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ON STABILITY OF NON-SELFCONJUGATED DIFFERENCE SCHEMES WITH M-MATRICES FOR EVOLUTIONARY BOUNDARY VALUE PROBLEMS WITH ELLIPTIC OPERATOR OVER THE WHOLE SPACE

M.A. Bochev (a.k.a. M.A.Botchev)

Introduction

Let us consider the following initial-boundary value problem in a two-dimensional domain Ω:

$$\begin{aligned} &\frac{\partial c}{\partial t} + L[c] = F(x, y, t);\\ &L[c] = -(D_1 c_x)_x - (D_2 c_y)_y + v_1 c_x + v_2 c_y + \lambda c;\\ &D_i = D_i(x,y) > 0, \quad v_i = v_i(x,y), \quad i=1,2; \quad \lambda = \lambda(x,y) \geq 0; \qquad (1)\\ &\left(\alpha c + \beta \frac{\partial c}{\partial n}\right)\Big|_{\partial\Omega} = \gamma\Big|_{\partial\Omega}, \quad \alpha,\ \beta,\ \gamma \text{ are functions of } x,\ y;\\ &c(x, y, 0) = c_0(x, y). \end{aligned}$$

The questions on numerical solving for such problems with a non-selfconjugated operator were studied in a number of works (see [1]–[7]). Specific difficulties arise here in the situation where the first derivatives dominate in $L[c]$ (see [2], [5], and [6]).

As a rule, the analysis of stability of difference schemes with a non-selfconjugated operator $A \simeq L[c]$ essentially uses the positivity of A in a certain Hilbert space (HS) (see [3]). For example, the papers [6], [7] treat schemes with matrix A, whose symmetric part $\frac{1}{2}(A+A^*)$ is being positive definite (i.e., there is valid the positivity of A in HS). Generally speaking, we cannot apply this property if A is an M-matrix.

The present article is dealing with stability analysis for implicit two-layer schemes with M-matrix. Sufficient conditions for stability are obtained in uniform network norm. We consider also the schemes of variable directions.

1. Approximation of operator $L[c]$. Properties of difference operator

We choose a uniform mesh $\Omega_h = \{(x_p, y_k)\}$ with steps h_1 (along x) and h_2 (along y). We shall approximate the second derivatives in a standard way with second order approximation, while the first derivatives — by differences "against the stream" with first order approximation. Then the difference equations, which approximate the operator $L[c]$ in every interior point of the mesh domain Ω_h, have the following form:

$$(L[c])_{p,k} \simeq W'_{p,k} c_{p-1,k} + S'_{p,k} c_{p,k-1} + C'_{p,k} c_{p,k} + N'_{p,k} c_{p,k+1} + E'_{p,k} c_{p+1,k};$$

$$W'_{p,k} = -h_1^{-2}(D_1)_{p-1/2,k} - 0.5h_1^{-1}[(v_1)_{p,k} + |v_1|_{p,k}];$$

$$S'_{p,k} = -h_2^{-2}(D_2)_{p,k-1/2} - 0.5h_2^{-1}[(v_2)_{p,k} + |v_2|_{p,k}];$$

$$C'_{p,k} = h_1^{-2}[(D_1)_{p-1/2,k}+(D_1)_{p+1/2,k}]+h_2^{-2}[(D_2)_{p,k-1/2}+(D_2)_{p,k+1/2}] + \qquad (2)$$
$$+ h_1^{-1}|v_1|_{p,k}+h_2^{-1}|v_2|_{p,k}+\lambda_{p,k};$$
$$N'_{p,k} = -h_2^{-2}(D_2)_{p,k+1/2}+0.5h_2^{-1}[(v_2)_{p,k}-|v_2|_{p,k}];$$
$$E'_{p,k} = -h_1^{-2}(D_1)_{p+1/2,k}+0.5h_1^{-1}[(v_1)_{p,k}-|v_1|_{p,k}].$$

Let us assume that, after discretization of certain form, the boundary value conditions of problem (1) will take the view of linear combinations of meanings of unknown function c in a given node of Ω_h abutting the boundary (we denote this meaning by $\hat{c}$) and in every boundary node of its stencil (we denote this meaning by $\breve{c}$):

$$a_\theta\breve{c}+b_\theta\hat{c} = g_\theta. \qquad (3)$$

Here the symbol θ must be replaced by one of symbols W, S, N, or E in dependence on the disposition of boundary nodes of the stencil with respect to given boundary abutting node $\hat{c}$. The meanings a_θ, b_θ, g_θ are defined by coefficients α, β, and γ of the boundary value conditions of problem (1).

By means of difference equations (2) in boundary abutting points of Ω_h and boundary value conditions after discretization (3) we now obtain the difference relations valid for all points Ω_h (excluding the boundary ones):

$$(L[c]-F)^m_{p,k} \simeq W_{p,k}c_{p-1,k}+S_{p,k}c_{p,k-1}+C_{p,k}c_{p,k}+N_{p,k}c_{p,k+1}+E_{p,k}c_{p+1,k}-f^m_{p,k};$$
$$W_{p,k} = (1-\delta_W)W'_{p,k};\quad S_{p,k} = (1-\delta_S)S'_{p,k};\quad N_{p,k} = (1-\delta_N)N'_{p,k};\quad E_{p,k} = (1-\delta_E)E'_{p,k};$$
$$C_{p,k} = C'_{p,k}-\delta_W\frac{b_W}{a_W}W'_{p,k}-\delta_S\frac{b_S}{a_S}S'_{p,k}-\delta_N\frac{b_N}{a_N}N'_{p,k}-\delta_E\frac{b_E}{a_E}E'_{p,k}; \qquad (4)$$
$$f^m_{p,k} = F^m_{p,k}-\delta_W\frac{g_W}{a_W}W'_{p,k}-\delta_S\frac{g_S}{a_S}S'_{p,k}-\delta_N\frac{g_N}{a_N}N'_{p,k}-\delta_E\frac{g_E}{a_E}E'_{p,k}.$$

Here δ_θ $(\theta=W,S,N,E)$ is the function equaling unit at those boundary abutting points of Ω_h, whose stencils have boundary node on the side θ; at all other points of Ω_h it vanishes. The coefficients $W'_{p,k}$, $S'_{p,k}$, $N'_{p,k}$, $E'_{p,k}$, and $C'_{p,k}$ are defined by formulas (2).

Setting intrinsic order of enumeration of points of the mesh domain Ω_h $((p,k) = (1,1),(1;2),\ldots)$, we then obtain

$$L[c]-F \simeq A\bar{\mathbf{c}}-\bar{\mathbf{f}}^m; \qquad (5)$$
$$\bar{\mathbf{c}} = (c_{1,1},c_{1,2},\ldots)\in\mathbb{R}^n,\quad \bar{\mathbf{f}}^m = (f^m_{1,1},f^m_{1,2},\ldots)\in\mathbb{R}^n,$$

where n is the summary number of interior and boundary abutting points of Ω_h, A is a real $n\times n$-matrix with positive elements $C_{p,k}$ standing on its main diagonal. In addition to $C_{p,k}$, every line of A contains no more than four non-trivial non-positive elements: $W_{p,k}$, $S_{p,k}$, $N_{p,k}$ and $E_{p,k}$.

Let $\rho(A)$ denote the spectral radius of the matrix A and I — the unit $n\times n$-matrix. All vector and matrix inequalities are understood as element-by-element ones.

Let us cite necessary definitions.

Definition 1. A matrix A is termed *M-matrix* (see [8], p.85) if $A=sI-B$, where $B\geq 0$, $s>\rho(B)$. In the case where $s=\rho(B)$, the matrix is called *degenerate M-matrix* (see [9]).

Definition 2. A *regular splitting* of matrix A (see [8], p.88) is representation of A in the form $A=P-Q$, where $\det P \neq 0$, $P^{-1} \geq 0$, $Q \geq 0$.

We need also the following propositions.

1. If M-matrix A does not degenerate, then $A^{-1} \geq 0$ (see [8]).
2. If a matrix $A=(a_{ij})$ satisfies conditions

$$a_{ii}>0,\ a_{ij}\leq 0\ (i\neq j),\ i,j=1,\dots,n;\ \sum_j a_{ij}\geq 0,\ i=1,\dots,n, \tag{6}$$

then A is an M-matrix (maybe, degenerate one) [10]. If the matrix A is, in addition, irreducible, and even one of the line sums in (6) is positive, then A is a non-degenerate M-matrix (see [8]).

3. Let A be a matrix from (5), and let boundary value condition (3) satisfy

$$b_\theta/a_\theta \geq -1, \quad \theta = W, S, N, E. \tag{7}$$

Then (see [12]) A is a non-degenerate M-matrix under condition $\lambda|_{\Omega_h} \not\equiv 0$, or under condition that even one of inequalities (7) is strict.

4. If in (1) $\alpha\equiv 0$, $\beta\equiv 1$, $\gamma\equiv 0$, $\lambda\equiv 0$ (i.e., we solve degenerated von Neumann problem), and $a_\theta=-b_\theta$, $g_\theta=0$ in (3), then the matrix A in (5) is a degenerate M-matrix (see [11]) and

$$\operatorname{Ker} A = \{\bar{c}\in\mathbb{C}^n \mid c_1=\dots=c_n\}.$$

5. For any regular splitting $A=P-Q$ of a non-degenerate M-matrix A we have $\rho(P^{-1}Q)<1$ (see [8]). If A is a degenerate M-matrix, then $\rho(P^{-1}Q)\leq 1$ (see [9]).

2. Construction of difference schemes. Stability analysis

We assume in what follows that the matrix A in (5) satisfies conditions (6). The propositions 3), 4) of [10]–[12] supply sufficient conditions for fulfillment of such an assumption. Let us consider that the matrix norm means its maximal line norm, i.e., $\|A\| = \max_i \sum_j |a_{ij}|$, and the vector norm means the norm in $C(\Omega_h)$ (or the l_∞-norm), i.e., $\|\bar{c}\|=\max_i |c_i|$. As it is known, $\|A\bar{c}\| \leq \|A\|\cdot\|\bar{c}\|$ for any real matrix A and $\bar{c}\in\mathbb{R}^n$ (see [13], p.357).

Let problem (1) be approximated by the scheme

$$\frac{\bar{c}^{m+1}-\bar{c}^m}{\tau} + M\bar{c}^{m+1} + N\bar{c}^m = \bar{f}^m, \quad m\geq 0, \tag{8}$$

$$A = M+N, \tag{9}$$

where A ($\simeq L[c]$) is the matrix from (5).

Definition 3. Let the matrices M and N in (9) inherit properties (6) of the matrix A if only M (or N) in (9) is not trivial. Then we call (9) *replicative splitting* of A (and the scheme (8) — *scheme of replicative splitting*).

The canonical notation for scheme (8) is

$$B\frac{\bar{c}^{m+1}-\bar{c}^m}{\tau} + A\bar{c}^m = \bar{f}^m \quad (B = I+\tau M), \quad m\geq 0.$$

Theorem 1. *If the time step satisfies condition*

$$\tau \leq [\max_i n_{ii}]^{-1} \tag{10}$$

(here n_{ii} are elements of main diagonal of N), then the scheme of replicative splitting (8) is stable:

$$\|\bar{\mathbf{c}}^{m+1}\| \leq \|\bar{\mathbf{c}}^{m}\|+\tau\|\bar{\mathbf{f}}^{m}\|, \quad m \geq 0, \tag{11}$$

$$\|S\| \leq 1, \quad S = (I+\tau M)^{-1}(I-\tau N), \tag{12}$$

$$\det A \neq 0 \Longrightarrow \rho(S)<1, \tag{13}$$

and monotone, i.e., for $\bar{\mathbf{c}}^0 \geq 0$ *we have*

$$\bar{\mathbf{f}}^k \geq 0 \ (k \leq m) \Longrightarrow \bar{\mathbf{c}}^m \geq 0, \quad m \geq 0. \tag{14}$$

Proof. We put

$$s = \max_i m_{ii}, \quad B = sI-M, \tag{15}$$

and estimate

$$\|I+\tau M\| = \|(1+\tau s)I-\tau B\| \geq \big|(1+\tau s)\|I\|-\tau\|B\|\big| \geq 1+\tau(s-\|B\|). \tag{16}$$

By virtue of choice of s and fulfillment of condition (6) for M we have $s \geq \|B\|$, and (16) yields

$$\|(I+\tau M)^{-1}\| \leq \|I+\tau M\|^{-1} \leq 1, \quad \tau>0. \tag{17}$$

Then we obtain under restriction (10) that

$$\|S\| \leq \|I-\tau N\| = \max_i\{|1-\tau n_{ii}|+\sum_{j\neq i}|1-\tau n_{ij}|\} = 1-\tau \min_i\{\sum_j n_{ij}\} \leq 1. \tag{18}$$

Thus, implication (12) has been proved. Inequality (11) can be easily obtained from (12) and (16) by means of triangle inequality in relation

$$\bar{\mathbf{c}}^{m+1} = S\bar{\mathbf{c}}^{m}+\tau(I+\tau M)^{-1}\bar{\mathbf{f}}^{m}, \quad m \geq 0. \tag{19}$$

In addition, it obviously follows from (11) that

$$\|\bar{\mathbf{c}}^{m+1}\| = \|\bar{\mathbf{c}}^{0}\|+\tilde{C} \max_{0\leq k \leq m-1} \|\bar{\mathbf{f}}^{k}\|, \quad \tilde{C} = m\tau, \quad m \geq 0.$$

Furthermore, for

$$P = I+\tau M, \quad Q = I-\tau N$$

the representation $P-Q=\tau A$ is a regular splitting (here P is an M-matrix according to propositions 1), 2), and $Q \geq 0$ by virtue of (10)). Consequently, in following 5) we have $\rho(P^{-1}Q)=\rho(S) \leq 1$, and the inequality is strict if $\det A \neq 0$.

Since $P^{-1} \geq 0$ (proposition 1)) and $S=P^{-1}Q \geq 0$, therefore (19) implies monotonicity of the scheme, i.e., inequality (14) and Theorem 1 have been proved.

Remark 1. By means of (16)–(18) we have for $\bar{\mathbf{f}}^m \equiv 0$ that

$$\|\bar{\mathbf{c}}^{m}\| \leq r\|\bar{\mathbf{c}}^{m-1}\|, \quad r = \frac{1-\tau \min_i \sum_j n_{ij}}{1+\tau(s-\|B\|)}, \tag{20}$$

where s and B are defined in (15).

3. The schemes of variable directions in terms of replicative splitting A

We consider the following difference analog of (1):

$$\begin{cases} \dfrac{\bar{\mathbf{c}}^{m+1/p}-\bar{\mathbf{c}}^{m}}{\zeta}+M\bar{\mathbf{c}}^{m+1/p}+N\bar{\mathbf{c}}^{m} = \bar{\mathbf{f}}^{m}; \\ \dfrac{\bar{\mathbf{c}}^{m+1}-\bar{\mathbf{c}}^{m+1/p}}{\xi}+M\bar{\mathbf{c}}^{m+1/p}+N\bar{\mathbf{c}}^{m+1} = \bar{\mathbf{f}}^{m+1/p}, \end{cases} \tag{21}$$

$$\zeta+\xi = \tau, \quad p = \frac{\tau}{\zeta}, \quad m \geq 0,$$

where M and N define replicative splitting of A (see (8)).

Theorem 2. *Let time step* (21) *satisfy the relation*

$$\tau = \zeta+\xi, \quad \zeta \le [\max_i n_{ii}]^{-1}, \quad \xi \le [\max_i m_{ii}]^{-1}, \tag{22}$$

where m_{ii} and n_{ii} are elements of the main diagonals of M and N.

Then scheme (21) *is stable*:

$$\|\bar{\mathbf{c}}^{m+1}\| \le \|\bar{\mathbf{c}}^{m}\| + \tau \max\{\|\bar{\mathbf{f}}^{m}\|, \|\bar{\mathbf{f}}^{m+1/p}\|\}, \quad m \ge 1, \tag{23}$$

$$\|S\| \le 1, \quad S = (I+\xi N)^{-1}(I-\xi N)(I+\zeta M)^{-1}(I-\zeta N), \tag{24}$$

$$\det A \ne 0 \Longrightarrow \rho(S) < 1 \tag{25}$$

(here S is the transfer matrix of (21)) *and monotone, i.e., for $\bar{\mathbf{c}}^0 \ge 0$ we have*

$$\bar{\mathbf{f}}^{k}, \bar{\mathbf{f}}^{k+1/p} \ge 0 \ (k \le m) \Longrightarrow \bar{\mathbf{c}}^{m+1} \ge 0, \quad m \ge 0. \tag{26}$$

Proof. Inequality (24) with regard to restriction (22) follows immediately from (16)-(18) in connection with $(I+\xi N)^{-1}$ ($(I+\zeta M)^{-1}$) and $I-\xi M$ (or $I-\zeta N$). Then we obtain (23) by repetition of the proof of Theorem 1. Let us also note that an obvious consequence of (23) is

$$\|\bar{\mathbf{c}}^{m}\| \le \|\bar{\mathbf{c}}^{0}\| + \tilde{C} \max_{0 \le k \le m-1} \{\|\bar{\mathbf{f}}^{k}\|, \|\bar{\mathbf{f}}^{k+1/p}\|\}, \quad \tilde{C} = m\tau, \quad m \ge 1.$$

In order to prove (25), let us note that (see [3]):

$$(I-\xi M)(I+\zeta M)^{-1} = (I+\zeta M)^{-1}(I-\xi M), \quad \zeta, \xi > 0,$$

and, hence,

$$S = [(I+\zeta M)(I+\xi N)]^{-1}(I-\xi M)(I-\zeta N) = P^{-1}Q,$$

where $P-Q=\tau A$ is a regular splitting, since $\rho(S)=\rho(P^{-1}Q)<1$ for $\det A \ne 0$ (see propositions 1), 5)). The monotonicity (26) of scheme (21) follows from element-wise non-negativity of matrices S, $(I+\xi N)^{-1}$, and $I-\xi M$ (see proposition 1)). Theorem 2 has been proved.

Remark 2. If $\zeta=\xi=\tau/2$ in (21), then under conditions of Theorem 2 restriction (22) turns into the following one

$$\tau \le 2[\max_i d_{ii}]^{-1}, \quad d_{ii} = \max\{m_{ii}, n_{ii}\}.$$

Remark 3. Let $\det A \ne 0$, $F=\mathrm{const}(t)$. Then by virtue of Theorems 1 and 2 the transfer matrices S for schemes (8) and (21) satisfy inequality $\rho(S)<1$. In this case, (8) and (21) are convergent iterative schemes (see [14], p.33) for solving a system of linear algebraic equations

$$A\bar{\mathbf{c}} = \bar{\mathbf{f}},$$

where A and $\bar{\mathbf{f}}$ are defined in (5). In order to bound the rate of convergence, one can apply (20).

4. Examples

Let us cite several examples of replicative splittings of A.

a) Explicit scheme. In (8), (9) we have

$$M = 0, \quad N = A \tag{27}$$

with condition of stability (10)

$$\tau \le [\max_{(p,k)} C_{p,k}]^{-1}, \tag{28}$$

where $C_{p,k}$ are elements of the main diagonal of A (see (4)). In [15], the explicit scheme (8), (27) was considered for solving problem (1) with

$$D_i \equiv 0 \ (i=1,2), \ \lambda \equiv 0, \ \alpha \equiv 1, \ \beta \equiv 0, \ \gamma \equiv 0. \tag{29}$$

In the case where $v_i \equiv \text{const}$ $(i=1,2)$, the authors of [15] by means of Neumann's spectral test of stability [1] obtained the necessary condition for stability

$$\tau \leq [|v_1|/h_1 + |v_2|/h_2]^{-1}, \tag{30}$$

which coincides with our sufficient condition (28). Indeed, due to (29) and (4) we have

$$C_{p,k} = |v_1|_{p,k}/h_1 + |v_2|_{p,k}/h_2. \tag{31}$$

In the case where $v_i \neq \text{const}$ $(i=1,2)$, it was proposed in [15] to change in (30) the $|v_i|$ by $\max\limits_{\Omega}|v_i|$ $(i=1,2)$. By virtue of (31), the result of such a change coincides with the proved above condition (28).

b) Completely implicit scheme. In this case we have in (9)

$$M = A, \quad N = 0,$$

and, as it follows from (18),

$$\|S\| \leq \|I-\tau N\| = 1, \quad \tau > 0,$$

i.e., completely implicit scheme (8) is unconditionally stable.

c) Longitudinal-transversal scheme. Let us put in (9)

$$\begin{aligned} M\bar{\mathbf{c}} &\simeq -(D_1 c_x)_x + v_1 c_x + \lambda c, \\ N\bar{\mathbf{c}} &\simeq -(D_2 c_y)_y + v_2 c_y, \end{aligned} \tag{32}$$

where approximation is done in accordance with (2)-(5). Then, as one can easily verify, relations (6) are fulfilled for M, N and for A simultaneously, i.e., (8), (32) is a replicative splitting. Scheme (8), (31) is rather effective, because for enumeration of points of Ω_h in the order $(p,k)=(1,1),(2,1),\ldots$ (i.e., p changes first) the matrix M is three-diagonal one.

d) Schemes of splitting with respect to physical processes. Let the matrices A_1 and A_2 approximate diffusive and convective terms of $L[c]$, respectively, in accordance with (2)-(5) (see [6]):

$$A_1\bar{\mathbf{c}} \simeq -(D_1 c_x)_x - (D_2 c_y)_y + \lambda c,$$

$$A_2\bar{\mathbf{c}} \simeq v_1 c_x + v_2 c_y.$$

Obviously, the sum A_1+A_2 gives us the matrix A from (5). One can verify that $A=A_1+A_2$ is a replicative splitting (for A_1, A_2, and A, relation (6) fulfills simultaneously). By putting in (9)

$$M = A_1, \quad N = A_2,$$

we obtain diffusion-implicit scheme (8) (beginning from the next layer with respect to time we take differences corresponding to diffusion terms of $L[c]$). Restriction (10) on a time step turns into the following one:

$$\tau \leq [\max_{(p,k)}\{|v_1|_{p,k}/h_1 + |v_2|_{p,k}/h_2\}]^{-1}.$$

Analogously, for $M=A_2$ and $N=A_1$, scheme (8) is a convective-implicit scheme with stability restriction (10) in the form

$$\tau \le \left[\max_{(p,k)}\left\{\frac{(D_1)_{p-1/2,k}+(D_1)_{p+1/2,k}}{h_1^2}+\frac{(D_2)_{p,k-1/2}+(D_2)_{p,k+1/2}}{h_2^2}+\lambda_{p,k}\right\}\right]^{-1}.$$

e) Triangular schemes. We represent A as follows

$$A = L+U, \tag{33}$$

where L and U are the lower and the upper triangular matrices, whose off-diagonal elements coincide with corresponding elements of A. We denote diagonal elements of L (or U) by $L_{p,k}$ ($U_{p,k}$, respectively), $(p,k)\in\Omega_h$. Let

$$\varphi = \max_{(p,k)}\{-W_{p,k}-S_{p,k}\} - \max_{(p,k)}\{-N_{p,k}-E_{p,k}\},$$

$$L_{p,k} = \begin{cases}-W_{p,k}-S_{p,k}, & \varphi<0;\\ C_{p,k}-U_{p,k} & \varphi\ge 0,\end{cases}$$
$$U_{p,k} = \begin{cases}C_{p,k}-L_{p,k}, & \varphi<0;\\ -N_{p,k}-E_{p,k} & \varphi\ge 0,\end{cases} \tag{34}$$

where $\theta_{p,k}$ ($\theta = W,S,C,N,E$) are elements of A from (4). We define splitting (9) as follows

$$N = \begin{cases}L, & \varphi<0;\\ U, & \varphi\ge 0,\end{cases} \qquad M = A-N,$$

what yields a restriction onto step τ of scheme (8) in the form

$$\tau \le [\min\{\max_{(p,k)}\{-W_{p,k}-S_{p,k}\},\ \max\{-N_{p,k}-E_{p,k}\}\}]^{-1}. \tag{35}$$

One can show, that (9), (33), and (34) give the least rigid restriction (35) on the time step of scheme (8) among all the possible triangular replicative splittings of A.

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Rostov State University