Port-Based Modeling of a Flexible Link

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***Abstract*—In this paper, a simple way to model flexible robotic links is presented. This is different from classical approaches and from the Euler–Bernoulli or Timoshenko theory, in that the proposed model is able to describe large deflections in 3-D space and does not rely on any finite-dimensional approximation (e.g., modal approximation). The model has been formulated within the port Hamiltonian formalism because intuitive considerations on the geometric behavior of the elastic link naturally define a Stokes–Dirac structure, the kernel of a port Hamiltonian system. Moreover, port Hamiltonian systems can be easily interconnected, thus allowing the description of complex systems as a composition of parts in an object-oriented way. By combining rigid bodies, springs, dampers, joints and, finally, flexible links, it is virtually possible to model and mathematically describe whatever complex mechanical structure formed by beams. In order to demonstrate the dynamical properties of the model and how complex mechanisms can be obtained by port interconnection, simulations of 1-DoF and 2-DoF serial manipulators and of a 2-DoF flexible closed kinematic chain are presented.**

***Index Terms*—Flexible structures, modeling, port Hamiltonian systems, simulation.**

I. Introduction

THIS paper is devoted to the modeling and simulation of an elastic beam suitable for describing the dynamics of a flexible link for lightweight manipulators or large articulated structures. A common way to describe a flexible link is by means of the Euler-Bernoulli equation [1], [2]. This model is valid under the assumptions that the link is slender with uniform geometric and inertial characteristics, that the link is flexible in the lateral directions and stiff with respect to axial forces and to torsion and bending forces due to gravity, and that nonlinear deformation and friction can be neglected. Then, the Euler–Bernoulli beam is a linear infinite-dimensional model which takes into account only perpendicular deformation with respect to an unstressed reference configuration. If also the rotational deformation of the cross section has to be considered, the Timoshenko model can be adopted [3]–[5]. In any case, both models are valid only for small deformations.

The use of distributed parameters models makes the dynamical equations of a multi-body system rather difficult and the development of control strategies quite complex [3], [6]. This is the reason why, starting from the Euler-Bernoulli model, a finite-dimensional approximation of the elastic behavior is usually deduced from a combination of modal analysis and system identification, [1], [7]–[11]. The resulting model is still finite-dimensional and the development of simulation software and of control methodologies can be carried out by referring to standard techniques. Clearly, this approximation is only valid when deformations are infinitesimal. Same considerations hold when the model derives from classical Euler–Lagrange or Newton-Euler approaches, in which the configuration of the flexible link is the composition of a pair of homogeneous transformations. The first one is related to the motion of the unstressed configuration, treated as a rigid body, and the second one represents the tip deflection with respect to the configuration at rest [12]–[14].

In this paper, a nonlinear model of a flexible link able to describe finite deformations in the 3-D space is presented. As in [15], the position and orientation of the cross section of the link are described by elements of the Lie group of actions $SE(3)$ and the dynamics is written as the evolution of elements of the Lie algebras $se(3)$ and $se^*(3)$. So, the proposed model is formulated in the *robotic language*, thus allowing a natural integration with well-established modeling and simulation techniques for rigid manipulators. A similar approach has been discussed in [16], in which a static description of the elastic behavior has been provided, and in [17], where a dynamical model of a planar beam has been introduced under the same hypotheses of this paper and, here, generalized to the 3-D case.

As in [17], the model is written as a distributed port Hamiltonian system [18]–[20]. There are several reasons for this. At first, intuitive considerations concerning the geometric behavior of the elastic beam immediately lead to the definition of what is called a Stokes–Dirac structure. Dirac [21], [22] and Stokes–Dirac structures [18], [23] are the geometric structures behind finite [22], [24] and infinite-dimensional [18], [23] port Hamiltonian systems respectively. These objects describe the interconnection structure of the system, that is the way in which power flows within the system and between the system and its environment. The dynamics follows from the Stokes–Dirac structure combined with the constitutive equations, i.e., with the Hamiltonian (energy) function, [4], [5], [25], [26]. Besides the port Hamiltonian approach, what is quite unusual is the way in which the elastic behavior is described. Roughly speaking, differently than in classic Hamiltonian and Lagrangian frameworks, position and orientation of the cross section are not treated as state variables. In fact, it seems more natural to adopt the strain measure that appears in the constitutive equations.

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This choice leads to a model in which a nonlinear coupling between rotational and translational deformation is superposed to a transmission phenomenon related to waves propagation. Simpler mathematical models (e.g., Timoshenko beam) follow by linearization of this nonlinear effect around the unstressed configuration. Clearly, the information on the position of the cross section in space is lost but it can be recovered, with some redundancy, by time integration of the *velocity* of the cross-section (a co-energy variable).

On the other hand, the interconnection of *physical* systems can be naturally carried out within the port Hamiltonian framework. In fact, a port Hamiltonian system is able to interact with the *environment* by means of power ports like in electrical networks and the interconnection with other components is possible once a power-conserving relation between the port variables is specified. This is what it is meant with *port-based modeling*. The dynamical model of a mechanical structure can be modeled as the power conserving interconnection of a set of simpler components (e.g., rigid bodies, joints, springs, dampers and, finally, flexible links) written in port Hamiltonian form. The interconnection equations, [27] and [28] are, basically, an extension to the mechanical domain of the Kirchhoff laws. A systematic procedure for modeling articulated structures made of rigid links has been presented in [27] and [29] successively modified in [30] in order to take flexible links into account.

Within the port Hamiltonian framework, the model has deep connection with the physics, in the sense that the dynamics relates the variation of the energy variables to the "gradient" of the Hamiltonian. Therefore, energy and energetic properties are crucial. Consequently, control applications can be developed by taking into account the effects of the controller in terms of energy dissipation and/or shaping [31], while discretization techniques should preserve the geometric structure (Dirac structure) and the energy balance properties of the infinite-dimensional system. Moreover, a finite-dimensional description of the link should deal with eventually varying boundary conditions taking into account the torques/forces applied at the extremities of the link by actuators.

In [32], a spatial discretization procedure based on a particular type of mixed finite element and able to provide a finite-dimensional input–output approximated system has been introduced. From the analysis of the geometric structure of an infinite-dimensional port Hamiltonian system, a finite-dimensional approximation still in port Hamiltonian form can be obtained. Recently, this methodology has been extended [33] and applied to the model of the flexible link. Then, a bond graph [34] of the finite-dimensional approximation has been obtained and simulated by means of the package 20Sim©, [35]. The properties of the model in terms of dynamical description of elasticity and also in terms of composition with others are illustrated by means of simulations of 1-DoF and 2-DoF serial manipulators and of a 2-DoF closed kinematic chain.

This paper is organized as follows. In Section II, the notation is briefly introduced and a short background on Lie groups (Section II-A), Dirac and Stokes–Dirac structures (Section II-B) and finite- and infinite-dimensional port Hamiltonian systems (Section II-C) is given. The distributed parameter model of the link is presented in Section III, while simulations results are illustrated in Section IV. Finally, conclusions and future developments are discussed in Section V.

II. Background

A. Lie Groups, Lie Algebras and Group Operations

In order to present the notation adopted in this paper, some basic concepts on Lie groups and Lie algebras are briefly discussed. For a more detailed and complete treatment, please refer to [29], [36]. Denote by $\mathbb{E}_i$ and $\mathbb{E}_j$ a couple of 3-D Euclidean spaces and define a pair of rigid bodies B_i and B_j as subsets of $\mathbb{E}_i$ and $\mathbb{E}_j$ respectively. The relative position of $\mathbb{E}_i$ with respect to $\mathbb{E}_j$ is an element $h_i^j \in SE_i^j(3)$, being $SE_i^j(3)$ the set of positive isometries from $\mathbb{E}_i$ to $\mathbb{E}_j$.

If $\mathcal{I}$ is an open interval of $\mathbb{R}$, it is possible to consider curves in $SE_i^j(3)$ parametrized by [1] $\tau \in \mathcal{I}$. The differentiable function $h_i^j : \mathcal{I} \to SE_i^j(3)$ is a relative motion and its derivative with respect to τ is $\dot{h}_i^j \in T_{h_i^j} SE_i^j(3)$. It is convenient to transport this vector to the tangent space at the identity of the group, i.e., to a Lie algebra. More precisely, it is possible to map $\dot{h}_i^j$ either to an element $t_i^{i,j} \in se(3)$ or to an element $t_i^{j,j} := t_i^j \in se(3)$. If τ is interpreted as time, $t_i^{i,j}$ and t_i^j represent the twist of body B_i with respect to B_j expressed in $\mathbb{E}_i$ and $\mathbb{E}_j$ respectively. These representations are intrinsic, since they do not depend on the relative configuration h_i^j, and related through the adjoint map. In fact

$$t_i^j = \mathrm{Ad}_{h_i^j} t_i^{i,j}. \tag{1}$$

Once $t_i^{i,j} \in se(3)$ is given, it is possible to compute the relative motion $h_i^j(\tau)$ of space $\mathbb{E}_i$ with respect to $\mathbb{E}_j$ that passes through $h_i^j(0)$ for $\tau = 0$ with a *velocity* equal to $t_i^{i,j}$. The motion is given by

$$h_i^j(\tau) = h_i^j(0) \circ e^{t_i^{i,j}\tau} \tag{2}$$

where e denotes the group exponential. The group exponential maps an element of the algebra to an element of the group. Consequently, for any $t_i^{i,j} \in se(3)$, it is possible to define a linear map within the Lie algebra as the differential of $\mathrm{Ad}_{e^{t_i^{i,j}\tau}}$ in $\tau = 0$

$$\mathrm{ad}_{t_i^{i,j}} = \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{Ad}_{e^{t_i^{i,j}\tau}} \Big|_{\tau=0}. \tag{3}$$

The same considerations hold for $t_i^j \in se(3)$.

Given $h_i^j \in SE_i^j(3)$, it is also possible to consider the co-vectors belonging to $T^*_{h_i^j} SE_i^j(3)$ which, applied to elements of $T_{h_i^j} SE_i^j(3)$, result in a scalar. Based on this duality property, it is possible to define the *"forces"*. More precisely, the generalized force between two Euclidean spaces $\mathbb{E}_i$ and $\mathbb{E}_j$ in h_i^j is an element of $T^*_{h_i^j} SE_i^j(3)$. This element can be intrinsically mapped to $w_i^{i,j} \in se^*(3)$ or to $w_i^j \in se^*(3)$, which

[1]Usually, the independent variable τ represents time. In this paper, it can represent time, and also, the spatial parametrization of the link. In fact, as discussed in Section III, the spatial domain of the distributed parameter system is the interval $\mathcal{Z} = [0, L] \subset \mathbb{R}$, where L is the length of the link in the unstressed configuration. Which of the two meanings for τ is meant is clear from the context.

represent the wrench between B_i and B_j expressed in $\mathbb{E}_i$ and $\mathbb{E}_j$ respectively. We have that

$$w_i^{i,j} = \mathrm{Ad}^*_{h_i^j} w_i^j. \tag{4}$$

B. Dirac and Stokes–Dirac Structures

Denote by $\mathcal{F} \times \mathcal{E}$ the space of power variables, with $\mathcal{F}$ an n-dimensional linear space, the space of flows (e.g., velocities and currents) and $\mathcal{E} \equiv \mathcal{F}^*$ its dual, the space of efforts (e.g., forces and voltages), and by $\langle ef \rangle$ the power associated to the *port* $(f, e) \in \mathcal{F} \times \mathcal{E}$, with $\langle \cdot, \cdot \rangle$ the dual product between f and e. Based on the dual product, the following symmetric bilinear form (+pairing operator) is well defined.

$$\ll (f_1, e_1), (f_2, e_2) \gg = \langle e_1, f_2 \rangle + \langle e_2, f_1 \rangle \tag{5}$$

Consider a linear subspace $\mathbb{S} \subset \mathcal{F} \times \mathcal{E}$ of dimension m and denote by $\mathbb{S}^\perp$ its orthogonal complement with respect to (5). Based on (5), it is possible to give the following definition.

Definition 2.1: Consider the space of power variables $\mathcal{F} \times \mathcal{E}$ and the symmetric bilinear form (5). A (constant) Dirac structure on $\mathcal{F}$ is a linear subspace $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$ such that $\mathbb{D} = \mathbb{D}^\perp$.

Note 2.1: Suppose that $(f, e) \in \mathbb{D}$; from (5), we have that $\langle e, f \rangle = 0$, i.e., a Dirac structure on $\mathcal{F}$ defines a power-conserving relation between power variables $(f, e) \in \mathcal{F} \times \mathcal{E}$.

Example 2.1: Suppose that $\mathcal{F} \equiv \mathcal{E} = \mathbb{R}^n$. Then, it is easy to prove that, if $J = -J^{\mathrm{T}}$, the set $\mathbb{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} | -f = Je\}$ is a Dirac structure with respect to $\ll (f_1, e_1), (f_2, e_2) \gg = e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1$.

This definition can be generalized in order to cope with an infinite-dimensional space of power variables, thus leading to the definition of Stokes–Dirac structure, [18]. For simplicity, suppose that $\mathcal{Z} = [0, L] \subset \mathbb{R}$ is the spatial domain of the distributed parameter system and refer to [18] for the general case where $\mathcal{Z}$ is a compact subset of $\mathbb{R}^d$, $d \geq 1$. Given the linear spaces in duality $\mathcal{F}^q \equiv \mathcal{E}^{q^*}$ and $\mathcal{F}^p \equiv \mathcal{E}^{p^*}$, the space of *distributed* flows $\mathcal{F}^d$ can be defined as $\mathcal{F}^d = \Omega^1_{\mathcal{F}^q}(\mathcal{Z}) \times \Omega^1_{\mathcal{F}^p}(\mathcal{Z})$, where $\Omega^1_{\mathcal{F}^q}(\mathcal{Z})$ (resp. $\Omega^1_{\mathcal{F}^p}(\mathcal{Z})$) is the space of $\mathcal{F}^q$-valued (resp. $\mathcal{F}^p$-valued) one-forms on $\mathcal{Z}$, while the space of *distributed* efforts is $\mathcal{E}^d = \Omega^0_{\mathcal{E}^q}(\mathcal{Z}) \times \Omega^0_{\mathcal{E}^p}(\mathcal{Z})$, where $\Omega^0_{\mathcal{E}^q}(\mathcal{Z})$ (resp. $\Omega^0_{\mathcal{E}^p}(\mathcal{Z})$) is the space of $\mathcal{E}^q$-valued (resp. $\mathcal{E}^p$-valued) zero-forms on $\mathcal{Z}$, [37]. Roughly speaking, the flows are *densities* defined on $\mathcal{Z}$, while the efforts are simply *functions* on the same domain. In the case of the flexible link, it is assumed that $\mathcal{F}^q \equiv \mathcal{E}^p = se(3)$ and $\mathcal{F}^p \equiv \mathcal{E}^q = se^*(3)$.

In the simplest case, the distributed parameter system can interact with the environment through the boundary $\partial\mathcal{Z} = \{0, L\}$ of the spatial domain. Consequently, the system is equipped by a boundary port, that is by a pair $(f^B, e^B) \in \mathcal{F}^B \times \mathcal{E}^B$, with $\mathcal{F}^B$ and $\mathcal{E}^B$ the space of (boundary) flows and efforts. Due to the hypothesis on the dimension of the spatial domain, these sets are finite-dimensional linear spaces. Once the power associated to $(f^q, f^p, e^q, e^p) \in \mathcal{F}^d \times \mathcal{E}^d$ and to $(f^B, e^B) \in \mathcal{F}^B \times \mathcal{E}^B$ is defined, it is possible to generalize the +pairing operator to the distributed parameter case. Then the definition of Stokes–Dirac structure follows in the same way as Definition 2.1.

Example 2.2: Suppose that $\mathcal{F}^q \equiv \mathcal{F}^p \equiv \mathcal{F}^B = \mathbb{R}$. Then, it is possible to prove that

$$\mathbb{D} = \Bigg\{ (f^q, f^p, f^B, e^q, e^p, e^B) \in \mathcal{F}^d \times \mathcal{F}^B \times \mathcal{E}^d \times \mathcal{E}^B \mid -\begin{pmatrix} f^q \\ f^p \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} e^q \\ e^p \end{pmatrix}, \begin{pmatrix} f^B \\ e^B \end{pmatrix} = \begin{pmatrix} e^q|_{\partial\mathcal{Z}} \\ e^p|_{\partial\mathcal{Z}} \end{pmatrix} \Bigg\}$$

is a Stokes–Dirac structure with respect to

$$\begin{aligned} &\ll \left(f_1^q, f_1^p, f_1^B, e_1^q, e_1^p, e_1^B\right), \left(f_2^q, f_2^p, f_2^B, e_2^q, e_2^p, e_2^B\right) \gg \\ &= \int_{\mathcal{Z}} \left(f_1^q e_2^q + f_2^q e_1^q + f_1^p e_2^p + f_2^p e_1^p\right) \\ &\quad + e_1^{B^{\mathrm{T}}}(L) f_1^B(L) + e_2^{B^{\mathrm{T}}}(L) f_2^B(L) \\ &\quad - e_1^{B^{\mathrm{T}}}(0) f_1^B(0) - e_2^{B^{\mathrm{T}}}(0) f_2^B(0) \end{aligned}$$

where d denotes the exterior derivative [37] and $|_{\partial\mathcal{Z}}$ the restriction on the boundary of $\mathcal{Z}$; this can be seen with the use of integration by parts.

C. Port Hamiltonian Systems

A Dirac structure is a linear space that describes internal power flows and the exchange of energy between the system and its environment. The dynamics follows once the port behavior of the energy storing elements is specified. In finite dimensions, denote by $\mathcal{X}$ the space of energy variables and by $H : \mathcal{X} \to \mathbb{R}$ the energy function. Then, the dynamics of the system follows from the corresponding Dirac structure $\mathbb{D}$ once it is imposed that

$$f = -\dot{x} \qquad e = \frac{\partial H}{\partial x}$$

If the Dirac structure of Example 2.1 is considered, the dynamics is expressed by the following differential equation:

$$\dot{x} = J\frac{\partial H}{\partial x}, \qquad x(0) = x_0 \in \mathcal{X}.$$

It is easy to see that, in this case, $\dot{H} = 0$, i.e., energy is conserved. This is due to the fact that this Dirac structure does not consider any further port to which dissipative elements or sources can be interconnected. Refer to [22], [24] for a general formulation of port Hamiltonian systems.

Also in infinite dimensions, central issues are the identification of a proper space of energy variables and the definition of the energy (Hamiltonian) function. The energy variables are given by a pair $(\alpha_q, \alpha_p) \in \mathcal{X} := \Omega^1_{\mathcal{F}^q}(\mathcal{Z}) \times \Omega^1_{\mathcal{F}^p}(\mathcal{Z})$, i.e., by a pair of one-forms. Then, it is possible to define the energy density as $H : \mathcal{Z} \times \mathcal{X} \to \Omega^1_{\mathbb{R}}(\mathcal{Z})$ and the total energy as

$$\mathcal{H}(\alpha_q, \alpha_p) = \int_{\mathcal{Z}} H(z, \alpha_q, \alpha_p)$$

System dynamics follows by imposing that $f^q = -\dot{\alpha}_q$, $f^p = -\dot{\alpha}_p$, $e^q = \delta_{\alpha_q}\mathcal{H}$ and $e^p = \delta_{\alpha_p}\mathcal{H}$, where δ denotes the variational derivative, [18]. If the Stokes–Dirac structure introduced in Example 2.2 is considered, it follows that

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = e^{B^{\mathrm{T}}} f^B \Big|_{z=0}^{z=L} = e^{B^{\mathrm{T}}}(L) f^B(L) - e^{B^{\mathrm{T}}}(0) f^B(0)$$

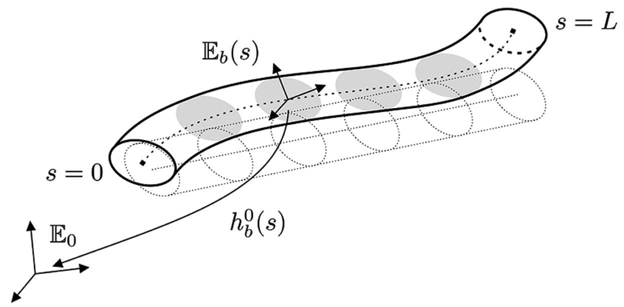

Fig. 1. Schematic representation of a flexible link in the deformed and unstressed configurations.

i.e., the variation of internal energy equals the difference between incoming (at $x = L$) and outgoing (at $x = 0$) power.

III. Flexible Link Model

A. Link Description and Deformation Analysis

Consider a slender flexible beam of length L and with an unstressed configuration which is *not* required to be a straight line. As illustrated in Fig. 1 and following [15] and [17], if $s \in [0, L]$ denotes the position along the link in the unstressed configuration, assume that the configuration in the space of the cross section with respect to an inertial reference $\mathbb{E}_0$ is given by $h_b^0(s) \in SE_b^0(3)$, where the subscript b denotes the *body reference* $\mathbb{E}_b(s)$ attached to the cross section. The motion of the cross section is due to a wrench $w_b^{b,0}$ and described by a twist $t_b^{b,0}(s)$. Both quantities are expressed in $\mathbb{E}_b(s)$.

Making use of (1), the relative motion between infinitesimally closed cross sections due to elasticity is given by

$$t_\delta^b(s) = \mathrm{Ad}_{h_0^b(s)} \mathrm{d} \left(\mathrm{Ad}_{h_b^0(s)} t_b^{b,0}(s) \right) \tag{6}$$

if expressed in body reference, where d is the exterior derivative which acts as a derivation in the spatial variable s. Note that thanks to the use of twists, in case in which a pure rigid body motion is taking place, $t_b^{b,0}(s)$ would be the same for all $s \in [0, L]$ and equal to the rigid body motion.

As shown in Fig. 2(a), the twist $t_b^{b,0}$ represents the "velocity" of the cross section in space and is defined in each point of the spatial domain $[0, L]$. On the other hand, $t_\delta^b(s)$ provides the relative "velocity" of the cross section in $s + \Delta s$ with respect to the one in s ($\Delta s \to 0$) and can be integrated along the spatial domain for the same instant in time to compute how the link deformation changes in time. Integrating $t_b^{b,0}$ in s does not provide any meaningful quantity. Even if $t_b^{b,0}$ and t_δ^b are both twists, they are different objects. Since $t_b^{b,0}$ is a function in $[0, L]$, it is a $se(3)$-valued zero-form, while t_δ^b is a $se(3)$-valued one-form. Roughly speaking, t_δ^b requires an infinitesimal Δs to make sense or, equivalently, can be integrated along the line.

The relative motion $t_\delta^b(s)$ is related to a deformation and, consequently, a wrench $w_b^{b,0}$ acting on the cross section due to the elastic forces is present. As reported in Fig. 2(b), the difference between the elastic forces acting on the cross sections in $s + \Delta s$ and in s generates a wrench w_δ^b on the link element $[s, s + \Delta s]$

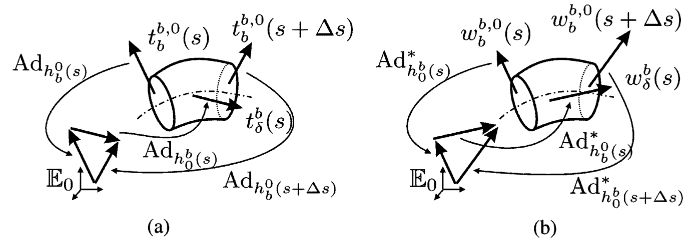

Fig. 2. Intuitive explanation of relations (6) and (7). (a) Computation of $t_\delta^b(s)$. (b) Computation of $w_b^{b,0}(s)$.

which is responsible of the motion. Making use of (4) and dually with respect to (6), we have that

$$w_\delta^b(s) = \mathrm{Ad}^*_{h_b^0(s)} \mathrm{d} \left(\mathrm{Ad}^*_{h_0^b(s)} w_b^{b,0}(s) \right) \tag{7}$$

where $w_b^{b,0}$ and w_δ^b are respectively $se^*(3)$-valued zero and one-forms. The integral over the spatial domain of w_δ^b provides the total wrench acting on the link.

The link can interact with the environment through a pair of ports at its extremities. The boundary port variables are the twist and wrench of the cross sections in $s = 0$ and $s = L$

$$\left(t_b^{b,0}(0), w_b^{b,0}(0) \right) \quad \text{and} \quad \left(t_b^{b,0}(L), w_b^{b,0}(L) \right) \tag{8}$$

Note that these quantities are expressed in body references $\mathbb{E}_b(0)$ and $\mathbb{E}_b(L)$.

B. Associated Stokes–Dirac Structure

Equations (6) and (7), together with the boundary terms (8), are fundamental in the definition of the Stokes–Dirac structure describing the energetic structure of the flexible link. Before providing such a Stokes–Dirac structure, as discussed in Section II-B, it is necessary to define the space of power variables. As far as the space of flows is concerned, let us assume

$$\mathcal{F}^d = \Omega^1_{se(3)}(\mathcal{Z}) \times \Omega^1_{se^*(3)}(\mathcal{Z})$$

while the space of efforts is

$$\mathcal{E}^d = \Omega^0_{se^*(3)}(\mathcal{Z}) \times \Omega^0_{se(3)}(\mathcal{Z}).$$

This choice will become more clear later. From (8), the *natural* space of boundary power variables is given by

$$\begin{aligned}\mathcal{F}^B \times \mathcal{E}^B &= \Omega^0_{se(3)}(\partial\mathcal{Z}) \times \Omega^0_{se^*(3)}(\partial\mathcal{Z}) \\ &= (se(3) \times se^*(3)) \times (se(3) \times se^*(3))\end{aligned}$$

since both the pairs twist/wrench in $s = 0$ and $s = L$ have to be taken into account.

Given $(f^q, f^p, f^B, e^q, e^p, e^B) \in \mathcal{F}^d \times \mathcal{F}^B \times \mathcal{E}^d \times \mathcal{E}^B$, the associated power (i.e., the duality product) can be computed as follows:

$$\begin{aligned}&\langle\langle (e^q, e^p, e^B)(f^q, f^p, f^B) \rangle\rangle \\ &\quad = \int_{\mathcal{Z}} *(\langle e^q * f^q \rangle + \langle * f^p e^p \rangle) + \langle e^B f^B \rangle |_{s=0}^{s=L}\end{aligned} \tag{9}$$

where $*$ is the Hodge star operator mapping one-forms to zero-forms and vice-versa, [37]. The duality product on the right side is the natural pairing between elements of $se(3)$ and $se^*(3)$. Note that (9) makes sense since $*f^q$ and $*f^p$ are zero-forms (i.e., functions) on $\mathcal{Z}$ and then $*(\langle e^q * f^q\rangle + \langle *f^p e^p\rangle)$ a one-form that can be integrated on $\mathcal{Z}$.

Propostion 3.1: The Set:

$$\mathbb{D} = \Bigg\{ (f^q, f^p, f^B, e^q, e^p, e^B) \in \mathcal{F}^d \times \cdots \times \mathcal{E}^B \,| \\ \begin{pmatrix} f^q \\ g^p \end{pmatrix} = -\begin{pmatrix} \mathrm{Ad}_{h_0^b(s)} \mathrm{d}\left(\mathrm{Ad}_{h_b^0(s)}\, e^p\right) \\ \mathrm{Ad}^*_{h_b^0(s)} \mathrm{d}\left(\mathrm{Ad}^*_{h_0^b(s)}\, e^q\right) \end{pmatrix} \\ \begin{pmatrix} f^B \\ e^B \end{pmatrix} = \begin{pmatrix} e^q|_{\partial\mathcal{Z}} \\ e^p|_{\partial\mathcal{Z}} \end{pmatrix} \Bigg\} \tag{10}$$

is a Stokes–Dirac structure with respect to the +pairing determined by the duality product (9).

Proof: As discussed in [17] in the planar case, it is possible to find a coordinate change that maps (10) into the Stokes–Dirac structure of Example 2.2, whose properties have been proved in [18]. An alternative method is based on integration by parts and takes advantage of the skew–adjoint properties of the differential operator that defines the subset (10). This method is extensively adopted in [19], [20], and [25] in similar proofs involving also higher order differential operators. ■

C. Constitutive Relations and Hamiltonian Function

The energy (state) variables associated with the flexible link are the infinitesimal "deformation" q, i.e., the strain, and momentum p, expressed in body reference. More in details, the state space is defined as

$$\mathcal{X} = \Omega^1_{se(3)}(\mathcal{Z}) \times \Omega^1_{se^*(3)}(\mathcal{Z})$$

with q and p $se(3)$-valued and $se^*(3)$-valued one-forms respectively. From a *physical* point of view, it is necessary that these quantities are 1-forms because they are *densities*.

Once the space of state variables has been defined, the last step is the definition of a Hamiltonian function, given by the sum of two contributions: the kinetic energy and the potential elastic one, due to deformation. As far as the kinetic energy is concerned, denote by I the inertia tensor which defines a quadratic form on $se(3)$. As discussed in [29], this tensor uniquely defines a bijection between elements in $se(3)$ and elements in $se^*(3)$. More precisely, given $t_b^{0,b} \in se(3)$, there is a unique $p \in se^*(3)$ such that $p(\bar{t}_b) = \langle t_b^{0,b} \bar{t}_b\rangle_I$, for every $\bar{t}_b \in se(3)$. The quantity p is the momentum of the cross section. Since I is non-singular, it is possible to define the following one-form which represents the kinetic energy density of the link

$$K(p) = \frac{1}{2} * \langle *p \,|\, *p\rangle_Y \tag{11}$$

where $Y = I^{-1}$. Note that $*p$ is a function of $s \in \mathcal{Z}$ with values in $se^*(3)$.

As far as the elastic energy contribution is concerned, denote by C the compliance tensor, with inverse C^{-1}, which defines a quadratic form on $se(3)$. Then, the elastic energy density is given by the following one-form [16], [18]:

$$W(q) = \frac{1}{2} * \langle *q \,|\, *q\rangle_{C^{-1}}. \tag{12}$$

Again, $*q$ is a function of $s \in \mathcal{Z}$ with values in $se(3)$. Relation (12) holds under the hypothesis of linear elasticity theory since the corresponding energy density is *quadratic* in the strain. If nonlinear effects have to be taken into account, (12) should be replaced by a map $W: \mathcal{Z} \times \Omega^1_{se(3)}(\mathcal{Z}) \to \Omega^1_{\mathbb{R}}(\mathcal{Z})$, in which the dependence on the spatial variables has been also considered. For simplicity, in the remaining part of the paper, a quadratic elastic energy density in the form (12) is assumed.

Finally, from (11) and (12), the total energy (Hamiltonian) function is given by

$$\begin{aligned} \mathcal{H}(p,q) &= \frac{1}{2}\int_{\mathcal{Z}} (K(p) + W(q)) \\ &= \frac{1}{2}\int_{\mathcal{Z}} *(\langle *p \,|\, *p\rangle_Y + \langle *q \,|\, *q\rangle_{C^{-1}}). \end{aligned} \tag{13}$$

D. Port Hamiltonian Model of the Link

As discussed in Section II-C, the port Hamiltonian model of a dynamical system follows from the corresponding Stokes–Dirac structure once the port behavior of the energy storing components (and in case of the resistive part) has been specified. Consequently, the port Hamiltonian model of the link results from (10) and from

$$-f^q = \frac{\partial q}{\partial t}\left(= t_\delta^b\right), \quad -f^p = \frac{\partial p}{\partial t} - p \wedge t_b^{b,0} \left(= w_\delta^b\right) \tag{14a}$$

$$e^q = \delta_q \mathcal{H}\left(= w_b^{b,0}\right), \quad e^p = \delta_p \mathcal{H}\left(= t_b^{b,0}\right). \tag{14b}$$

The second relation in (14a) is simply the second law of dynamics in body reference, with $p \wedge t_b^{b,0} \equiv \mathrm{ad}^*_{t_b^{b,0}} p$, [29].

The port Hamiltonian representation of the link follows immediately. Note how the results presented in [17] have been easily extended in order to cope with deformation in 3-D space.

$$\begin{cases} \partial_t q = \mathrm{Ad}_{h_0^b(s)} \mathrm{d}\left(\mathrm{Ad}_{h_b^0(s)} \delta_p \mathcal{H}\right) \\ \partial_t p = \mathrm{Ad}^*_{h_b^0(s)} \mathrm{d}\left(\mathrm{Ad}^*_{h_0^b(s)} \delta_q \mathcal{H}\right) + p \wedge \delta_p \mathcal{H}. \end{cases} \tag{15a}$$

Furthermore, the boundary terms are given by

$$\begin{aligned} f^B(0) &= \delta_p \mathcal{H}|_{s=0} \quad e^B(0) = \delta_q \mathcal{H}|_{s=0} \\ f^B(L) &= \delta_p \mathcal{H}|_{s=L} \quad e^B(L) = \delta_q \mathcal{H}|_{s=L}. \end{aligned} \tag{15b}$$

From the properties of a Dirac structure, see Note 2.1, the following energy balance relation easily follows:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \langle e^B(L), f^B(L)\rangle - \langle e^B(0), f^B(0)\rangle. \tag{16}$$

This relation states an obvious property of this physical system, i.e., the fact that the variation of internal energy equals the total power flow at its boundary. Since no dissipative effect is considered, if the boundary energy flow is set to zero (i.e., in the case of a flexible beam clamped at both its extremities) energy is conserved. At the end of this section, a way to include dissipative effects, but also distributed actuation or gravity, in terms of port interconnection is illustrated.

Note that the model (15a) depends on the configuration of the cross section with respect to the inertial reference $h_b^0(s)$, but the state variable of the system is the strain q related to the infinitesimal deformation. However, if $\hat{h}_b^0 : \mathcal{Z} \to SE_b^0(3)$ denotes the unstressed configuration, the "twist" $\hat{n} \in \Omega^1_{se(3)}(\mathcal{Z})$ describing how the unstressed configuration evolves in the spatial variable s and expressed in body reference is given by the following (see also [15]):

$$\hat{n} = \left(\hat{h}_b^0\right)^{-1} \mathrm{d}\hat{h}_b^0. \tag{17}$$

Then, for all $t \in se(3)$, relation (6) becomes

$$\mathrm{Ad}_{h_0^b} \mathrm{d}\left(\mathrm{Ad}_{h_b^0} t\right) = \mathrm{d}t + \mathrm{ad}_{(q+\hat{n})} t \tag{18}$$

and, dually, for every $w \in se^*(3)$, (7) can be written as

$$\mathrm{Ad}^*_{h_b^0} \mathrm{d}\left(\mathrm{Ad}^*_{h_0^b} w\right) = \mathrm{d}w - \mathrm{ad}^*_{(q+\hat{n})} w. \tag{19}$$

More details in Appendix A. By combining (15a) with (18) and (19), the dynamics of the flexible link assume the following equivalent expression:

$$\begin{cases} \partial_t q = \mathrm{d}\delta_p \mathcal{H} + \mathrm{ad}_{(q+\hat{n})} \delta_p \mathcal{H} \\ \partial_t p = \mathrm{d}\delta_q \mathcal{H} - \mathrm{ad}^*_{(q+\hat{n})} \delta_q \mathcal{H} + p \wedge \delta_p \mathcal{H}. \end{cases} \tag{20}$$

Note the invariance of the dynamics to group action. The boundary terms remain the same as in (15b).

E. Introducing A Distributed Port

In (20), it is not immediate to see how the effects of a gravity field can be taken into account since the configuration of the cross section with respect to the inertial reference $h_b^0(s)$ is not available. Moreover, at present stage it is not clear how dissipative effects can be taken into account and if it is possible to include distributed actuation along the spatial domain of the flexible link. As discussed in [18] for Maxwell's equations, in [5] for the Timoshenko beam and in [25] for piezo-electric materials, a possible solution can be to modify the Stokes–Dirac structure (10) by including a distributed port along the domain. In the case of flexible link, this distributed port is characterized by a space of power variables

$$\mathcal{F}^e \times \mathcal{E}^e = \Omega^0_{se(3)}(\mathcal{Z}) \times \Omega^1_{se^*(3)}(\mathcal{Z})$$

in which flows are twists, defined pointwise, and the efforts are wrench density. If it is supposed that these quantities are expressed in body reference, by taking into account (18) and (19), the Stokes–Dirac structure (10) modifies into

$$\mathbb{D} = \Bigg\{ (f^q, \dots, f^B, e^q, \dots, e^B) \in \mathcal{F}^d \times \cdots \times \mathcal{E}^B \Big| \\ \begin{pmatrix} f^q \\ f^p \end{pmatrix} = -\begin{pmatrix} \mathrm{d}e^p + \mathrm{ad}_{(q+\hat{n})} e^p \\ \mathrm{d}e^q - \mathrm{ad}^*_{(q+\hat{n})} e^q \end{pmatrix} - \begin{pmatrix} 0 \\ e^e \end{pmatrix} \\ f^e = e^p, \quad \begin{pmatrix} f^B \\ e^B \end{pmatrix} = \begin{pmatrix} e^q|_{\partial\mathcal{Z}} \\ e^p|_{\partial\mathcal{Z}} \end{pmatrix} \Bigg\} \tag{21}$$

Note that the boundary terms (f^B, e^B) defined in (10) remain the same. Thanks to this distributed port it is possible to act on the flexible link along its spatial domain by imposing a distributed "force" and, at the same time, to measure the "velocity" of the cross section in space.

This distributed port can be *terminated* on a distributed impedance in order to model dissipative effects, such as viscous friction. In the linear case, it is sufficient to introduce a quadratic form D on $se(3)$ describing dissipation in body frame and to impose that $e^e = -{*}Df^e = -{*}D\delta_p\mathcal{H}$. Note that the power flow is always less than zero. As far as concerns gravity, a possible solution can be to integrate in time $f^e = \delta_p \mathcal{H} \equiv t_b^{0,b}$ so that position and orientation in space of the cross section is known. Then, gravity is a distributed source of effort modulates by orientation information that has to be interconnected to e^e.

Finally, suppose that deformation is small, that is $q \simeq 0$. If the unstressed configuration is assumed to be a straight line along the x-axis of the fixed reference frame, (20) simplifies to the superposition of a Timoshenko beam dynamics in the y and z directions and of a transverse wave in the x direction. This can be proved by writing (20) in coordinates (see Section IV-A) and by taking into account the port Hamiltonian formulation of the Timoshenko beam equations presented in [4] and [5]. This result has been originally verified in [17] for the planar case by following a different procedure.

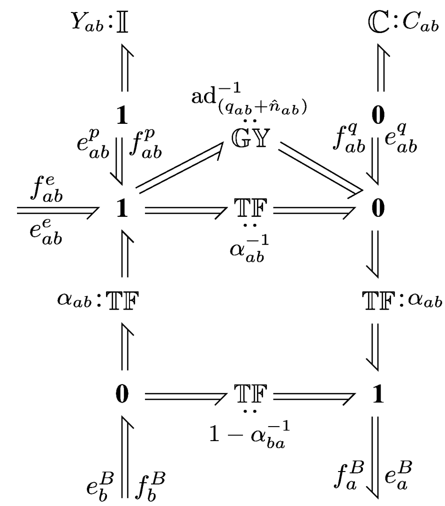

Fig. 3. Bond graph of the finite-dimensional approximation of the flexible link on $\mathcal{Z}_{ab} \subset \mathcal{Z}$.

IV. Simulations

In this section, the properties of the model, in terms of dynamical description of elasticity and in terms of composition

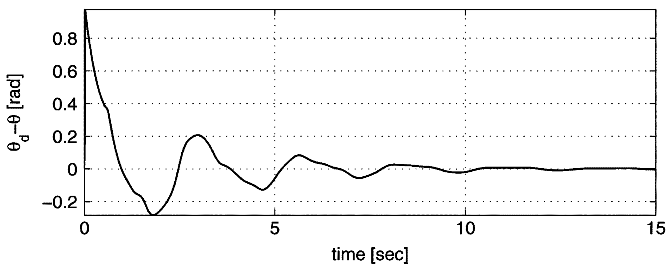

Fig. 4. Evolution in time of joint rotation error $\theta_d - \theta$, in radians.

TABLE I
FLEXIBLE LINK—PARAMETERS

Param.	Value	Param.	Value
J	$\mathrm{diag}(0.05,\ 0.005,\ 0.05)$	L	1
ρ	0.1	N	20
C_o	$5 \cdot 10^{-3}\mathrm{Id}_3$	α_{ab}	1/2
C_c	0	K_P	15
C_t	$\mathrm{diag}(10^{-5},\ 5 \cdot 10^{-3},\ 5 \cdot 10^{-3})$	$\mathbb{R}_{PD}$	3

with others, are illustrated. At first, in Section IV-A, the basic information for obtaining a finite-dimensional port Hamiltonian approximation of (20) in coordinates are presented. The behavior of a single link interconnected to a motor via a rotational joint is illustrated in Section IV-B. Then, a couple of flexible links is interconnected by rotational joints and a simple 2-DoF serial manipulator is obtained. Simulations are presented in Section IV-C. Finally, the model of a 2-DoF closed kinematic chain is implemented and its dynamical response shown in Section IV-D[2] .

A. Moving to Coordinates

The simulation of the flexible link (20) requires a finite-dimensional approximation. Since *energetic behavior* is a central issue in port Hamiltonian modeling, a spatial discretized approximation based on energy considerations has been developed. Based on a generalization of the approach proposed in [32] and recently extended to higher order Stokes–Dirac structures on one-dimensional spatial domain in [33], the resulting finite-dimensional model is again in port Hamiltonian form and satisfies the same energy balance relation of its infinite-dimensional counterpart. Basically, the spatial domain $\mathcal{Z}$ is divided into N parts $\mathcal{Z}_{ab} = [a, b] \subset \mathcal{Z}$ and on $\mathcal{Z}_{ab}$ the dynamics is approximated by a finite-dimensional port Hamiltonian system whose bond graph is in Fig. 3.

In this bond graph, the $\mathbb{I}$ element describes the motion of the portion of the link in space while $\mathbb{C}$ its elastic behavior due to deformation. Note the presence of the discrete approximation (f^e_{ab}, e^e_{ab}) of the distributed port introduced in (21) and the boundary ports (f^B_a, e^B_a) and (f^B_b, e^B_b), i.e., the pairs twist/wrench defined in $s = a$ and $s = b$. The internal interconnection structure approximates the propagation phenomenon inside the flexible link together with the nonlinear coupling due to deformation. The free parameter $\alpha_{ab} = 1 - \alpha_{ba}$ has to satisfy $0 < \alpha_{ab} < 1$. The final finite-dimensional approximation of the link results from the interconnection of the N elements on the boundary power ports.

[2]We have included supplementary color AVI files which contain the simulation results briefly discussed in this section. These will be available at http://ieeexplore.ieee.org.

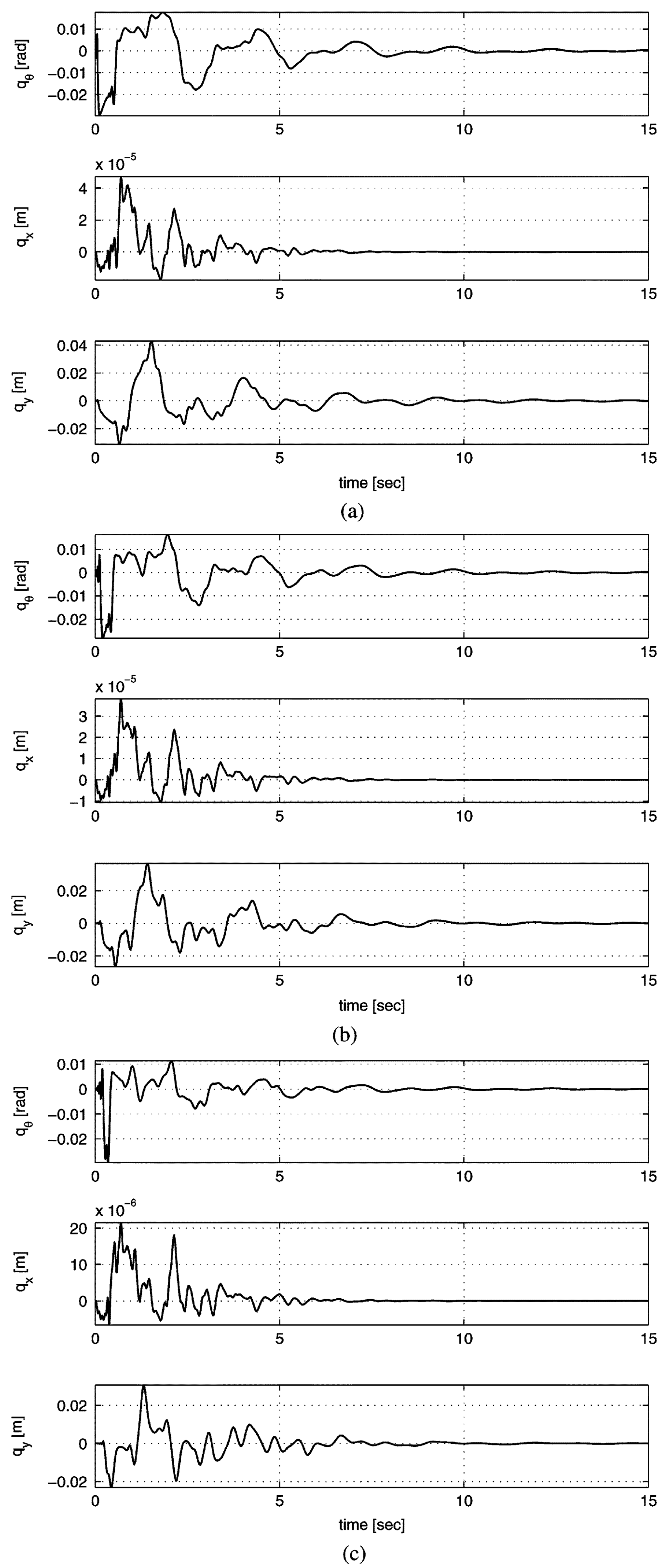

Fig. 5. Values of q_θ, q_x and q_y in s equal to $L/4$, $L/2$ and $3/4L$. Remember that these quantities are expressed in body reference. (a) $s = L/4$, (b) $s = L/2$, $s = 3/4L$.

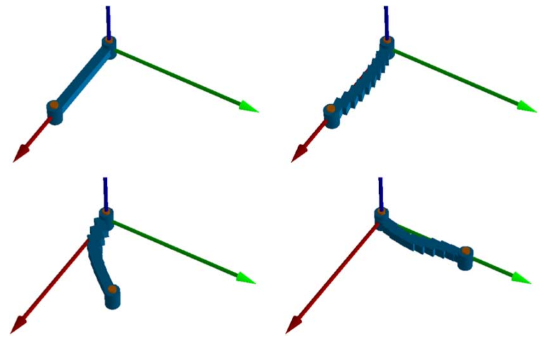

Fig. 6. Transient behavior of the single link.

In coordinates, strain q and momenta p can be described by $\mathbb{R}^6$ vectors whose components depend on time t and on the spatial variable $s \in \mathcal{Z}$. More precisely, if Q and P denote the coordinate description of such quantities, we write

$$Q = (Q_o, Q_t)^{\mathrm{T}} \quad P = (P_o, P_t)$$

where o and t denote the orientation and translation part of q and p. Clearly, Q_o, Q_t, P_o, and P_t belong to $\mathbb{R}^3$.

The kinetic and potential energies (11) and (12) can be written as follows. If J denotes the inertia tensor and ρ the mass density, a coordinate expression for Y is

$$Y = \underbrace{\begin{pmatrix} J & 0 \\ 0 & \rho\,\mathrm{Id}_3 \end{pmatrix}}_{=I}^{-1} \tag{22}$$

with Id_3 the 3×3 identity matrix, and then (11) becomes $K(P) = (1/2)P^{\mathrm{T}}YP$. As far as concerns the potential elastic energy due to deformation, the compliance tensor C can be written in coordinates as follows:

$$C = \begin{pmatrix} C_o & C_c \\ C_c^{\mathrm{T}} & C_t \end{pmatrix} \tag{23}$$

where C_o takes into account the rotational deformation, C_t the translational one while C_c the coupling between rotation and translation [38]. Finally, the adjoint of the exponential map has the following coordinate representation. Given $T = (T_o, T_t)^{\mathrm{T}} \in \mathbb{R}$ a coordinate description of a twist, i.e., of an element in $se(3)$, then

$$\mathrm{ad}_T = \begin{pmatrix} \tilde{T}_o & 0 \\ \tilde{T}_t & \tilde{T}_o \end{pmatrix}$$

where $\tilde{T}_{\{o,t\}}$ is the 3×3 skew–symmetric matrix associated to the vector $T_{\{o,t\}} \in \mathbb{R}^3$. In coordinates, the adjoint of this map is simply its transposed.

Once the coordinate description of (20) is written, the discrete approximation results in the bond graph of Fig. 3, in which flow and effort associated to each bond represent a vector in $\mathbb{R}^6$.

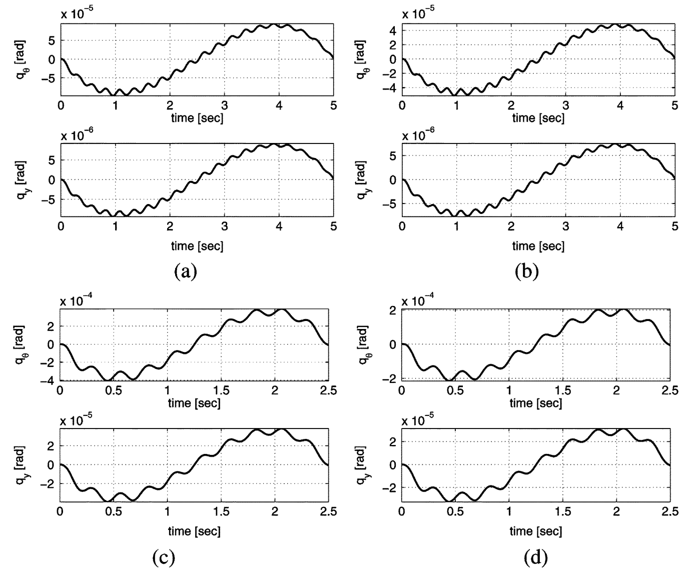

Fig. 7. PP-s and PP-f maneuvers. (a) PP-s, $s = L/4$. (b) PP-s, $s = L/2$. (c) PP-f, $s = L/4$. (d) PP-f, $s = L/2$.

B. Single Link

In this first set of simulations, a (single) flexible link is interconnected to a fixed base by means of a rotational joint allowing only rotations along the z axis. The rotation of the joint with respect to a given reference is denoted by θ. As far as concerns the boundary conditions in $s = L$, it is supposed that no interaction between the flexible link and the environment takes place. Then, the external wrench at $s = L$ is set equal to zero.

Simulations have been performed with the software package 20Sim© [35]. The idea is to show the behavior of the model and of its finite-dimensional approximation. In particular, we want to demonstrate how this model is able to describe large deformations and the nonlinear coupling between rotational and translational deflection. The parameters of the link and the desired joint trajectory θ_d have been chosen in such a way that large deflection in the mechanical structure appears during the transient. Same considerations hold for the controller gains. The controller is a simple PD on the joint variable θ which is the simplest solution for asymptotically stabilize such kind of systems once the derivative action is not null, [1]. Since the derivative action act as a dissipative effect, it has been implemented by a *physical* damper denoted by $\mathbb{R}_{\mathrm{PD}}$.

The desired joint trajectory θ_d is a unitary (in radians) step in $t = 0$. The physical parameters of the flexible link and of the rotational joint, together with the parameters K_P and $\mathbb{R}_{\mathrm{PD}}$ of the PD controller have been chosen as reported in Table I. In the same table, the parameters of the discrete approximation of the link, which results from the power conserving interconnection of $N = 20$ elements of Fig. 3, have been also indicated. Note that the parameters have been chosen in such a way that the motion corresponding to the given joint reference θ_d is planar and the deformation can be globally described by a rotation around the z-axis and by a pair of translations in the x and y directions.

Simulation results are presented in Figs. 4 and 5. In the first one, the evolution in time of the error $\theta_d - \theta$ on the joint variable is presented. As desired, the error slowly decreases to zero with a convergence rate that can be increased by augmenting the

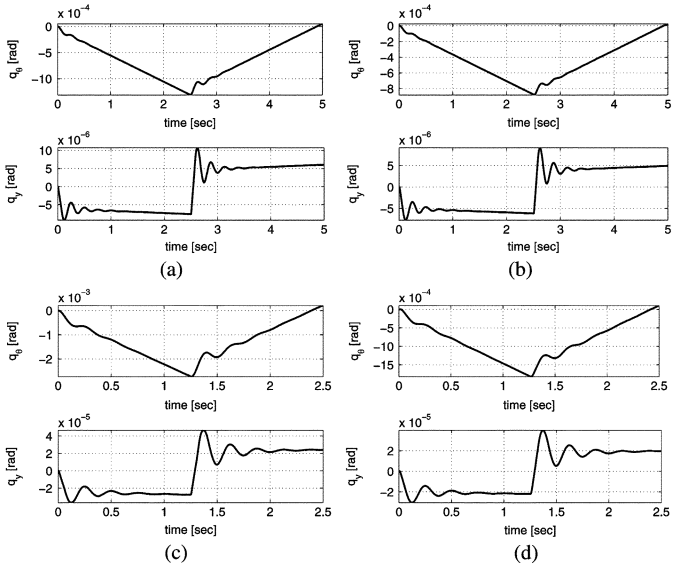

Fig. 8. ST-s and ST-f maneuvers (with damping). (a) ST-s, $s = L/4$. (b) ST-s, $s = L/2$. (c) ST-f, $s = L/4$. (d) ST-f, $s = L/2$.

derivative action in the controller. In Fig. 5, the deformation of the link in three different positions have been reported. Here, q_θ represents the rotational deformation (bending), while q_x and q_y the deformations along the x and y axes respectively (shear and normal strain). It is important to remember that the state variable q (or its spatial discretized approximation) represents the deformation of the cross section with respect to the unstressed configuration and is expressed in body reference. This is the reason why shear and normal strain in $s = L/2$ and $s = 3/4L$ [see Fig. 5(b) and (c)] are smaller than in $s = 1/4L$, reported in Fig. 5(a).

The initial part of the transient response is shown in Fig. 6. Each finite element is graphically represented by a small cube. As reported in Table I, the flexible link has been divided into $N = 20$ elements which approximate the dynamics (20). On the other hand, in order to speed up simulation and reduce the computational efforts, the 3-D graphical description of the link uses the information provided by ten elements only. Note the large deformations and that the step response is typical of a minimum phase system, as expected.

Furthermore, the same time domain analysis presented in [39] has been performed. More precisely, two test trajectories for θ have been considered, each one executed at high and low speed. The first one is a generic "pick-and-place" trajectory represented by a fifth-order polynomial and denoted by PP

$$\theta(t) = (\theta_f - \theta_0)\frac{t^3}{T^3}\left(10 - 15\frac{t}{T} + 6\frac{t^2}{T^2}\right) + \theta_0$$

for which $\theta_0 = \theta(0) = 0$ rad is the initial and $\theta_f = \theta(T) = 2$ rad the desired angle. The second trajectory is a step acceleration (ST) which aims to excite high-order vibration modes in the mechanical structure

$$\ddot{\theta}(t) = \begin{cases} A, & \text{if } 0 \le t < T/2 \\ -A, & \text{if } T/2 \le t < T \\ 0, & \text{if } t \ge T \end{cases}$$

with $A = 4(\theta_f - \theta_0)/T^2$. Both trajectories have been executed at slow velocity $(-s)$ with $T = 5$ s and at high-speed $(-f)$, with $T = 2.5$ s. The time evolution of q_θ and q_y at $s = L/4$ and $s = L/2$ are reported in Figs. 7 and in 8 for the PP and ST maneuvers, respectively.

Simulation results are comparable with the ones presented in [39]. In the case of ST maneuvers, viscous damping has been introduced in order to reduce oscillations and show how the (normal) strain follows the joint accelerations. Note that ST and PP maneuvers excite almost the same resonance modes: this is evident if the plots in Figs. 7(b) and 8(b) are compared.

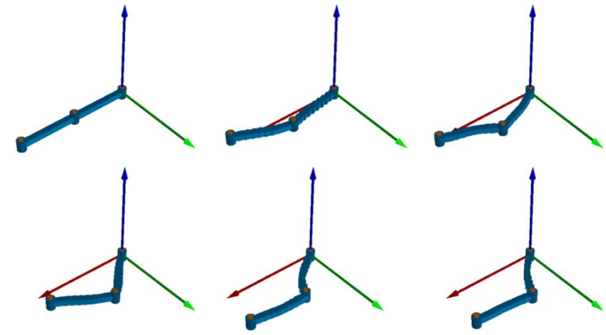

Fig. 9. Simulation results for the 2-DoF manipulator with flexible links.

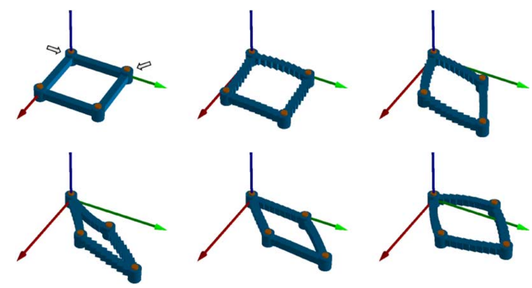

Fig. 10. Simulation results for the 2-DoF flexible closed kinematic chain.

C. 2-DoF Flexible Manipulator

A 2-DoF planar manipulator with flexible links and two rotational joints is now described. Aim of this section is to show how it is possible to create mechanical structures by properly interconnecting a set of elementary components (flexible links, joints, motors, environment).

In this simulation, the initial configuration of the mechanism is along the x-axis an the controller acts on joint torques. The flexible links are equal and with the physical properties summarized in Table I. The regulators and the assumptions concerning the discretization technique remains the same while τ_D has been changed into 0.75. The reference trajectory is again given in joint space. Starting from an initial configuration denoted by (0,0), the desired final configuration is $(0.7, -1)$. All the angles are in radiant. Part of the trajectory of the mechanism is reported in Fig. 9.

D. Simple 2-DoF Flexible Closed Kinematic Chain

Port-based modeling techniques allows to easily create also complex mechanical structure by properly specifying only the interconnection equations. In this section, a simulation of a closed kinematic chain is presented. More precisely, four flexible links have been interconnected trough rotational joints in order to obtain an articulated parallelogram. The links are assumed to be equal and with physical parameters given in Table I. Clearly, if the links were rigid, the resulting mechanism would have 2 DoF. Then, we suppose that only two joints are actuated, as in the 2-DoF case discussed in Section IV-C. These joint are indicated by an arrow in the first image in Fig. 10.

Starting from an initial configuration denoted by (0,0) in joint space, the PI regulators drive the system to the new state (0.6,1). As usual, all angles are in radiant. The behavior of the system is briefly represented in Fig. 10. Deformations are smaller than in the previous case since the mechanical structure is closed. As a matter of fact, the controller gains have been also increased to $K_P = 300$ in order to make deflections more evident.

V. Conclusions and Future Work

In this paper, the model of a flexible link suitable for robotic applications has been discussed. The model has been developed for such situations in which the infinite-dimensional nature of the elastic deformation dynamics has to be taken into account. The proposed model is characterized by a one-dimensional spatial domain but it is able to describe deformations in the three-dimensional space. These deformations are not required to be infinitesimal, as in the case of Euler–Bernoulli or Timoshenko beam, since the dynamical model is able to deal with large deflections and the unstressed configuration is not required to be a straight line. Moreover, Timoshenko beam and, consequently, Euler-Bernoulli beam, can be deduced from the proposed model under the small deformation hypothesis. The nonlinear coupling between rotation and translational deformation has been considered and nonlinear constitutive equations in the elastic deformation contribution can be taken into account.

The model has been developed within the port Hamiltonian formulation because simple geometric considerations provide the basic structure behind the dynamical model, i.e., its Stokes–Dirac structure. This mathematical object describes the power flows within the dynamical system and between the system and its environment. Then, once the constitutive equations have been specified, the mathematical model follows automatically. Moreover, in port Hamiltonian system, *interconnection* is a central notion. In other words, a key feature of port Hamiltonian systems (or, better, of port-based modeling techniques) is that each (sub-)model can be interconnected to others (sub-)models once (linear) relations on the port variables are given. Then, if rigid and flexible links, joints, springs, dampers and environment are modeled within this formalism, the dynamics of a complex mechanical structure follows from the power conserving interconnection of these atomic *components*. Examples of 1-DoF and 2-DoF manipulators with flexible links and of a closed kinematic chain have been provided.

The port-based approach simplifies also the simulation of complex mechanism. In order to be able to simulate a flexible link, it is necessary to provide a finite-dimensional description. Since energetic behavior is a central issue in port Hamiltonian modeling, a spatial discretized approximation of the flexible link based on energy considerations has been developed. The resulting finite-dimensional model is again in port Hamiltonian form and satisfies the same energy balance relation of its infinite-dimensional counterpart.

Future work will deal with modeling and control applications. As regard modeling, a systematic way for obtaining the dynamical model of a mechanism with rigid and flexible links is under study and some preliminary results have been presented in [30]. As far as control applications are concerned, the problem of developing new control schemes for robots with flexible link that take into account the infinite-dimensional nature of elasticity will be approached. Initial idea is to apply the general methodology presented in [5], [40] to the regulation of manipulators in which flexible links are modeled in the way illustrated here.

Appendix

A. Proof of (18) and (19)

Suppose to evaluate (6) around a generic $s = \bar{s} \in \mathcal{Z}$ and consider the "twist" $\hat{n} \in \Omega^1_{se(3)}(\mathcal{Z})$ describing how the unstressed configuration evolves in the spatial variable s defined in (17). With some abuse in notation, from (2), locally we can write that $h_b^0(\bar{s} + \Delta s) = h_b^0(\bar{s}) \circ e^{(q(\bar{s}) + \hat{n}(\bar{s}))\Delta s}$ and

$$\mathrm{Ad}_{h_b^0(\bar{s}+\Delta s)} = \mathrm{Ad}_{h_b^0(\bar{s})} \circ \mathrm{Ad}_{e^{(q(\bar{s})+\hat{n}(\bar{s}))\Delta s}}$$

Consequently, from (3)

$$\mathrm{d}\left(\mathrm{Ad}_{h_b^0(s)}\right)\Big|_{s=\bar{s}} = \mathrm{Ad}_{h_b^0(\bar{s})} \circ \mathrm{ad}_{(q(\bar{s})+\hat{n}(\bar{s}))}$$

which leads to (18). Dually, since

$$\begin{aligned} h_0^b(\bar{s} + \Delta s) &= \left(h_b^0(\bar{s} + \Delta s)\right)^{-1} \\ &= e^{-(q(\bar{s})+\hat{n}(\bar{s}))\Delta s} \circ h_0^b(\bar{s}) \end{aligned}$$

we have that $\mathrm{Ad}^*_{h_0^b(\bar{s}+\Delta s)} = \mathrm{Ad}^*_{h_0^b(\bar{s})} \circ \mathrm{Ad}^*_{-e^{(q(\bar{s})+\hat{n}(\bar{s}))\Delta s}}$ which proves (19). See also [29, p. 98].

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