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ON THE STRUCTURE OF SHOCK WAVES IN LIQUID-BUBBLE MIXTURES

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The structure of shock waves in liquids containing gas bubbles is investigated theoretically. The mechanisms taken into account are the steepening of compression waves in the mixture by convection and the effects due to the motion of the bubbles with respect to the surrounding fluid. This relative motion, radial and translational, gives rise to dissipation and to dispersion caused by the inertia of the radial flow associated with an expanding or compressed bubble. For not too thick shocks the dissipation by radial motion around the bubbles dominates over the dissipation by relative translational motion, in mixtures with low gas content. The overall thickness of the shock appears to be determined by the dispersion effect. Dissipation, however, is necessary to permit a steady shock wave. It is shown that, analogous to undular bores, a stationary wave train may exist behind the shock wave.

§ 1. Introduction

A mixture of liquid and small gas bubbles is a fluid which derives density mainly from the liquid content and compressibility from the gas content. For many purposes the dynamics of such a fluid can be described by considering this as an homogeneous one with an appropriate equation of state. Results from the dynamics of single phase gases can be applied then which suggests, among other phenomena, the existence of shock waves in these mixtures. Indeed, shock waves have been observed experimentally by various investigators. Campbell and Pitcher [1] reported measurements and photographs of shock waves in bubble-liquid mixtures and also established Hugoniot relations, i.e. relations that express conservation of mass and momentum across the shock. In ordinary gas-dynamics also energy is conserved across the shock. With shock waves in liquid-bubble mixtures the changes are almost exactly isothermal and then the conservation of mass and momentum is sufficient to express quantities behind the shock in terms of quantities in front of the shock.

In [1] the pressure ratio p_2/p_1 across the shock, the subscripts 1 and 2 referring to the regions in front of and behind the shock, is (by means of the Hugoniot relations) found to be related to the Mach number M_1 by

$$\frac{p_2}{p_1} = M_1^2. \tag{1.1}$$

By the Mach number the ratio speed of shock/speed of sound is indicated. The experimental results show good agreement with this relation. Just as in ordinary gasdynamics nothing can be said about the mechanism of the shock wave so far. It is the purpose of this paper to attempt an analysis of what happens in the shock wave and to predict the thickness of the shock wave. In ordinary gases the thickness of shock waves is of the order of the mean molecular free path, since this is the scale on which steepening of a compression wave by convection may be balanced by viscous diffusion. For a full account of this see Lighthill [2]. For a mixture of liquid and bubbles it is evident that length scales associated with the presence of the bubbles will determine the shock thickness. The homogeneous flow theory ignores the bubble character of the gas content and therefore cannot give insight in the structure of the shock wave. The motion of the bubbles with respect to the surrounding fluid gives rise to interesting mechanical effects. We will first discuss these effects briefly and subsequently investigate their importance for the generation of shock waves.

§ 2. Effects due to radial motion

When a bubble of radius R is immersed in an incompressible fluid with density ρ_f and viscosity μ, the radial motion in the fluid due to spherical expansion of the bubble can be described by the velocity potential

$$\phi = -\frac{R^2}{r^2}\frac{dR}{dt}, \tag{2.1}$$

where r is the distance from the center.

The velocity $v_r = \partial\phi/\partial r$ equals $\mathrm{d}R/\mathrm{d}t$ at $r = R$ and vanishes at infinity. Since $\nabla^2 v_r = 0$ Bernoulli's theorem is applicable. For the pressure p_g of the gas inside the bubble this yields, if we require the continuity of stresses at the bubble fluid interface,

$$p_g - p_\infty = \rho_f \left\{ R \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} + \frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 \right\} + \frac{4\mu}{R}\frac{\mathrm{d}R}{\mathrm{d}t}. \tag{2.2}$$

In this expression p_∞ is the pressure far from the bubble. According to (2.2) a difference in pressure between the gas in the bubble and the fluid at some distance is due to the inertia of the fluid (the first two terms on the right hand side of (2.2)) and viscous stress (the last term). In a dilute suspension of bubbles in water, say, we may replace p_∞ by the average pressure in the mixture at the location of the bubble. The effect of the inertia terms on the propagation of pressure waves through a bubble-liquid mixture has been investigated in some detail in [3]. It is shown there that the inertia effect results, incorporated in the dynamics of the mixture, in equations similar to the Boussinesq equations and (for waves in one direction only) the Korteweg-de Vries equation for long water waves. The only type of uniform waves that satisfy the Korteweg-de Vries equation are the so called conoidal waves. There is no solution of the type of a shock wave, i.e. a transition from one constant level to a different one. The addition of the viscous term in the r.h.s. makes, as we shall show, a transition of this kind possible.

§ 3. Relative translational motion

The importance of dissipation associated with translational motion of the bubble with respect to the fluid was, for pressure waves in a mixture, emphasized by Batchelor [4]. For moderately high Reynolds numbers the relative motion of a bubble can be determined from the irrotational inviscid flow around the bubble. This is a good approximation because the gas in the bubble is free to move about and therefore there is no constraint on the tangential velocity of the fluid at the bubble surface. Hence there is no velocity boundary layer. The dissipation takes place in the irrotational flow outside the bubble and can easily be calculated. Equating the dissipation rate to a frictional force times the relative velocity gives (Levich [5]) for the frictional force

$$F = 12\pi\mu R(v - u), \tag{3.1}$$

where v is the velocity of the bubble and u of the surrounding fluid. Eq. (3.1) holds when effects of surface active agents, which may lead to behaviour of the gas bubble as a solid, can be left out of account. According to [5], p. 448, there is support for this in the case of a bubble-liquid mixture. When (3.1) is adopted as resistance law the relative velocity $v - u$ for given u can be calculated. Neglecting the mass of the bubble we have, V being the volume of the bubble,

$$\tfrac{1}{2}\rho_f V \frac{\mathrm{d}}{\mathrm{d}t}(v-u) + 12\pi R\mu(v-u) = -V\frac{\mathrm{d}p}{\mathrm{d}x},$$

where $\mathrm{d}p/\mathrm{d}x$ is the pressure gradient in the fluid. With a small volumetric gas content the equation of motion for the fluid is

$$\rho_f \frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\mathrm{d}p}{\mathrm{d}x},$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(v-u) + \frac{24\pi R\nu}{V}(v-u) = 2\frac{\mathrm{d}u}{\mathrm{d}t}.$$

The kinematic viscosity μ/ρ is denoted by ν. For given $\mathrm{d}u/\mathrm{d}t$ it follows that

$$v - u = 2\int_{-\infty}^{t} \frac{\mathrm{d}u}{\mathrm{d}\tau} \exp\left\{-\frac{18\nu}{R^2}(t-\tau)\right\}\mathrm{d}\tau. \tag{3.2}$$

In a shock wave the velocity u is related (see section 4) to the volumetric gas content β and the shock velocity U by

$$u = U(\beta - \beta_1),$$

where β_1 is the gas content far in front of the shock. For $\beta \ll 1$, u is small with respect to U and we may write $U\,\mathrm{d}/\mathrm{d}x$ for $\mathrm{d}/\mathrm{d}t$, resulting in

$$v - u = 2\int_{-\infty}^{x} \frac{\mathrm{d}u}{\mathrm{d}\xi} \exp\left\{-\frac{18\nu}{UR^2}(x-\xi)\right\}\mathrm{d}\xi. \tag{3.3}$$

Batchelor [4] shows that for a pressure wave of frequency Ω, velocity of propagation a, with bubbles of radius R and volumetric gas content β, the ratio of the dissipation due to radial and translational bubble motion is of order Ω^2R^2/β^2a^2 or R^2/β^2l^2, where l is the wave length.

Even for small β, 10^{-2} say, this is not a large quantity for most sound waves. For shock waves it may be shown that the ratio of dissipation due to radial motion and to translational motion is of order R^2/β^2d^2, where d is the shock thickness. This can be confirmed by calculation of the dissipation with help of the expression for the viscous stress in (2.2) and the frictional resistance in (3.2) but also follows from replacing the wave length by shock thickness in Batchelor's result. The effect of radial motion is dominant for $d \ll R/\beta$. A lower bound for the shock thickness can be estimated in the following way. During the passage of a shock wave a bubble is reduced from radius R to a smaller radius. The time necessary for such a change is of order ω_B^{-1}, where ω_B is the resonance bubble frequency given by

$$\omega_B = \frac{1}{R}\left(\frac{3p}{\rho_f}\right)^{\frac{1}{2}}. \tag{3.4}$$

On the other hand the time of passage of a shock with thickness d is of order d/c, where c is the sound velocity in the mixture given (see e.g. [3]) by

$$c^2 = \frac{p}{\rho_f(1-\beta)\,\beta}. \tag{3.5}$$

Requiring that d/c is at least equal to ω_B^{-1} leads to

$$d > \frac{R}{\beta^{\frac{1}{2}}}. \tag{3.6}$$

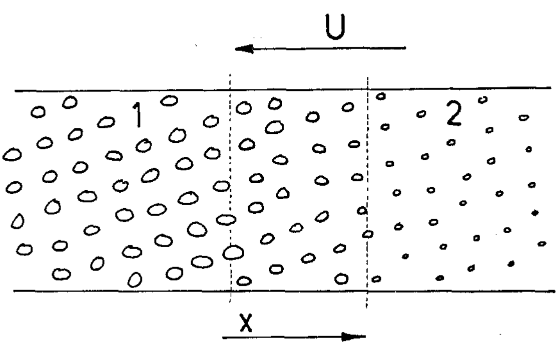

Fig. 1. Shock wave in bubble-liquid mixture, travelling with speed U from $x = \infty$ (side 2) to $x = -\infty$ (side 1).

This estimate of the shock thickness suggests that for $\beta \ll 1$, to which we shall restrict ourselves, the condition $d \ll R/\beta$ will be satisfied, in which case dissipation by radial motion dominates over dissipation by translational motion. We shall therefore in the following neglect in an early stage the effect of translational motion and after results for the shock thickness have been obtained inspect whether indeed this neglect can be justified.

§ 4. Equations for a stationary shock wave

We consider the situation sketched in Fig. 1. A shock wave passes with velocity U from right to left through a bubble-liquid mixture. The conditions far in front of the shock are denoted with 1, far behind with 2. The local density ρ is defined by

$$\rho = \rho_f(1 - \beta), \qquad (4.1)$$

when we neglect the density of the gas.

As before ρ_f is the density of the fluid, regarded as incompressible, and β is the volumetric gas content, related with the number density n of the bubbles and their radius R by

$$\beta = \tfrac{4}{3}\pi n R^3. \qquad (4.2)$$

The mixture in front of the shock is at rest. We denote the local velocity in the mixture with u and the pressure (as before) with p. This pressure is the average pressure in a small volume element. In the homogeneous fluid approach p equals the pressure p_g in the bubbles. As soon as the bubble character of the gas content is allowed for (see section 2) p_g differs from p. Mass conservation requires

$$\rho_f(1 - \beta)(u + U) = \rho_f(1 - \beta_1)\, U,$$

which for $\beta \ll 1$ reduces to

$$u = U(\beta - \beta_1). \qquad (4.3)$$

This shows that u is small with respect to U, so that for the practically important cases of small β (usually in experiments β does not exceed a few percent) some simplifications may be carried through. One of these is that we may consider the number density n as a constant. Since the mass of each bubble remains constant, there is conservation of the number of bubbles passing through a unit surface normal to the flow. When v denotes the velocity of the bubble,

$$(v + U)\, n = \text{constant.}$$

Because v is of the same order of magnitude as u and u is small with respect to U, n is approximately constant,

$$n = \text{constant.} \tag{4.4}$$

Next we formulate the conservation of momentum in a frame moving with the shock. Here we consider the momentum of the mixture $\rho_f(1 - \beta)(u + U)^2$ and the Kelvin impulse (Lamb [6], § 119) associated with the motion of the bubbles with respect to the fluid. For one bubble in a fluid the Kelvin impulse equals the product of the virtual or added mass times the relative velocity. The rate of change of the Kelvin impulse equals the external force on the bubble. In the mixture the number of bubbles passing through unit surface normal to the flow direction is $n(U + v)$, which is approximately nU because $v \ll U$. The virtual mass is $\frac{1}{2}\rho_f V$. Taking into account both the momentum of the fluid and the Kelvin impulse we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\rho_f(1 - \beta)(U + u)^2 + \tfrac{2}{3}\pi n U R^3 \rho_f(v - u) + p\} = 0. \tag{4.5}$$

For $\beta \ll 1$ this reduces with help of (3.3) and (4.3) to

$$p = p_1 + \rho_f U^2 \cdot \left\{(\beta_1 - \beta) - \beta \int_{-\infty}^{x} \frac{\mathrm{d}(\beta - \beta_1)}{\mathrm{d}\xi} \exp\left\{-\frac{18\nu}{UR^2}(x - \xi)\right\} \mathrm{d}\xi\right\}. \tag{4.6}$$

The relation (3.3) holds for constant R and cannot be used in (4.5), strictly speaking. However, taking R constant will introduce a small error only, since normally the change in R over the shock wave is small. An equation for β is obtained by expressing p in terms of the pressure p_g in the gas and the rate of change of the bubble radius by means of (2.2). Since it is known that bubbles in an oscillating pressure field behave isothermally in a wide range of frequencies (see [7]) we take

$$p_g = \frac{p_1 \beta_1}{\beta}, \tag{4.7}$$

which expresses isothermal changes provided (4.4) holds. By using (4.2) and (4.4) and by approximating as before $\mathrm{d}/\mathrm{d}t$ by $U\,\mathrm{d}/\mathrm{d}x$ the viscous term in (2.2) becomes $\frac{4}{3}\mu\beta^{-}U\,\mathrm{d}\beta/\mathrm{d}x$.

For clarity we first disregard the inertia terms in (2.2) and write using (4.7)

$$p = \frac{p_1\beta_1}{\beta} - \frac{4}{3}\,\frac{\mu U}{\beta}\,\frac{\mathrm{d}\beta}{\mathrm{d}x}.$$

Inserting this in (4.6) yields

$$\frac{p_1\beta_1}{\beta} - \frac{4}{3}\,\frac{\mu U}{\beta}\,\frac{\mathrm{d}\beta}{\mathrm{d}x} =$$

$$= p_1 + \rho_f U^2\left\{(\beta_1-\beta) - \beta\int\limits_{-\infty}^{x}\frac{\mathrm{d}(\beta-\beta_1)}{\mathrm{d}\xi}\exp\left\{-\sigma\left(\frac{x-\xi}{R}\right)\right\}\mathrm{d}\xi,\right.$$

where (4.8)

$$\sigma = \frac{18\nu}{UR}. \tag{4.9}$$

Using the expression (3.5) for the velocity of sound this can be written as

$$c_1^2(\beta_1-\beta)\left\{\frac{\beta_1}{\beta} - \frac{U^2}{c_1^2}\right\} +$$

$$+ U^2\beta\int\limits_{-\infty}^{x}\frac{\mathrm{d}(\beta-\beta_1)}{\mathrm{d}\xi}\exp\left\{-\sigma\left(\frac{x-\xi}{R}\right)\right\}\mathrm{d}\xi = \frac{4}{3}\,\frac{\nu U}{\beta}\,\frac{\mathrm{d}\beta}{\mathrm{d}x}. \tag{4.10}$$

Taking in (4.6) $x \to \infty$ gives the relation between conditions far upstream and downstream from the shock. Using (3.5) it then follows that

$$\frac{\beta_1}{\beta_2} = \frac{U^2}{c_1^2},$$

a relation also found by Campbell and Pitcher [1]. Introducing this into (4.10) gives

$$(\beta_1-\beta)(\beta-\beta_2) = -\frac{4}{3}\,\frac{\nu}{U}\,\frac{\mathrm{d}\beta}{\mathrm{d}x} +$$

$$+ \beta^2\int\limits_{-\infty}^{x}\frac{\mathrm{d}(\beta-\beta_1)}{\mathrm{d}\xi}\exp\left\{-\sigma\left(\frac{x-\xi}{R}\right)\right\}\mathrm{d}\xi. \tag{4.11}$$

Upon partial integration of the last term we finally obtain

$$(\beta_1 - \beta)(\beta - \beta_2) = -\frac{4}{3}\frac{\nu}{U}\frac{\mathrm{d}\beta}{\mathrm{d}x} + \beta^2(\beta - \beta_1) +$$

$$+ \frac{\sigma}{R}\beta^2 \int_{-\infty}^{x} (\beta - \beta_1) \exp\left\{-\sigma\left(\frac{x-\xi}{R}\right)\right\} \mathrm{d}\xi. \qquad (4.12)$$

This equation enables us to estimate the influence of the relative translational motion. For $\beta \ll \beta_1$, the term $\beta^2(\beta - \beta_1)$, associated with the spreading of momentum due to relative translational motion, is small with respect to the left hand side of (4.12) and can therefore be neglected. The last term on the right hand side of (4.12) is with a shock thickness d (which is an unknown quantity as yet) of the order of magnitude $\sigma\beta^3 d/R$, whereas the viscous term arising from the radial motion is of order $\nu\beta/Ud$. Using (4.9) for σ shows that the ratio of the effects of radial and translational motion is of order $R^2/\beta^2 d^2$, confirming the estimate made in section 3. We shall assume that for shock waves in mixtures of small β this quantity is large enough to permit the neglect of the terms associated with the translational motion and inspect afterwards the validity of this. Before including the inertia effects of the radial motion in the analysis we consider (4.12) with the last two terms on the right hand side discarded,

$$(\beta_1 - \beta)(\beta - \beta_2) = -\frac{4}{3}\frac{\nu}{U}\frac{\mathrm{d}\beta}{\mathrm{d}x}. \qquad (4.13)$$

Apart from the restriction $\beta \ll 1$ this equation would exactly describe the shock structure when this is determined by viscous diffusion due to radial motion of the fluid in the vicinity of the bubbles. Equation (4.13) can be solved exactly. The solution for β tending to β_1 for $x \to -\infty$ and to β_2 for $x \to +\infty$ is

$$\beta = \frac{\beta_1 + \beta_2}{2} - \frac{\beta_1 - \beta_2}{2} \tanh\left\{\frac{3U\beta_1\left(1 - \dfrac{p_1}{p_2}\right)x}{8\nu}\right\}. \qquad (4.14)$$

The argument of the tanh in (4.14) can be written in various ways. Here we have chosen quantities that are measured during experiments. Campbell and Pitcher [1] give a photograph and a pressure recording of a shock under the conditions: $p_1/p_2 = \frac{1}{6}$, $\beta_1 = 5 \times 10^{-2}$. They do not report the value of p_1. However, since the shock was created by first evacuating the space above a mixture and subsequently admitting atmospheric pressure, we may assume that $p_2 = 10^5$ N/m^2. The fluid used in [1] was an aqueous solution of glycerine with density 1.145×10^3 kg/m^3. To obtain the value of the kinematic viscosity ν, not reported by Campbell and Pitcher, we first note that since a 100% solution of glycerine has density 1.266 kg/m^3 the concentration used in [1] was apparently

$$\frac{0.145}{0.266} = 55\%.$$

Measurement of viscosity as a function of concentration at 20°C gave, for 55% concentration, $\nu = 1.2 \times 10^{-5}$ m^2/s. Unfortunately, in [1] the temperature was not reported, so some uncertainty remains. Taking the mentioned values for the pertinent quantities we obtain from (3.5) $c_1 = 17$ m/s and from (1.1) $U = 41$ m/s. Further we define the thickness of the shock as twice the value of x for which the tanh in (4.14) assumes the value 0.99. The argument of the tanh is about 3 in that case. The thickness of the shock wave calculated in this way from (4.14) turns out to be 1.1×10^{-4} m. The bubble size in the experiments in [1] is 10^{-4} m. Judged from the photograph in [1] the shock wave covers some bubble diameters and therefore our calculated value is too small. Comparison with the lower limit for the shock thickness obtained in (3.6) shows that the mechanism incorporated in (4.13) leads to too small values for the shock thickness The estimate of (3.6) $R/\beta^{\frac{1}{2}}$ amounts to a thickness of about 2 bubble diameters for the bubble diameter of 10^{-4} m and $\beta = 5 \times 10^{-2}$.

§ 5. Shock wave with inertia effects around bubbles

We proceed now to incorporate the inertia terms of (2.2) in the analysis. These terms are highly nonlinear and this makes any progress difficult. Linearization however will be permitted for weak shocks and also for the outskirts of stronger shocks since there the local value of β differs only to a small amount from the asymptotic values β_1 and β_2. The linearized form of the term $\rho_f R\, d^2R/dt^2$ is $(\rho_f \bar{R}^2 U^2/3\bar{\beta})\, d^2\beta/dx^2$, where the values of R and β with respect to which linearization is carried out are indicated with a bar. The term

$\rho_f(dR/dt)^2$ is neglected in a linear approximation. The other terms being as before, the form of (2.2) appropriate for our purpose is

$$p = \frac{p_1\beta_1}{\beta} - \frac{\rho_f\bar{R}^2U^2}{3\bar{\beta}}\frac{d^2\beta}{dx^2} - \frac{4}{3}\frac{\mu U}{\bar{\beta}}\frac{d\beta}{dx}. \tag{5.1}$$

Inserting this into (4.6) and neglecting the effect of relative translational motion gives after manipulations analogous to those leading from (4.6) to (4.12),

$$(\beta_1 - \beta)(\beta_2 - \beta) = \frac{\bar{R}^2}{3}\frac{d^2\beta}{dx^2} + \frac{4}{3}\frac{\nu}{U}\frac{d\beta}{dx}. \tag{5.2}$$

Compared with (4.12), including inertia effects leads to the additional term $\bar{R}^2/3\, d^2\beta/dx^2$. It must be borne in mind, however, that (5.2) involves a linear approximation, whereas (4.12) is only subject to the condition $\beta \ll 1$. If we leave out the viscous term in (5.2) and differentiate with respect to x we obtain

$$\frac{\bar{R}^2}{3}\frac{d^3\beta}{dx^3} + (\beta_1 + \beta_2)\frac{d\beta}{dx} - \frac{d}{dx}(\tfrac{1}{2}\beta^2) = 0. \tag{5.3}$$

This equation has a form to which the Korteweg-de Vries equation, discussed in [3] in the context of pressure waves in a bubble-liquid mixture, reduces when solutions representing stationary waves are assumed. It is known that this equation has no shock wave type solution. Neglecting in (5.2) the inertia term (first term on right hand side) results as we have shown in section 4 in a transition of the required kind, though with too small a shock thickness. Equation (5.2) combines the effects of inertia and viscosity in the structure of the shock [11]. In the central part $d^2\beta/dx^2$ is negligibly small and there the viscosity term will be dominant. Away from the center the influence of the inertia, or dispersion term (because of its importance for dispersion of pressure waves) will be important. Since (5.2) allows no exact solution we shall estimate the shock thickness from analyzing the outskirts of the wave. The referee of the present paper proposed to find the shock thickness by estimating the maximum value of $d\beta/dx$ from a graphical solution of (5.2) in the β, $d\beta/dx$ plane, by means of the method of isoclines. However it turns out that this gives not a good approximation to the shock thickness, because only over a very short part of the interval $\beta_1 - \beta_2$, $d\beta/dx$ is near $|d\beta/dx|_{max}$.

First we consider the low pressure region where β is near β_1. Introducing

$$y = \beta_1 - \beta$$

and linearizing the convection term on the left hand side of (5.2) with respect to y we obtain

$$\frac{R_1^2}{3}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{4}{3}\,\frac{\nu}{U}\,\frac{\mathrm{d}y}{\mathrm{d}x} - (\beta_1 - \beta_2)\,y = 0. \tag{5.4}$$

This equation describes the balance of dispersion, viscous diffusion, and convection at the low pressure side of the shock. The solution which tends to zero for $x \to -\infty$ is

$$y \sim \exp\left[-\frac{2\nu x}{UR_1^2} + x\left\{\left(\frac{2\nu}{UR_1^2}\right)^2 + 3\left(\frac{\beta_1 - \beta_2}{R_1^2}\right)\right\}^{\frac{1}{2}}\right]. \tag{5.5}$$

For the experiment of Campbell and Pitcher [1] $2\nu/UR_1^2 = O(10^2)$ and $\{3(\beta_1 - \beta_2)/R_1^2\}^{\frac{1}{2}} = O(10^3)$. This means that in this particular experiment the slope of the front part of the wave is determined mainly by the balance of convection and dispersion. Just as in the case of bores some viscous dissipation is needed to permit the existence of a stationary shock, because without dissipation no shock exists (5.3). The width of this part of the shock is of order $R_1/(\beta_1 - \beta_2)^{\frac{1}{2}}$, which amounts to 10^{-3} m or about 10 bubble radii. It follows further from (5.4) that for very weak shocks for which

$$\left(\frac{\beta_1 - \beta_2}{R_1^2}\right)^{\frac{1}{2}} \ll \frac{\nu}{UR_1^2}$$

viscosity dominates the slope of the front part. In that case we find by expansion of the square root in (5.4)

$$y \sim \exp\frac{3Ux(\beta_1 - \beta_2)}{4\nu}, \tag{5.6}$$

corresponding to the result earlier obtained in analyzing the shock without dispersion effects (4.13). So we obtain the interesting result that, unless the shock is very weak, its front is determined by the balance between convection and dispersion. Next we investigate the high pressure side were β is near β_2. Introducing

$$\bar{y} = \beta - \beta_2,$$

we obtain here, analogous to (5.4),

$$\frac{R_2^2}{3}\frac{d^2\bar{y}}{dx^2}+\frac{4\nu}{3U}\frac{d\bar{y}}{dx}+(\beta_1-\beta_2)\,y=0. \tag{5.7}$$

The solution of (5.7) which tends to zero for $x \to \infty$ is

$$\bar{y}\sim\exp\left[-\frac{2\nu x}{UR_2^2}+x\left\{\left(\frac{2\nu}{UR_2^2}\right)^2-\frac{3(\beta_1-\beta_2)}{R_2^2}\right\}^{\frac{1}{2}}\right]. \tag{5.8}$$

This expression shows some interesting features of the high pressure part of the shock wave.

For $\nu/UR_2 \gg (\beta_1-\beta_2)^{\frac{1}{2}}$ (5.8) reduces to

$$\bar{y}\sim\exp-\frac{3Ux(\beta_1-\beta_2)}{4\nu}.$$

The behaviour is like (5.6) and the corresponding shock thickness is as calculated in section 4. However when

$$\frac{2\nu}{UR_2}<\{3(\beta_1-\beta_2)\}^{\frac{1}{2}},$$

the expression between { } in (5.8) is negative and in consequence of this waves appear at this side of the shock. The wave length is given by

$$\lambda=\frac{2\pi R_2}{\left\{3(\beta_1-\beta_2)-\left(\frac{2\nu}{UR_2}\right)^2\right\}^{\frac{1}{2}}}. \tag{5.9}$$

The damping of the waves is expressed by the factor $\exp\{-2\nu x/UR_2^2\}$. Apparently stationary waves can exist at the rear part of the shock. For order of magnitude purposes we may, concerning the experiments in [1], disregard the difference between R_1 and R_2, (their quotient is $6^{\frac{1}{3}}$). Then it follows from the values of the various quantities given in the discussion on the front part that waves with a length of order 10^{-3} m and damped in a distance 10^{-2} m may be expected. Indeed, the pressure recording in [1] shows this type of undulations. Of course, on the basis of one recording no definite conclusions can be drawn. The waves described here bear a strong resemblance with the waves in the undular bore. Also for the undular bore to exist as a stationary wave, some dissipation is necessary. The dissipation mechanism, that is to say the way in which viscosity affects the structure, may be quite different in both cases. For the shock wave a possible structure has been worked out in the present investigation where emphasis is laid on the viscous stress associated with the relative radial motion of the bubbles. Most existing theories Chester [8], Peregrine [9], Meyer [10] for the undular bore, deal in the analytic part with dispersion only, with the exception of Chesters' model for the undular bore in which the viscous shear stress in the nonuniform flow between bottom and free surface provides the means for a transition between two different levels.

§ 6. Discussion

The picture of a shock wave in a bubble-liquid mixture evolving from the foregoing analysis is, that, provided $\nu U/R \ll (\beta_1 - \beta_2)^{\frac{1}{2}}$, the rise in pressure takes mainly place over a distance of order $R/(\beta_1 - \beta_2)^{\frac{1}{2}}$, while waves with wavelength given by (5.9) exist at the back of the shock. We found earlier that the effect of translational relative motion is negligible for $d \ll R/\beta$. From the result $d \sim R/(\beta_1 - \beta_2)^{\frac{1}{2}}$ it follows that for $\beta \ll 1$ this condition is satisfied except for very weak shocks. Waves in a bubble-liquid mixture are highly dispersive. For wave number k and frequency ω the dispersion relation is (see e.g. [3] or [4])

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + \dfrac{k^2c^2}{\omega_B^2}}, \tag{6.1}$$

the bubble frequency ω_B being given by (3.4). The phase velocity ω/k is larger than the group velocity $d\omega/dk$. The stationary waves behind the shock have a phase velocity U and since energy propagates with the group velocity energy is radiated away from the shock, similar to the radiation of energy associated with the wave train behind the undular bore. From (6.1) it follows that

$$k = \frac{\omega}{c\left\{1 - \left(\dfrac{\omega}{\omega_B}\right)^2\right\}^{\frac{1}{2}}}$$

This shows that no wave propagation is possible for $\omega > \omega_B$. When ω approaches ω_B the bubbles will perform oscillations with frequency ω_B. No waves occur, the bubbles are convected with the fluid. This will happen for strong shock waves as may be shown as follows. Associated with the wavelength (5.9) is a frequency $2\pi U/\lambda$. Inserting λ from (5.9) and using (3.5) and the shock relation $U^2/c_1^2 = p_2/p_1$ we may write this as

$$\omega = \frac{1}{R_2}\left\{\frac{3p_2\left(1-\dfrac{\beta_2}{\beta_1}\right)}{\rho_f}\right\}^{\frac{1}{2}}. \tag{6.2}$$

Upon comparison with ω_B for the high pressure side (deduced from (3.4)),

$$\omega_B = \frac{1}{R_2}\left(\frac{3p_2}{\rho_f}\right)^{\frac{1}{2}}, \tag{6.3}$$

we conclude that for weak shocks ω is always fairly below ω_B. For very strong shocks, $\beta_2/\beta_1 \to 0$, ω approaches ω_B and then no wave propagation is possible. Under those circumstances the storage of energy in the resonance oscillations may play an important role in the shock mechanism. Here an interesting difference with the undular bore arises, where breaking occurs as the bore gets strong and no energy can be radiated away in waves. The aspects of resonance oscillations in relation with shocks will be discussed elsewhere. Of course, for strong shocks, more violent mechanisms as the breaking up of bubbles in several parts may also be of importance. In conclusion we expect the present theory to hold for weak and moderately strong shocks.

A future paper will treat experiments on this type of shock waves.

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