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Geometrical Interpretation of the Bilinear Equations for the KP Hierarchy

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Abstract. The Grassmann manifold approach to the KP hierarchy, in the spirit of Segal and Wilson, is used to define the wavefunction ψ_W and its adjoint $\psi_{W^\perp}$. From the fact that ψ_W and $\psi_{W^\perp}$ are the orthogonal, we derive the bilinear equations. The modified equations are treated at the same time.

0. Introduction

In this Letter we give a geometrical interpretation of the bilinear form of the equations of the KP hierarchy and its modified versions. Apart from [1], this formula is derived using generalized Plücker coordinates [2, 3, 4]. (Only [3] contains the modified versions.)

Now one can associate to each subspace W in the Grassmann manifold of $L^2(S^1, \mathbb{C})$ a wavefunction ψ_W which is a solution of the linear system connected to the KP equations. In a natural way, we will link a wavefunction $\psi_{W^\perp}$ to the orthocomplement $W^\perp$ of W. Expressing $\psi_{W^\perp}$ in a τ-function, we prove that the bilinear equations boil down to the orthogonality relations for ψ_{W_1} and $\psi_{W_2^\perp}$ for $W_1 \subset W_2$.

Contrary to [1–4], our framework is analytic.

1. The Grassmann Manifold and Wavefunctions

Let H be $L^2(S^1, \mathbb{C})$. The Fourier series of $f \in H$ we denote by $\Sigma_{i\in\mathbb{Z}}\, a_i \lambda^i$, $a_i \in \mathbb{C}$. For each l in $\mathbb{Z}$, we write

$$H_l = \left\{ \sum_{i \geq l} a_i \lambda^i \in H \right\}.$$

The spaces H_0 and $H_0^\perp$ are also denoted by H_+, resp. H_-, as in [5, 6]. To the decomposition $H = H_+ \oplus H_-$ is associated the Grassmann manifold $\mathrm{Gr}(H)$ of H. It is a homogeneous space of the group $\mathrm{Gl}_{\mathrm{res}}(H)$, which consists of the automorphisms g of H that decompose w.r.t. $H = H_+ \oplus H_-$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \text{ and } d \text{ Fredholm, } b \text{ and } c \text{ Hilbert–Schmidt}.$$

The connected components $\mathrm{Gr}^{(l)}(H)$, $l \in \mathbb{Z}$, are determined by

$$\mathrm{Gr}^{(l)}(H) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}(H_+) \mid \text{index}\,(a) = l\right\} = \mathrm{Gl}^l_{\mathrm{res}} \cdot H_+$$

$$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}(H_l),\ \text{index}\,(a) = 0\right\}.$$

As in [5], one shows that $\mathrm{Gl}^0_{\mathrm{res}}(H)$ acts transitively on each of the $\mathrm{Gr}^{(l)}(H)$.

Let $\Lambda\colon H \to H$ be the 'multiplication by λ'. The group of flows connected to the KP hierarchy is

$$\Gamma_+ = \left\{\exp\left(\sum_{i \geqslant 1} t_i \Lambda^i\right) \in \mathrm{Gl}^0_{\mathrm{res}}(H),\ \sum_{i \geqslant 1} |t_i|\,(1+\varepsilon)^i < \infty \text{ for some } \varepsilon > 0\right\}.$$

For each W_l in $\mathrm{Gr}^{(l)}(H)$, let $\Gamma_+^{W_l}$ be given by

$$\Gamma_+^{W_l} = \{\gamma \in \Gamma_+ \mid \text{orthogonal projection } \gamma^{-1} W_l \to H_l \text{ is a bijection}\}.$$

As in [6], one shows that $\Gamma_+^{W_l}$ is nonempty. To W_l one associates a wavefunction $\psi_{W_l}(\gamma, \zeta)$, where $\gamma \in \Gamma_+^{W_l}$ and $1 < |\zeta| < 1 + \varepsilon$, of the form

$$\psi_{W_l}(\gamma, \zeta) = \zeta^l\left(1 + \sum_{k>0} a_k(\gamma)\zeta^{-k}\right)\exp\left(\sum_{i \geqslant 1} t_i \zeta^i\right)$$

$$= \hat{\psi}_{W_l}(\gamma, \zeta)\exp\left(\sum_{i \geqslant 1} t_i \zeta^i\right)$$

$$= \zeta^l\left\{\sum_{j \in \mathbb{Z}} \alpha_j(\gamma)\zeta^j\right\},$$

which has the property that its L^2-boundary value $\psi_{W_l}(\gamma, \lambda)$, $\lambda \in S^1$, belongs to W_l for all γ in $\Gamma_+^{W_l}$. The wavefunction is uniquely determined by this requirement.

Next, we introduce the adjoint wavefunction of ψ_{W_l}. A direct verification shows that if $W_l \in \mathrm{Gr}^{(l)}(H)$, then the orthogonal projection of $W_l^\perp$ to H_- is a Fredholm operator of index $-l$.

Moreover, we have for each γ in Γ_+ and $W_l \in \mathrm{Gr}^{(l)}(H)$ that $(\gamma^{-1} W_l)^\perp = (\gamma^*)W_l^\perp$, where γ^* is the adjoint of γ. Now we exchange the role of H_+ and H_- and the Grassmann manifold connected to this decomposition of H contains all $W_l^\perp$. On the new Grassmann manifold we consider the adjoint flow-group, i.e.

$$\Gamma_- = \left\{\exp\left(\sum_{i \geqslant 1} s_i \Lambda^{-i}\right) \in \mathrm{Gl}^0_{\mathrm{res}}(H),\ \sum_{i \geqslant 1} |s_i|\,(1+\varepsilon)^i < \infty \text{ for some } \varepsilon > 0\right\}$$

and introduce analogously

$$\Gamma_-^{W_l^\perp} = \{\gamma \in \Gamma_-, \text{ the orthogonal projection } \gamma_1^{-1} W_l^\perp \to H_l^\perp \text{ is a bijection}\}$$

$$= \{(\gamma^*)^{-1}, \gamma \in \Gamma_+^{W_l}\}.$$

The adjoint wavefunction $\psi_{W_l^\perp}(\gamma, \eta)$, with

$$\gamma \in \Gamma_-^{W_l^\perp} \quad \text{and} \quad \frac{1}{1+\varepsilon} < |\eta| < 1,$$

has the form

$$\begin{aligned}\psi_{W_l^\perp}(\gamma, \eta) &= \eta^{l-1}\left(1 + \sum_{k>0} b_k(\gamma)\eta^k\right)\exp\left(\sum_{i\geqslant 1} s_i \eta^{-i}\right)\\ &= \hat{\psi}_{W_l^\perp}(\gamma, \eta)\exp\left(\sum_{i\geqslant 1} s_i\eta^{-i}\right)\\ &= \eta^{l-1}\sum_{j\in\mathbb{Z}} \beta_j(\gamma)\eta^j\end{aligned}$$

and is uniquely determined by the property that the L^2-boundary value $\psi_{W_l^\perp}(\gamma, \lambda)$, $\lambda \in S^1$, belongs to $W_l^\perp$ for all γ in $\Gamma_-^{W_l^\perp}$.

2. The Bilinear Equations

For k and $l \in \mathbb{Z}$, $k \geqslant l$, the Japanese school discussed in [3] the bilinear equations of the (k, l) modified KP hierarchy. We will show that these relations are nothing but the following observation: Assume $W_l \in \mathrm{Gr}^{(l)}(H)$ and $W_k \in \mathrm{Gr}^{(k)}(H)$ are such that $W_l \supseteq W_k$. This happens, e.g., for $W_l = g \cdot H_l$ and $W_k = g \cdot H_k$ with $g \in \mathrm{Gl}^0_{\mathrm{res}}(H)$. Then $W_l^\perp \subset W_k^\perp$ and due to the defining properties of $\psi_{W_l^\perp}$ and ψ_{W_k}, namely $\psi_{W_k}(\gamma, \lambda) \in W_k$ and $\psi_{W_l^\perp}(\gamma, \lambda) \in W_l^\perp$, we have

$$\begin{aligned}\int_{|\lambda|=1} \psi_{W_k}(\gamma, \lambda)\overline{\psi_{W_l^\perp}(\gamma, \lambda)}\,\mathrm{d}\lambda &= 0\\ = \sum_{j\in\mathbb{Z}} \alpha_j(\gamma)\bar{\beta}_{j+1+k-l}(\gamma) &= 0.\end{aligned} \tag{1}$$

We rewrite (1) such that the integrand becomes analytic in λ. For

$$\overline{\psi_{W_l^\perp}(\gamma, \lambda)} = \lambda^{1-l}(1 + \sum \bar{b}_k(\gamma)\lambda^{-k})\exp\left(\sum_{i\geqslant 1} \bar{s}_i\lambda^i\right).$$

Thus (1) becomes

$$\oint \lambda^{k-l}\left(1 + \sum_{i\geqslant 1} a_i(\gamma)\lambda^{-i}\right)\left(1 + \sum_{j\geqslant 1} \bar{b}_j(\gamma)\lambda^{-j}\right)\exp\left(\sum_{i\geqslant 1}(t_i + \bar{s}_i)\lambda^i\right)\mathrm{d}\lambda \tag{2}$$

with $\mathrm{d}\lambda$ such that for analytic f in $1 < |\lambda| < 1 + \varepsilon$

$$\oint f(\lambda)\,\mathrm{d}\lambda = \mathrm{Res}(f)_{\lambda=0}.$$

Next we give the expression of $\hat{\psi}_{W_l}$ and $\hat{\psi}_{W_l^\perp}$ in terms of the τ-function associated to W_l. Substituting this into (2) gives you the formula (2.4) occurring in [3]. Let w_l be a continuous embedding of H_l into H with image $W_l \in \mathrm{Gr}^{(l)}(H)$ and denote its decomposition w.r.t. $H = H_l \oplus H_l^\perp$ by $\binom{(w_l)_+}{(w_l)_-}$. As shown in [5], we may assume that $(w_l)_+$ has the form 'identity + trace-class'. To such an embedding we associate a function τ_{w_l} on Γ_+ by

$$\tau_{w_l}(\gamma) = \tau_{w_l}\left(\exp\left(\sum_{i>0} t_i\Lambda^i\right)\right) := \det((w_l)_+ + a^{-1}b(w_l)_-)$$

where $\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$ is the decomposition of γ^{-1} w.r.t. $H_l \oplus H_l^\perp$. Using the multiplicativity of det, one shows that if the orthogonal projection of $\gamma_1^{-1} W_l$ to H_l is bijective then

$$\frac{\tau_{w_l}(\gamma_1\gamma_2)}{\tau_{w_l}(\gamma_1)} = \tau_{w(l_1\gamma_1)}(\gamma_2) \quad \text{with} \quad w(l_1\gamma_1) = \begin{pmatrix} \mathrm{Id} \\ (\gamma^{-1}{}_1 \circ w_l)_- \circ (\gamma_1^{-1} \circ w_l)_+^{-1}\end{pmatrix}. \tag{3}$$

For $|\zeta| > 1$, let q_ζ be

$$1 - \frac{\Lambda}{\zeta} = \exp\left(-\sum_{i\geq 1} \frac{1}{i\zeta^i}\Lambda^i\right) \in \Gamma_+ .$$

Let $\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$ be its decomposition w.r.t. $H_l \oplus H_l^\perp$. Then a direct computation shows that b is given by: for all $t < l$

$$\lambda^t \mapsto \zeta^t \cdot \sum_{k=l}^{\infty} \left(\frac{\lambda}{\zeta}\right)^k .$$

Hence, $a^{-1}b$ maps $\Sigma_{t<l} a_t\lambda^t$ to $\{\Sigma_{t<l} a_t\zeta^t\} \cdot \lambda^l/\zeta^l$. Now we apply formula (3) with $\gamma_1 \in \Gamma_+$ and $\gamma_2 = q_\zeta$. Denote for simplicity $(\gamma_1^{-1} \circ w_l)_- \circ (\gamma_1^{-1} \circ w_l)_+^{-1}$ by w_-. By definition, we know that

$$\hat{\psi}_{W_l}(\gamma_1, \lambda) = \lambda^l + w_-(\lambda^l) = \lambda^l + \sum_{k<l} w_{kl}(\gamma_1)\lambda^k .$$

Now $\tau_w(q_\zeta) = 1 + \mathrm{Trace}(a^{-1}bw_-) = 1 +$ coefficient of λ^l in $(a^{-1bw_-})\lambda^l = 1 + \zeta^{-l}\Sigma_{k<l} w_{kl}(\gamma_1)\zeta^k$. Thus, we get

$$\hat{\psi}_{W_l}(\gamma_1, \zeta) = \zeta^l \cdot \frac{\tau_{w_l}(\gamma_1 q_\zeta)}{\tau_{w_l}(\gamma_1)} = \zeta^l \cdot \frac{\tau_{w_l}\left(\exp\left(\Sigma\left(t_i - \frac{1}{i}\zeta^{-i}\right)\Lambda^i\right)\right)}{\tau_{w_l}(\exp(\Sigma\, t_i\Lambda^i))} .$$

The formula (2.4) in [3] already indicates the direction we have to look for in the case of $\hat{\psi}_{W_l^\perp}$: instead of q_ζ consider $(q_\zeta)^{-1}$. Therefore, we start with the computation of $\tau_w(q_\zeta^{-1})$ with $w = \binom{\mathrm{Id}}{w_-}$. Let $\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$ be the decomposition of q_ζ w.r.t. $H = H_l \oplus H_l^\perp$. Then b is given by

$$\sum_{j<\lambda} c_j\lambda^j \mapsto -c_{l-1} \cdot \frac{1}{\zeta} \cdot \lambda^l$$

and $a^{-1}b$ by

$$\sum c_j \lambda^j \mapsto \frac{-c_{l-1}}{\zeta} \cdot \sum_{k=0}^{\infty} \frac{\lambda^{k+l}}{\zeta^k} = \frac{-c_{l-1} \cdot \zeta^l}{\zeta} \sum_{m \geqslant l} \frac{\lambda^m}{\zeta^m} .$$

In particular, its image is one-dimensional and, therefore,

$$\det(\mathrm{Id} + a^{-1}bw_-) = 1 + \mathrm{Trace}(a^{-1}bw_-) .$$

Now let (w_{sm}), $s < 1$ and $m \geqslant l$, be the matrix coefficients of w_- w.r.t. the basis $\{\lambda^j\}$, i.e.

$$w_-(\lambda^m) = \sum_{s<l} w_{sm} \lambda^s , \quad m \geqslant l .$$

Then

$$a^{-1}bw_-(\lambda^m) = -w_{l-1m} \zeta^{l-1} \sum_{k \geqslant l} \frac{\lambda^k}{\zeta^k} .$$

Hence,

$$\mathrm{Trace}(a^{-1}bw_-) = \left\{ \sum_{m \geqslant l} - w_{l-1m} \zeta^{-m} \right\} \zeta^{l-1}$$

and

$$\tau_w(q_\zeta^{-1}) = 1 + \left\{ \sum_{m \geqslant l} - w_{l-1m} \zeta^{-m} \right\} \zeta^{l-1} .$$

If $\gamma \in \Gamma_+^{W_l}$, then $\gamma = (\gamma^*)^{-1} \in \Gamma_-^{W_l^\perp}$. Let w_- be $(\gamma^{-1} \circ w_l)_- \circ (\gamma^{-1} \circ w_l)_+^{-1}$ then one verifies directly that $\binom{-(w_-)^*}{\mathrm{Id}}$ is an embedding of $H_l^\perp$ into H with image $\gamma^* W_l^\perp$. Hence, by definition

$$\psi_{W_l^\perp}(\gamma, \lambda) = \lambda^{l-1} + \{(-w_-)^*\}(\lambda^{l-1}) .$$

Let (w_{sm}) be as above then the (m, s)th matrix coefficient of $(-w_-)^*$ is $-\overline{w}_{sm}$. In particular, we have

$$(-w_-)^*(\lambda^{l-1}) = \sum_{m \geqslant l} -\overline{w}_{l-1m} \lambda^m .$$

Therefore,

$$\psi_{W_l^\perp}(\gamma, \eta) = \eta^{l-1} + \sum_{m \geqslant l} - \overline{w}_{l-1m}(\gamma) \eta^m .$$

Hence, in the integrand of (2), we get

$$\zeta^{1-l} + \sum_{m \geqslant l} - w_{l-1m}(\gamma) \zeta^{-m} = \zeta^{1-l} \tau_w(q_\zeta^{-1})$$

$$= \zeta^{1-l} \frac{\tau_w(\gamma q_\zeta^{-1})}{\tau_w(\gamma)} = \zeta^{1-l} \frac{\tau_w\left(\left(t_i + \frac{1}{i} \zeta^{-1}\right)\right)}{\tau_w(\gamma)} .$$

Collecting all the expressions computed above, we have exactly formula $(2.4)_{k,l}$ in [3]. Indeed, if $\gamma' = \exp(\Sigma\, t_i' \Lambda^i) \in \Gamma_+^{W_l}$, then $(\gamma'^*)^{-1} = \exp(-\Sigma\, t_i' \Lambda^{-i})$ $(= \exp(\Sigma\, s_i \Lambda^{-i})$ above), and (2) becomes, after multiplication by $\tau_{w_l}(\gamma) \cdot \tau_{w_l}(\gamma')$,

$$\oint \lambda^{k-l} \tau_{w_k}\left(\left(t_i - \frac{1}{i}\,\lambda^{-i}\right)\right) \tau_{w_l}\left(\left(t_i' + \frac{1}{i}\,\lambda^{-i}\right)\right) \exp\left(\sum_{j \geqslant 1} (t_j - t_j')\lambda^j\right) \mathrm{d}\lambda = 0\,,$$

$$(k, l \in \mathbb{Z}, k \geqslant l)\,.$$

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