $\psi$  (defined in (4.5)) with  $\psi(\bar{x}) = 0$ . We start this section with a general discussion of first- and second-order optimality conditions for the following unconstrained optimization problem:

$$(P) \quad \text{Min } \chi(x) \quad \text{s.t. } x \in \mathbb{R}^n,$$

where  $\chi$  is a real valued function which is locally Lipschitz continuous and second-order (parabolically) directionally differentiable at  $\bar{x} \in \mathbb{R}^n$ , where the latter condition means that  $\chi$  is directionally differentiable at  $\bar{x}$  and the limit

$$\chi''(\bar{x}; d, u) := \lim_{t \downarrow 0} \frac{\chi(\bar{x} + td + \frac{1}{2}t^2u) - \chi(\bar{x}) - t\chi'(\bar{x}, d)}{\frac{1}{2}t^2} \tag{5.1}$$

exists for any  $d, u \in \mathbb{R}^n$ . It is well known that directional differentiability and local Lipschitz continuity imply that

$$\chi'(\bar{x}, d) = \lim_{\substack{t \downarrow 0 \\ d' \rightarrow d}} \frac{\chi(\bar{x} + td') - \chi(\bar{x})}{t},$$

that is,  $\chi$  is directionally differentiable at  $\bar{x}$  in the sense of Hadamard. Then, a first-order necessary optimality condition for  $\bar{x}$  to be a local minimizer of (P) is

$$\chi'(\bar{x}, d) \geq 0 \quad \text{for all } d \in \mathbb{R}^n. \tag{5.2}$$

Denoting the cone of critical directions at  $\bar{x}$  by

$$C(\bar{x}) = \{d \in \mathbb{R}^n \mid \chi'(\bar{x}, d) = 0\}, \tag{5.3}$$



a second-order necessary optimality condition is

$$\chi''(\bar{x}; d, u) \geq 0 \quad \text{for all } d \in C(\bar{x}) \text{ and all } u \in \mathbb{R}^n.$$

According to (5.1), local optimality in the latter condition is verified along parabolic paths. Now, we recall the following sufficient optimality condition for  $\bar{x}$ : The *second-order growth condition* is said to hold at  $\bar{x}$  if there exist a constant  $c > 0$  and a neighborhood  $N$  of  $\bar{x}$  such that

$$\chi(x) \geq \chi(\bar{x}) + c\|x - \bar{x}\|^2 \quad \text{for all } x \in N.$$

In particular, the second-order growth condition implies that

$$\inf_{u \in \mathbb{R}^n} \chi''(\bar{x}; d, u) > 0 \quad \text{for all } d \in C(\bar{x}) \setminus \{0\}, \tag{5.4}$$

but, in general, (5.4) does not imply the second-order growth condition. In [9], the following regularity condition is introduced which implies the equivalence of the second-order growth condition and (5.4) if (5.2) is assumed:  $\chi$  is called *second-order epiregular* at  $\bar{x}$  if for any  $d$  and for  $t \geq 0$  and any path  $u(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  such that  $tu(t) \rightarrow 0$  as  $t \downarrow 0$ , the following holds:

$$\chi(\bar{x} + td + \frac{1}{2}t^2u(t)) \geq \chi(\bar{x}) + t\chi'(\bar{x}, d) + \frac{1}{2}t^2\chi''(\bar{x}; d, u(t)) + o(t^2)$$

( $\mathbb{R}_+$  denotes the set of nonnegative reals).

**Proposition 5.1** (Bonnans et al. [9]). *Suppose that (5.2) holds and that  $\chi$  is second-order epiregular at  $\bar{x}$ . Then, the second-order growth condition holds at  $\bar{x}$  if and only if (5.4) is satisfied.*

For a more detailed study on this kind of optimality conditions we refer to, e.g., [4,11,72].

### 5.2. Second-order optimality conditions for GSIP

Now, we return to GSIP and apply the foregoing general results to the function  $\psi$  defined in (4.5). As mentioned in [74], a general discussion on the second-order epiregularity of the optimal value function  $\varphi$  is too complicated with the techniques currently at hand. Therefore, we will restrict ourselves to a particular subclass of GSIP and we will assume the following conditions (A1)–(A4) at  $\bar{x} \in M$ .

(A1) LICQ holds at each  $y \in Y_0(\bar{x})$ .

(A1) implies for each  $\bar{y} \in Y_0(\bar{x})$  that the set  $\text{KKT}(\bar{x}, \bar{y})$  is a singleton, say  $\text{KKT}(\bar{x}, \bar{y}) = \{\gamma(\bar{y})\}$  and that, restricted to a neighborhood of  $\bar{y}$ , the set

$$\Omega(\bar{x}, \bar{y}) = \{y \in \mathbb{R}^m \mid v_\ell(\bar{x}, y) = 0, \ell \in L_0(\bar{x}, \bar{y})\}$$

is a subset of  $Y(\bar{x})$  and a smooth manifold. For a given  $d \in \mathbb{R}^n$  define the following nonempty and compact subset of  $Y_0(\bar{x})$ :

$$Y_1(\bar{x}, d) := \arg \max_{y \in Y_0(\bar{x})} D_x \mathcal{L}(\bar{x}, y, \gamma(y))d.$$

(A2) SCS holds at each  $\bar{y} \in Y_1(\bar{x}, d)$ , i.e.  $\gamma_\ell(\bar{y}) > 0, \ell \in L_0(\bar{x}, \bar{y})$ .

(A1) and (A2) mean that, locally in a neighborhood of  $\bar{y}$ , the active inequality constraints  $v_\ell, \ell \in L_0(\bar{x}, \bar{y})$  can be treated as equality constraints which implies that  $Y_0(\bar{x})$  is a subset of  $\Omega(\bar{x}, \bar{y})$ .

(A3) For each  $\bar{y} \in Y_1(\bar{x}, d)$ , the set  $Y_0(\bar{x})$ , locally in a neighborhood of  $\bar{y}$ , is a smooth submanifold of  $\Omega(\bar{x}, \bar{y})$ .

Consider the tangent space:

$$T_{\bar{y}}\Omega = \{y \in \mathbb{R}^m \mid Dv_\ell(\bar{x}, \bar{y})y = 0, \ell \in L_0(\bar{x}, \bar{y})\}$$

of the smooth manifold  $\Omega(\bar{x}, \bar{y})$  at  $\bar{y}$  and the tangent space  $T_{\bar{y}}Y_0(\bar{x})$  of the smooth submanifold  $Y_0(\bar{x})$ . Obviously, we have  $T_{\bar{y}}Y_0(\bar{x}) \subset T_{\bar{y}}\Omega$ .

(A4) At each  $\bar{y} \in Y_1(\bar{x}, d)$  the following SOSC holds:

$$D_{yy}^2 \mathcal{L}(\bar{x}, \bar{y}, \gamma(\bar{y}))|_{[T_{\bar{y}}Y_0(\bar{x})]^\perp \cap T_{\bar{y}}\Omega} \prec 0.$$

By [78], (A4) is a necessary and sufficient condition for the second-order growth condition (for the lower level problem) to hold locally around  $\bar{y}$ .

Assuming conditions (A1)–(A4) we obtain that the optimal value function  $\varphi$  is locally Lipschitz continuous, second-order directionally differentiable and second-order epiregular at  $\bar{x}$ .

**Proposition 5.2** (Bonnans et al. [9], Bannans and Shapiro [10], Shapiro [78]). (i) If condition (A1) holds, then  $\varphi$  is locally Lipschitz continuous and directionally differentiable at  $\bar{x}$  with (4.3) for all  $d \in \mathbb{R}^n$ .

(ii) If, in addition, conditions (A2)–(A4) hold for a given direction  $d \in \mathbb{R}^n$ , then  $\varphi$  is second-order directionally differentiable and second-order epiregular at  $\bar{x}$  with

$$\varphi''(\bar{x}; d, u) = \max_{y \in Y_1(\bar{x}, d)} \{D_x \mathcal{L}(\bar{x}, y, \gamma(y))u + E_d(\bar{x}, y)\},$$

where

$$E_d(\bar{x}, y) = \max_{\eta \in Z_d(\bar{x}, y)} \{\eta D_{yy}^2 \mathcal{L}(\bar{x}, y, \gamma(y))\eta + 2d^T D_{xy}^2 \mathcal{L}(\bar{x}, y, \gamma(y))\eta + d^T D_{xx}^2 \mathcal{L}(\bar{x}, y, \gamma(y))d\}$$

and

$$Z_d(\bar{x}, y) = \{\eta \in \mathbb{R}^m \mid D_y v_\ell(\bar{x}, y)\eta + D_x v_\ell(\bar{x}, y)d = 0, i \in L_0(\bar{x}, \bar{y})\}.$$

A careful calculation shows that a second-order necessary optimality condition for  $\bar{x}$  to be a local minimizer of GSIP is that for any  $d \in C(\bar{x})$  ( $C(\bar{x})$  is defined as in (5.3) with  $\chi = \psi$ ) the optimal value of the following problem is nonnegative.

$$\text{Min } r \quad \text{s.t. } (r, w) \in M^3$$

with

$$M^3 = \left\{ (r, w) \in \mathbb{R} \times \mathbb{R}^n \mid \begin{array}{l} D_x \mathcal{L}(\bar{x}, y, \gamma(y))w + E_d(\bar{x}, y) \leq r, \quad y \in Y_1(\bar{x}, d) \\ Df(\bar{x})w + d^T D^2 f(\bar{x})d \leq r, \quad \text{in case } Df(\bar{x})d = 0 \end{array} \right\},$$

where the last constraint does not appear if  $Df(\bar{x})d < 0$ . This latter optimization problem is a linear semi-infinite programming problem which satisfies a corresponding constraint qualification of Mangasarian–Fromovitz type such that its dual problem has the same optimal value. This dual problem provides the following necessary and sufficient second-order optimality conditions for  $\bar{x}$  to be local minimizer of GSIP.

**Theorem 5.3** (Rückmann and Shapiro [74]). Assume that a Fritz–John first-order optimality condition as in Theorem 4.1(i) holds at  $\bar{x}$ . Furthermore, let conditions (A1) as well as (A2)–(A4) for any  $d \in C(\bar{x})$  be satisfied. Then:

(i) If  $\bar{x}$  is a local minimizer of GSIP, then for any  $d \in C(\bar{x})$  there exist finitely many  $y^j \in Y_1(\bar{x}, d)$ ,  $j = 1, \dots, p$  and  $\mu_i \geq 0$ ,  $i = 0, \dots, p$  with  $\sum_{i=0}^p \mu_i = 1$  such that

$$\mu_0 Df(\bar{x}) + \sum_{j=1}^p \mu_j D_x \mathcal{L}(\bar{x}, y^j, \gamma(y^j)) = 0 \tag{5.5}$$

and

$$\mu_0 d^T D^2 f(\bar{x})d + \sum_{j=1}^p \mu_j E_d(\bar{x}, y^j) \geq 0.$$

(ii) The following conditions are necessary and sufficient for the second-order growth condition to hold at  $\bar{x}$  with  $\psi$  defined in (4.5). For any  $d \in C(\bar{x}) \setminus \{0\}$  there exist finitely many  $y^j \in Y_1(\bar{x}, d)$ ,  $j = 1, \dots, p$  and  $\mu_i \geq 0$ ,  $i = 0, \dots, p$  with  $\sum_{i=0}^p \mu_i = 1$  such that (5.5) holds and

$$\mu_0 d^T D^2 f(\bar{x})d + \sum_{j=1}^p \mu_j E_d(\bar{x}, y^j) > 0.$$

According to (A3), the set  $Y_0(\bar{x})$  is locally in a neighborhood of  $\bar{y}$  a smooth manifold; [74] contains a corresponding example where  $Y_0(\bar{x})$  is locally a one-dimensional manifold which means in particular that the reduction ansatz does not hold at  $\bar{x}$ . Furthermore, [74] provides also a discussion for second-order optimality conditions for another particular subclass of GSIP where  $Y_0(\bar{x})$  is a finite set but, in general, the reduction ansatz does not hold at  $\bar{x}$ .

We conclude this section with the following:

**Remark 5.4.** It is not difficult to see that under the assumptions in Section 5.1 the case  $C(\bar{x}) = \{0\}$  (see (5.3)) implies that  $\bar{x}$  is a local minimizer of order one, that is, with some  $c > 0$  in a neighborhood  $N$  of  $\bar{x}$  the relation

$$\chi(x) \geq \chi(\bar{x}) + c\|x - \bar{x}\| \quad \text{for all } x \in N$$

holds. In case of GSIP under the condition, that  $\varphi(x)$  is directionally differentiable in the sense of Hadamard, then the assumption  $C(\bar{x}) = \{0\}$  for a feasible  $\bar{x}$  also implies that  $\bar{x}$  is a local minimizer of order one of GSIP (see, e.g., [88]).

## 6. Numerical methods

Up to the present, numerical algorithms for general GSIP problems have been developed mainly from a conceptual viewpoint. For some special structured problems, however, practical experiments have been achieved. In [51] an ill-posed model with one-dimensional index set is treated. In [90] design centering problems and robust optimization models have been solved numerically via the approach described at the end of Section 6.3 (see also [101]).

Roughly speaking, numerical methods for GSIP are based on the following two concepts:

- An extension of methods for common SIP to the GSIP case.
- A transformation (explicit or implicit) of GSIP into a common SIP.

Beginning with the first approach, in Sections 6.1, 6.2 and 6.3 we discuss how the primal, dual, and the discretization method for SIP, respectively, can be adapted to work for GSIP. To do so, for the primal and the discretization method the assumptions have to be strengthened considerably. In Section 6.4 we describe the second approach (we also refer to [60]).

For an extensive overview on numerical methods in standard semi-infinite optimization we refer the reader to [69].

In this exposition we cannot deal explicitly with all solution methods. We only mention here the homotopy method for solving GSIP. This method is based on a structural and generical analysis of one-parametric families of GSIP problems (cf. [49] for SIP and [33] for GSIP). We also only cite a new approach for computing the global minimizer of a semi-infinite problem (cf. [8]). This method is based on lower and upper bounding procedures.

### 6.1. Primal methods, methods of feasible directions

The basic idea of the approach is as follows. Given  $x^k \in M$  we compute a strictly feasible descent direction  $d^k$ , that is,

$$Df(x^k)d^k < 0 \quad \text{and} \quad x^k + \tau d^k \in M \quad \text{for all } \tau > 0 \text{ small enough.}$$

Then we proceed to a new point  $x^{k+1} = x^k + \tau d^k \in M$  with  $f(x^{k+1}) < f(x^k)$ .

*Conceptual method of feasible directions:*

*Step k:* Given  $x^k \in M$ .

- (1) Compute a strictly feasible descent direction  $d^k$ .
- (2) Compute the solution  $\tau_k$  of  $\min_{\tau>0} \{f(x^k + \tau d^k) \mid x^k + \tau d^k \in M\}$  and update  $x^{k+1} = x^k + \tau_k d^k$ .

We expect that (each convergent subsequence of)  $x^k$  converges to a point  $\bar{x}$  satisfying the Fritz–John condition. For SIP such a result holds under weak assumptions if  $d^k$  is obtained as solution of the linear SIP (see [62]):

$$\begin{aligned} \min_{d,z} z \quad \text{s.t. } & Df(x^k)d - z \leq 0, \\ & D_x g(x^k, y)d - z \leq -g(x^k, y) \quad \text{for all } y \in Y, \\ & \pm d_i \leq 1, \quad i = 1, \dots, n. \end{aligned}$$

Unfortunately the situation for GSIP is more difficult. The reason is that the (globally defined) function  $D_x g(x^k, y)$  in SIP must be replaced by the gradient  $D_x \mathcal{L}(x^k, y, \gamma)$  of the Lagrangian function which depends on  $y$  in a more complicated way. To circumvent this problem the primal method could be considered under the stronger assumptions of the reduction ansatz.

*Method of feasible directions under the reduction ansatz:* Let us consider a candidate minimizer  $\bar{x}$  of GSIP and assume that the reduction ansatz holds at all points near  $\bar{x}$  (see Section 3.3). Then some method of feasible directions can be applied to the locally reduced equivalent standard finite problem (cf. (3.5))

$$P_{\text{red}}(\bar{x}): \quad \min f(x) \quad \text{s.t. } \varphi_i(x) := g(x, y^i(x)) \leq 0, \quad i = 1, \dots, p. \tag{6.1}$$

In such a method the derivative  $D\varphi_i(x)$  is given by  $D_x L(x, y^i(x), \gamma^i(x))$  (see (3.4)).

There are different types of feasible direction methods. Apart from the classical Topkis–Veinnot variant there exist more recent methods which use second-order information and have a better convergence behavior. As an example we mention the Polak–He algorithm and refer the reader to [68, Section 2.6] for details. This feasible direction approach enjoys the following general convergence property (see e.g., [17, Theorem 12.5] for the Topkis–Veinnot variant and [68, Theorem 2.6.2] for the Polak–He algorithm).

**Theorem 6.1.** *Suppose that the reduction ansatz holds at  $\bar{x}$  and that  $x^k \rightarrow \bar{x}$ , where the iterates  $x^k$  are generated by the method of feasible directions applied to the finite problem  $P_{\text{red}}(\bar{x})$ . Then at  $\bar{x}$  the Fritz–John condition (4.2) is satisfied.*

### 6.2. Dual methods, KKT methods

In this approach, some variant of the Newton method is applied to compute a solution of the KKT optimality conditions for GSIP (i.e. (4.2) holds with  $\mu_0 = 1$ ).

This approach is essentially based on the reduction ansatz and it can directly be carried over from SIP to the GSIP case. For more details, we refer to [34] (for SIP) and to [91,62] (for GSIP). Therefore, here we only sketch two variants of this approach.

*SQP-method based on the reduction ansatz:*

*Step k:* Given  $x^k$  (not necessarily feasible).

- (1) Determine the local maxima  $y^1, \dots, y^{p_k}$  of  $Q(x^k)$  (see (3.3)).
- (2) Apply  $N_k$  steps of a SQP-solver (for finite programs) to the problem (see (6.1))

$$P_{\text{red}}(x^k): \quad \min_x f(x) \quad \text{s.t. } \varphi_i(x) := g(x, y^i(x)) \leq 0, \quad i = 1, \dots, p_k,$$

leading to iterates  $x^{k,i}, i = 1, \dots, N_k$ . Set  $x^{k+1} = x^{k,N_k}$  and  $k = k + 1$ .

For a discussion of such a method combining global convergence and local superlinear convergence we refer to [34].

In the case where the lower level problem  $Q(\bar{x})$  in (3.3) is a convex program (in the  $y$  variable) the following alternative approach is especially useful (see [90] for details).

Recall that for  $\bar{x} \in M$  any active index  $y \in Y_0(\bar{x})$  is a (global) maximizer of the lower level problem  $Q(\bar{x})$  and (under some constraint qualification)  $y \in Y_0(\bar{x})$  must satisfy the KKT condition  $D_y \mathcal{L}(\bar{x}, y, \gamma) = 0$  with some multiplier vector  $\gamma \geq 0$ . So we can consider the following relaxation of GSIP:

$$\begin{aligned}
 \min_{x,y,\gamma} f(x) \quad & \text{s.t. } g(x, y) \leq 0, \\
 & D_y g(x, y) - \gamma^T D_y v(x, y) = 0, \\
 & \gamma^T v(x, y) = 0, \\
 & \gamma \geq 0, \quad -v(x, v) \geq 0.
 \end{aligned} \tag{6.2}$$

In fact, under a constraint qualification for  $Q(x)$ , the feasible set of GSIP is contained in the (projection of the) feasible set of (6.2). In particular, any solution  $(x, y, \gamma)$  of (6.2) with the property that  $y$  is a minimizer of  $Q(x)$  yields a solution  $x$  of the original program. If in addition to the constraint qualification, the problem  $Q(x)$  is convex, then (6.2) is equivalent with the original GSIP program.

In the form (6.2), GSIP is transformed into a nonlinear program with complementarity constraints. To solve (6.2) a smoothing (interior point) approach can be used. Here, the complementarity constraints  $\gamma^T v(x, y) = 0$  or (in view of  $\gamma^T, -v(x, y) \geq 0$ ) equivalently  $\gamma_\ell v_\ell(x, y) = 0, \ell = 1, \dots, L$ , are replaced by the conditions  $\gamma_\ell v_\ell(x, y) = \tau, \ell = 1, \dots, L$ , where  $\tau > 0$  is a small perturbation parameter. The corresponding perturbed problem represents a common finite program which can be solved by standard methods. Convergence results for this approach and many numerical experiments are given in [90]. We also refer to [101] where this technique is used to solve numerically a special type of design centering problem appearing in the manufacturing of maximal volume gems.

We emphasize, that these dual methods are based of the reduction ansatz, so that the practicability of the method strongly depends on the generic structure of the problems as discussed in Section 3.3.

### 6.3. Discretization method

This method also can, in principle, be extended from SIP to GSIP. The additional problem arising in GSIP is that the index set  $Y(x)$  and thus its discretization  $Y^*(x)$  depends on  $x$ . So, to ensure the closedness of the feasible set of the discretized problems the discretizations have to be constructed in such a way that the grids  $Y^*(x)$  depend (at least) continuously on  $x$ . For simplicity we make the following assumptions.

**Assumption A<sub>1</sub>.** Let the feasible set  $M$  of GSIP be compact and assume that the set-valued mapping  $Y$  is continuous on  $M$  and satisfies (6.4).

As in SIP, in a discretization method we chose a finite grid  $Y^*(x)$  of  $Y(x)$  and solve the problem

$$P(Y^*): \quad \min f(x) \quad \text{s.t. } x \in M(Y^*) := \{x \mid g(x, y) \leq 0 \text{ for all } y \in Y^*(x)\}.$$

But different from SIP this problem is not a standard finite program since the index set  $Y^*(x)$  changes with  $x$ . Even for continuous index mappings  $Y(x)$  (i.e., the feasible set  $M(Y) = M$  is closed) the grid mapping  $Y^*(x)$  is not automatically continuous and thus the feasible set  $M(Y^*)$  may be nonclosed (see [91] for a simple example).

To avoid this problem we assume that the mapping  $Y^*$  satisfies:

**Assumption A<sub>2</sub>.** Let assumption A<sub>1</sub> hold and let the grid  $Y^*(x) \subset Y(x)$  be defined by

$$Y^*(x) = \{y_i^*(x) \mid i = 1, \dots, i^*\},$$

where  $y_i^* : M \rightarrow C_0$  are continuous (smooth) functions. (How such a grid  $Y^*(x)$  can be constructed is indicated in [92].)

Note that under the assumption  $A_2$  the problem  $P(Y^*)$  becomes a finite (continuous or smooth) program:

$$P(Y^*): \quad \min f(x) \quad \text{s.t.} \quad g_i^*(x) := g(x, y_i^*(x)) \leq 0, \quad i = 1, \dots, i^*.$$

If we define the meshsize of a discretization  $Y^*(x) \subset Y(x)$  by the maximum norm of the Hausdorff distance,

$$d_H(Y^*, Y) = \max_{x \in M} \left\{ \max_{y \in Y(x)} \min_{y^* \in Y^*(x)} \|y - y^*\| \right\},$$

then the discretization method for SIP can be extended to GSIP as follows:

*Conceptual discretization method:*

*Step k:* Given a grid  $Y^k(x) \subset Y(x)$  and a fixed small number  $\alpha > 0$ .

- (1) Compute a solution  $x^k$  of  $P(Y^k)$ .
- (2) Stop, if  $x^k$  is feasible within the accuracy  $\alpha > 0$ , i.e.  $g(x^k, y) \leq \alpha$  for all  $y \in Y(x^k)$ . Otherwise, select a (finer) discretization  $Y^{k+1}(x)$  and continue with step  $k + 1$ .

**Theorem 6.2** (Still [92]). *Let assumption  $A_1$  hold for GSIP and let a sequence of discretizations  $Y^k(x)$  of  $Y(x)$  be chosen such that for each  $Y^k$  assumption  $A_2$  is satisfied. Suppose that  $d_H(Y^k, Y) \rightarrow 0$  holds for  $k \rightarrow \infty$ . Then, the sequence of solutions  $x^k$  of  $P(Y^k)$  has an accumulation point  $\bar{x}$  and each such point is a solution of GSIP.*

The so-called exchange methods can be extended to GSIP as well (see [92] for details). In general, exchange methods are more efficient than the pure discretization approach as sketched above.

Due to the extra assumption that (at least locally) the grids  $Y^k(x) \subset Y(x)$  have to be constructed to be continuously (smoothly) depending on  $x$ , these discretization methods are not easy to implement and practical experiences have not yet been achieved.

#### 6.4. Transformation of GSIP into SIP

In principle, under appropriate assumptions, each GSIP can be transformed into an equivalent SIP. However such a transformation is of practical value only in cases where this transformation is defined globally.

*Globally defined transformation:* The ideal situation is that the index set  $Y(x)$  is defined as follows. Suppose we have given a nonempty, compact set  $Y_0 \subset \mathbb{R}^m$  and a function  $T_x(y) \in C^1(\mathbb{R}^n \times Y_0, \mathbb{R}^m)$  such that for all  $x \in \mathbb{R}^n$ ,  $T_x(Y_0) = Y(x)$ . Then obviously, the feasibility relation  $g(x, y) \leq 0$  for all  $y \in Y(x)$  can equivalently be described by the SIP feasibility condition

$$\hat{g}(x, y) := g(x, T_x(y)) \leq 0 \quad \text{for all } y \in Y_0.$$

For a one-dimensional index set  $Y(x)$  such a transformation can be constructed easily. Suppose that the index set is defined as  $Y(x) = [a(x), b(x)]$  with  $C^1$ -functions  $a(x) < b(x)$ ,  $x \in \mathbb{R}^n$ . Then we can chose

$$T_x(y) = yb(x) + (1 - y)a(x), \quad y \in Y_0 := [0, 1].$$

Also for higher dimensional index sets  $Y(x)$  such a global transformation exists if the set-valued mapping  $Y$  satisfies the following star-shaped properties.

**Assumption  $A_*$ .** Let the feasible set  $M$  be compact. Assume that the sets  $Y(x)$  are star-shaped in the following sense: There exist continuous functions  $c : M \rightarrow \mathbb{R}^m$ ,  $r : M \times S^m \rightarrow \mathbb{R}$ , satisfying for all  $x \in M$ ,  $b \in S^m := \{y \in \mathbb{R}^m \mid \|y\| = 1\}$

$$c(x) + \tau r(x, b)b \begin{cases} \in \text{int } Y(x) & \text{if } \tau \in [0, 1), \\ \notin Y(x) & \text{if } \tau \in (1, \infty). \end{cases}$$

Here,  $\text{int } Y(x)$  denotes the interior of  $Y(x)$ . Note that since  $Y(x)$  is closed this implies that for any  $x \in M$  the function  $c(x) + r(x, b)b$ ,  $b \in S^m$  gives a parameterization of the boundary  $\partial Y(x)$ .

**Theorem 6.3** (Still [91]). *Suppose that assumption  $A_s$  is fulfilled. Then the feasible set  $M$  of GSIP can be described as*

$$M = \{x \mid \hat{g}(x, y) := g(x, T_x(y)) \leq 0 \text{ for all } y \in Y_0\}, \tag{6.3}$$

where  $Y_0 = \{y = (\tau, b) \in [0, 1] \times S^m\}$  and  $T_x(y) = c(x) + \tau r(x, b)b$ . If moreover, for any fixed  $x$ , the function  $g(x, y)$  is convex in  $y$ , then  $M$  is given by (6.3) with  $Y_0 = S^m$  and  $T_x(y) = c(x) + r(x, y)y$ .

The following lemma gives a sufficient condition for  $A_s$ .

**Lemma 6.4** (Still [91]). *Let  $M$  be compact.  $A_s$  is fulfilled if the following holds:*

(i) *The set-valued mapping  $Y$  satisfies*

$$Y : M \rightrightarrows C_0 \quad \text{with } C_0 \subset \mathbb{R}^m \text{ compact.} \tag{6.4}$$

(ii) *For any  $x \in M$  the functions  $v_\ell(x, y)$  are convex functions of  $y \in C_0$  and  $v_\ell \in C(M \times C_0, \mathbb{R})$ ,  $\ell \in L$ .*

(iii) *For any  $x \in M$  the Slater condition is satisfied, i.e. there exists a point  $y_x \in \mathbb{R}^m$  such that  $v_\ell(x, y_x) < 0$ ,  $\ell \in L$ . (This in particular implies that  $Y(x)$  has interior points.)*

**Remark 6.5.** We emphasize that in practice it is not necessary to compute the transformation  $T_x(y)$  explicitly. It suffices to compute the values of the function  $T_x(y)$  (and/or their derivatives) for points  $x, y$  attained by the actual algorithm.

*Locally defined transformation:* The next result shows that, in theory, a GSIP can be transformed into a SIP problem under much weaker assumptions. However, the transformations are based on locally defined transformations which are then *glued together* by some partition of the unity argument (see [28,44,91,99] for details).

**Theorem 6.6.** *Assume that the feasible set  $M$  of GSIP is compact.*

(a) *(Under LICQ [99]). Suppose that for any  $x \in M$  the condition LICQ is satisfied at all points  $y \in Y(x)$ . Then, there are finitely many  $C^1$ -functions  $G^j(x, z)$  and sets  $Z^j = [a_1^j, b_1^j] \times \dots \times [a_m^j, b_m^j]$  in  $\mathbb{R}^m$ ,  $j = 1, \dots, P$ , such that*

$$x \in M \iff G^j(x, z) \leq 0 \text{ for all } z \in Z^j, \quad j = 1, \dots, P.$$

(b) *(Under MFCQ [91]). Suppose that for any  $x \in M$  the condition MFCQ is satisfied at all points  $y \in Y(x)$ . Then, there are finitely many Lipschitz continuous functions  $g^j(x, b, \tau)$ ,  $j = 1, \dots, N$ , such that*

$$x \in M \iff g^j(x, b, \tau) \leq 0 \text{ for all } b \in S^m, \tau \in [0, 1], \quad j = 1, \dots, N.$$

### 7. Final remarks

In analogy to standard semi-infinite programs, a long standing conjecture for generalized semi-infinite programs is that generically the reduction ansatz holds at all local minimizers of GSIP. As the feasible set is then locally described by finitely many smooth constraints, the optimality conditions are those of the locally reduced finite optimization problem. Once this conjecture is proven, general purpose numerical methods could base on the latter optimality conditions as termination criteria.

As we have pointed out, the reduction ansatz cannot be expected to hold everywhere in the feasible set. Still it may be possible to give a classification of the possible generic local structures of the feasible set, including re-entrant corner points and local nonclosedness. Also, a simple description for the closure of the feasible set in GSIP is desirable.

An ubiquitous problem of numerical semi-infinite programming, be it in standard or generalized form, is the computation of the active index set of a feasible point. As we have seen, the computation of this set amounts to finding *all global solutions* of the lower level problem, that is, to a demanding challenge for numerical reliability.

All these questions are currently under investigation and will be the subjects of future publications.

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## References

- [1] J.M. Abadie, On the Kuhn–Tucker theorem, in: J. Abadie (Ed.), *Nonlinear Programming*, Wiley, New York, 1967, pp. 21–36.
- [2] B. Bank, J. Guddat, D. Klatt, B. Kummer, K. Tammer, *Non-linear Parametric Optimization*, Birkhäuser, Basel, 1983.
- [3] M.S. Bazaraa, H.D. Sherali, C.M. Shetty, *Nonlinear Programming. Theory and Algorithms*, Wiley, New York, 1993.
- [4] A. Ben-Tal, Second-order and related extremality conditions in nonlinear programming, *J. Optim. Theory Appl.* 31 (1980) 143–165.
- [5] A. Ben-Tal, L. El Ghaoui, A. Nemirovski, Robustness, in: H. Wolkowicz et al. (Eds.), *Handbook of Semidefinite Programming*, Kluwer, Dordrecht, 2000, pp. 139–162.
- [6] A. Ben-Tal, A. Nemirovski, Robust solutions of uncertain linear programs, *Oper. Res. Lett.* 25 (1999) 1–13.
- [7] C. Berge, *Topological Spaces*, Oliver and Boyd, Edinburgh, 1963.
- [8] B. Bhattacharjee, P. Lemonidis, W.H. Green, P.I. Barton, Global solution of semi-infinite programs, *Math. Programming* 103 (2005) 283–307.
- [9] J.F. Bonnans, R. Cominetti, A. Shapiro, Second order optimality conditions based on parabolic second-order tangent sets, *SIAM J. Optim.* 9 (1999) 466–492.
- [10] J.F. Bonnans, A. Shapiro, Optimization problems with perturbations, a guided tour, *SIAM Rev.* 40 (1998) 228–264.
- [11] J.F. Bonnans, A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.
- [12] T. Bröcker, L. Lander, *Differentiable Germs and Catastrophes*, London Mathematical Society Lecture Notes Series, vol. 17, Cambridge University Press, Cambridge, 1975.
- [13] B. Brosowski, *Parametric Semi-infinite Optimization*, Peter Lang, Frankfurt a.M., 1982.
- [14] E.W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.
- [15] J.M. Danskin, *The Theory of Max–Min and its Applications to Weapons Allocation Problems*, Springer, New York, 1967.
- [16] V.F. Demyanov, L.V. Vasilev, *Nondifferentiable Optimization*, Optimization Software Inc., Publications Division, New York, 1985.
- [17] U. Faigle, W. Kern, G. Still, *Algorithmic Principles of Mathematical Programming*, Kluwer, Dordrecht, 2002.
- [18] A.V. Fiacco, K.O. Kortanek (Eds.), *Semi-infinite Programming and Applications*, Lecture Notes in Economics and Mathematical Systems, vol. 215, Springer, Berlin, 1983.
- [19] A.V. Fiacco, G.P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, Wiley, New York, 1968.
- [20] J. Gauvin, F. Dubeau, Differential properties of the marginal function in mathematical programming, *Math. Programming Study* 19 (1982) 101–119.
- [21] Y.B. Germeyer, *Einführung in die Theorie des Operations Research*, Akademie Verlag, Berlin, 1976.
- [22] K. Glashoff, S.A. Gustafson, *Linear Optimization and Approximation*, Springer, Berlin, 1983.
- [23] M.A. Goberna, M.A. López, *Linear Semi-infinite Optimization*, Wiley, Chichester, 1998.
- [24] M.A. Goberna, M.A. López (Eds.), *Semi-infinite Programming—Recent Advances*, Kluwer, Dordrecht, 2001.
- [25] E.G. Gol’stein, *Theory of Convex Programming*, Translations of Mathematical Monographs, vol. 36, American Mathematical Society, Providence, RI, 1972.
- [26] T.J. Graettinger, B.H. Krogh, The acceleration radius: a global performance measure for robotic manipulators, *IEEE J. Robot. Automat.* 4 (1988) 60–69.
- [27] P. Gritzmann, V. Klee, On the complexity of some basic problems in computational convexity. I. Containment problems, *Discrete Math.* 136 (1994) 129–174.
- [28] J. Guddat, H.Th. Jongen, J.-J. Rückmann, On stability and stationary points in nonlinear optimization, *J. Austral. Math. Soc. B* 28 (1986) 36–56.
- [29] F. Guerra Vázquez, J.-J. Rückmann, Extensions of the Kuhn–Tucker constraint qualification to generalized semi-infinite programming, *SIAM J. Optim.* 15 (3) (2005) 926–937.
- [30] M. Gugat, Semi-infinite terminal problems: a Newton-type method, *Optimization* 44 (1998) 25–48.
- [31] R. Hettich (Ed.), *Semi-infinite Programming*, Lecture Notes in Control and Information Sciences, vol. 7, Springer, Berlin, 1976.
- [32] R. Hettich, H.Th. Jongen, Semi-infinite programming: conditions of optimality and applications, in: J. Stoer (Ed.), *Optimization Techniques, Part 2*, Lecture Notes in Control and Information Sciences, vol. 7, Springer, Berlin, 1978, pp. 1–11.
- [33] R. Hettich, H.Th. Jongen, O. Stein, On continuous deformations of semi-infinite optimization problems, in: M. Florenzano, J. Guddat, M. Jimenez, H.Th. Jongen, G. Lopez Lagomasino, F. Marcellan (Eds.), *Approximation and Optimization in the Caribbean II*, Peter Lang Verlag, Frankfurt a.M., 1995, pp. 406–424.
- [34] R. Hettich, K.O. Kortanek, Semi-infinite programming: theory, methods, and applications, *SIAM Rev.* 35 (1993) 380–429.
- [35] R. Hettich, G. Still, Semi-infinite programming models in robotics, in: J. Guddat, H.Th. Jongen, B. Kummer, F. Nožička (Eds.), *Parametric Optimization and Related Topics II*, Akademie Verlag, Berlin, 1991, pp. 112–118.
- [36] R. Hettich, G. Still, Second order optimality conditions for generalized semi-infinite programming problems, *Optimization* 34 (1995) 195–211.
- [37] R. Hettich, P. Zencke, *Numerische Methoden der Approximation und semi-infiniten Optimierung*, Teubner, Stuttgart, 1982.
- [38] A. Hoffmann, R. Reinhardt, On reverse Chebyshev approximation problems, Technical University of Ilmenau, Preprint No. M08/94, 1994.
- [39] W.W. Hogan, Point-to-set maps in mathematical programming, *SIAM Rev.* 15 (1973) 591–603.



- [40] W.W. Hogan, Directional derivatives for extremal value functions with applications to the completely convex case, *Oper. Res.* 21 (1973) 188–209.
- [41] R. Horst, H. Tuy, *Global Optimization*, Springer, Berlin, 1990.
- [42] F. John, Extremum problems with inequalities as subsidiary conditions, in: *Studies and Essays, R. Courant Anniversary Volume*, Interscience, New York, 1948, pp. 187–204.
- [43] H.Th. Jongen, P. Jonker, F. Twilt, Critical sets in parametric optimization, *Math. Programming* 34 (1986) 333–353.
- [44] H.Th. Jongen, P. Jonker, F. Twilt, *Nonlinear Optimization in Finite Dimensions*, Kluwer, Dordrecht, 2000.
- [45] H.Th. Jongen, J.-J. Rückmann, On stability and deformation in semi-infinite optimization, in: R. Reemtsen, J.-J. Rückmann (Eds.), *Semi-Infinite Programming*, Kluwer, Boston, 1998, pp. 29–67.
- [46] H.Th. Jongen, J.-J. Rückmann, O. Stein, Disjunctive optimization: critical point theory, *J. Optim. Theory Appl.* 93 (1997) 321–336.
- [47] H.Th. Jongen, J.-J. Rückmann, O. Stein, Generalized semi-infinite optimization: a first-order optimality condition and examples, *Math. Programming* 83 (1998) 145–158.
- [48] H.Th. Jongen, J.-J. Rückmann, G.-W. Weber, One-parametric semi-infinite optimization: on the stability of the feasible set, *SIAM J. Optim.* 4 (1994) 637–648.
- [49] H.Th. Jongen, O. Stein, On generic one-parametric semi-infinite optimization, *SIAM J. Optim.* 7 (1997) 1103–1137.
- [50] C. Kaiser, W. Krabs, Ein Problem der semi-infiniten Optimierung im Maschinenbau und seine Verallgemeinerung, Working paper, Darmstadt University of Technology, Department of Mathematics, 1986.
- [51] A. Kaplan, R. Tichatschke, On a class of terminal variational problems, in: J. Guddat, H.Th. Jongen, F. Nožička, G. Still, F. Twilt (Eds.), *Parametric Optimization and Related Topics IV*, Peter Lang, Frankfurt a.M., 1997, pp. 185–199.
- [52] D. Klatte, Stability of stationary solutions in semi-infinite optimization via the reduction approach, in: W. Oettli, D. Pallaschke (Eds.), *Advances in Optimization*, Lecture Notes in Economics and Mathematical Systems, vol. 382, Springer, Berlin, 1992, pp. 155–170.
- [53] D. Klatte, Stable local minimizers in semi-infinite optimization: regularity and second-order conditions, *J. Comput. Appl. Math* 56 (1994) 137–157.
- [54] M. Kočvara, J. Zowe, A. Nemirovski, Cascading—an approach to robust material optimization, *Comput. Struct.* 74 (2000) 431–442.
- [55] W. Krabs, On time-minimal heating or cooling of a ball, *International Series of Numerical Mathematics*, vol. 81, Birkhäuser, Basel, 1987, pp. 121–131.
- [56] H.W. Kuhn, A.W. Tucker, Nonlinear programming, in: J. Neyman (Ed.), *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1951, pp. 481–492.
- [57] J. Kyriaris, On uniqueness of Kuhn–Tucker multipliers in nonlinear programming, *Math. Programming* 32 (1985) 242–246.
- [58] P.-J. Laurent, *Approximation et Optimisation*, Hermann, Paris, 1972.
- [59] F. Lempio, H. Maurer, Differential stability in infinite-dimensional nonlinear programming, *Appl. Math. Optim.* 6 (1980) 139–152.
- [60] E. Levitin, Reduction of generalized semi-infinite programming problems to semi-infinite or piece-wise smooth programming problems, Preprint No. 8-2001, University of Trier, 2001.
- [61] E. Levitin, R. Tichatschke, A branch-and-bound approach for solving a class of generalized semi-infinite programming problems, *J. Global Optim.* 13 (1998) 299–315.
- [62] M. López, G. Still, Semi-infinite programming, *European J. Oper. Res.* 180 (2007) 491–518.
- [63] O.L. Mangasarian, Set containment characterization, *J. Global Optim.* 24 (2002) 473–480.
- [64] O.L. Mangasarian, S. Fromovitz, The Fritz John necessary optimality conditions in the presence of equality and inequality constraints, *J. Math. Anal. Appl.* 17 (1967) 37–47.
- [65] V.H. Nguyen, J.J. Strodiot, Computing a global optimal solution to a design centering problem, *Math. Programming* 53 (1992) 111–123.
- [66] D.W. Peterson, A review of constraint qualifications in finite-dimensional spaces, *SIAM Rev.* 15 (1973) 639–654.
- [67] E. Polak, An implementable algorithm for the optimal design centering, tolerancing and tuning problem, *J. Optim. Theory Appl.* 37 (1982) 45–67.
- [68] E. Polak, *Optimization, Algorithms and Consistent Approximations*, Springer, Berlin, 1997.
- [69] R. Reemtsen, S. Görner, Numerical methods for semi-infinite programming: a survey, in: R. Reemtsen, J.-J. Rückmann (Eds.), *Semi-Infinite Programming*, Kluwer, Boston, 1998, pp. 195–275.
- [70] R. Reemtsen, J.-J. Rückmann (Eds.), *Semi-Infinite Programming*, Kluwer, Boston, 1998.
- [71] R.T. Rockafellar, Directional differentiability of the optimal value function in a nonlinear programming problem, *Math. Programming Study* 21 (1984) 213–226.
- [72] R.T. Rockafellar, R.J.B. Wets, *Variational Analysis*, Springer, Berlin, 1998.
- [73] J.-J. Rückmann, A. Shapiro, First-order optimality conditions in generalized semi-infinite programming, *J. Optim. Theory Appl.* 101 (1999) 677–691.
- [74] J.-J. Rückmann, A. Shapiro, Second-order optimality conditions in generalized semi-infinite programming, *Set-Valued Anal.* 9 (2001) 169–186.
- [75] J.-J. Rückmann, O. Stein, On linear and linearized generalized semi-infinite optimization problems, *Ann. Oper. Res.* 101 (2001) 191–208.
- [76] J.-J. Rückmann, O. Stein, On convex lower level problems in generalized semi-infinite optimization, in: M.A. Goberna, M.A. López (Eds.), *Semi-infinite Programming—Recent Advances*, Kluwer, Dordrecht, 2001, pp. 121–134.
- [77] A. Shapiro, On optimality conditions in quasidifferentiable optimization, *SIAM J. Control Optim.* 22 (1984) 610–617.
- [78] A. Shapiro, Perturbation theory of nonlinear programs when the set of optimal solutions is not a singleton, *Appl. Math. Optim.* 18 (1988) 215–229.
- [79] A. Shapiro, First and second-order optimality conditions and perturbation analysis of semi-infinite programming problems, in: R. Reemtsen, J.-J. Rückmann (Eds.), *Semi-Infinite Programming*, Kluwer, Boston, 1998, pp. 103–134.

- [80] O. Stein, On parametric semi-infinite optimization, Thesis, Shaker, Aachen, 1997.
- [81] O. Stein, The reduction ansatz in absence of lower semi-continuity, in: J. Guddat, R. Hirabayashi, H.Th. Jongen, F. Twilt (Eds.), *Parametric Optimization and Related Topics V*, Peter Lang, Frankfurt a.M., 2000, pp. 165–178.
- [82] O. Stein, On level sets of marginal functions, *Optimization* 48 (2000) 43–67.
- [83] O. Stein, The feasible set in generalized semi-infinite programming, in: M. Lassonde (Ed.), *Approximation, Optimization and Mathematical Economics*, Springer, 2001, pp. 313–331.
- [84] O. Stein, First order optimality conditions for degenerate index sets in generalized semi-infinite programming, *Math. Oper. Res.* 26 (2001) 565–582.
- [85] O. Stein, *Bi-level Strategies in Semi-infinite Programming*, Kluwer, Boston, 2003.
- [86] O. Stein, On constraint qualifications in non-smooth optimization, *J. Optim. Theory Appl.* 121 (2004) 647–671.
- [87] O. Stein, A semi-infinite approach to design centering, in: S. Dempe, S. Kalashnikow (Eds.), *Optimization with Multivalued Mappings*, Springer, 2006, pp. 209–228.
- [88] O. Stein, G. Still, On optimality conditions for generalized semi-infinite programming problems, *J. Optim. Theory Appl.* 104 (2000) 443–458.
- [89] O. Stein, G. Still, On generalized semi-infinite optimization and bilevel optimization, *European J. Oper. Res.* 142 (2002) 444–462.
- [90] O. Stein, G. Still, Solving semi-infinite optimization problems with interior point techniques, *SIAM J. Control Optim.* 42 (2003) 769–788.
- [91] G. Still, Generalized semi-infinite programming: theory and methods, *European J. Oper. Res.* 119 (1999) 301–313.
- [92] G. Still, Generalized semi-infinite programming: numerical aspects, *Optimization* 49 (2001) 223–242.
- [93] G. Still, Discretization in semi-infinite programming: the rate of approximation, *Math. Programming* 91 (2001) 53–69.
- [94] R.A. Tapia, M.W. Trosset, An extension of the Karush-Kuhn-Tucker necessity conditions to infinite programming, *SIAM Rev.* 36 (1994) 1–17.
- [95] R.H. Tütüncü, M. Koenig, Robust asset allocation, *Ann. Oper. Res.* 132 (2004) 157–187.
- [96] L. Vandenberghé, S. Boyd, Semidefinite programming, *SIAM Rev.* 38 (1996) 49–95.
- [97] L. Vandenberghé, S. Boyd, Connections between semi-infinite and semi-definite programming, in: R. Reemtsen, J.-J. Rückmann (Eds.), *Semi-Infinite Programming*, Kluwer, Boston, 1998, pp. 277–294.
- [98] G.-W. Weber, *Generalized Semi-Infinite Optimization and Related Topics*, Habilitation Thesis, Darmstadt University of Technology, 1999.
- [99] G.-W. Weber, Generalized semi-infinite optimization: on some foundations, *J. Comput. Technol.* 4 (1999) 41–61.
- [100] W. Wetterling, Definitheitsbedingungen für relative Extrema bei Optimierungs- und Approximationsaufgaben, *Numer. Math.* 15 (1970) 122–136.
- [101] A. Winterfeld, Maximizing volumes of lapidaries by use of hierarchical GSIP-models, Diploma thesis, Technische Universität Kaiserslautern and Fraunhofer Institut für Techno- und Wirtschaftsmathematik, 2004.
- [102] H. Wolkowicz, R. Saigal, L. Vandenberghé (Eds.), *Handbook of Semidefinite Programming: Theory, Algorithms and Applications*, Kluwer, Dordrecht, 2000.
- [103] J.J. Ye, S.-Y. Wu, First order optimality conditions for generalized semi-infinite programming problems, Preprint, Department of Mathematics, University of Victoria, 2006.
- [104] G. Zwier, *Structural Analysis in Semi-Infinite Programming*, Thesis, University of Twente, 1987.