JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 133, 449–460 (1988)

# Inequalities for Charlier Polynomials with Application to Teletraffic Theory

MADAN LAL MEHTA*

*Service de Physique Théorique, Centre d'Études Nucléaires de Saclay, 91191 Gif-sur-Yvette, France*

AND

ERIK A. VAN DOORN

*Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500AE Enschede, The Netherlands*

*Submitted by E. Stanley Lee*

Received February 17, 1987

## 1. INTRODUCTION

A basic teletraffic model consists of a group of $M$ servers (trunk lines) where customers (calls) arrive according to a Poisson process with intensity $\lambda$. If upon the arrival of a customer at least one of the servers is free, the customer seizes an arbitrary free server and keeps it occupied during the customer's service time. If all servers are busy, however, then the customer is immediately transferred to an *overflow group* of $N$ servers where he seizes an arbitrary free server, if available, and keeps it occupied during his service time. If no free server is available in the overflow group either then the customer is lost forever. Service times are mutually independent and exponentially distributed random variables with mean $\mu^{-1}$; they are also independent of the arrival process.

Brockmeyer [1] was the first to present an analysis for this model and therefore it is sometimes referred to as a *Brockmeyer system* (see Fig. 1).

An important performance measure of a Brockmeyer system is the (equilibrium) *time congestion* $T \equiv T(\lambda, \mu, M, N)$ of the overflow group, that is, the long-run proportion of time during which all $N$ servers of the

* Member of C.N.R.S.



0022-247X/88 $3.00

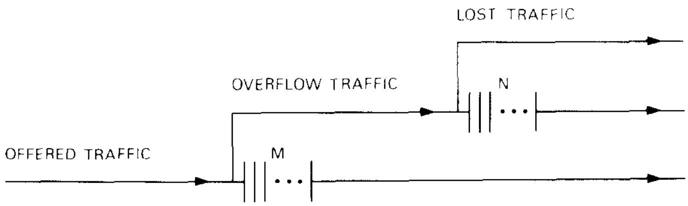


FIG. 1. A Brockmeyer system.

overflow group are busy. The explicit expression for $T$ given in [1] can be formulated conveniently as

$$T(\lambda, \mu, M, N) = \frac{c_M(-N-1, \lambda/\mu)}{c_M(-N, \lambda/\mu)} B(M+N, \lambda/\mu). \tag{1}$$

Here $c_m(x, a)$, $x \in \mathbb{R}$, $a > 0$, $m = 0, 1, ...,$ are Charlier polynomials,

$$c_m(x, a) \equiv \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{x}{i} \frac{i!}{a^i} \tag{2}$$

(see, e.g., [2, p. 226]), and $B(n, a)$, $a > 0$, $n = 0, 1, ...,$ is the Erlang loss function,

$$B(n,a) \equiv c_n^{-1}(-1, a) = \frac{a^n}{n!} \left(1 + \frac{a}{1} + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!}\right)^{-1} \tag{3}$$

(see, e.g., [4]). We note that $B(n, a)$ is a well-tabulated function [12, 13]. so that calculation of $T$ for specific values of the parameters essentially amounts to calculation of the ratio of Charlier polynomials in (1).

For several reasons, which will be mentioned in Section 3, it is of interest to have simple upper and lower bounds for $T$. This provides the motivation for this study. In Section 2 we present a number of inequalities for Charlier polynomials. Then, in Section 3, we show that some bounds for $T$ that have appeared in the literature follow directly from these inequalities, and that the inequalities yield some new and better bounds for $T$ as well.

## 2. INEQUALITIES FOR CHARLIER POLYNOMIALS

Our interest in this section focuses on the functions $q_m(x, a)$, $m = 0, 1, ...,$ defined by

$$q_m(x, a) \equiv c_m(-x-1, a)/c_m(-x, a), \qquad x \geq 0, a > 0, \tag{4}$$

and

$$q_m(x,\ 0) \equiv \lim_{a \downarrow 0} q_m(x, a), \qquad x > 0. \tag{5}$$

LEMMA 1. *For $x > 0$, $a > 0$, and $m = 0, 1, ...,$ one has*

(i) $q_m(x, a) \geqslant 1$ (*with equality only if* $m = 0$);

(ii) $q_0(x, a) = 1$, $q_n(x, a) \to \infty$ $(n \to \infty)$;

(iii) $q_m(0, a) = B^{-1}(m, a)$, $q_m(y, a) \to 1$ $(y \to \infty)$;

(iv) $q_m(x, 0) = 1 + mx^{-1}$, $q_m(x, b) \to 1$ $(b \to \infty)$.

*Proof.* The second part of (ii), which is stated here for convenience, is an easy consequence of the left-hand inequality in (14). The other results follow readily from (2). ▮

Before stating and proving our main results pertaining to the functions $q_m(x, a)$, we recall some recurrence relations for Charlier polynomials that will be used in what follows. Namely, from, e.g., [2, p. 227] we have

$$ac_m(x+1, a) + (m - a - x)\, c_m(x, a) + xc_m(x-1, a) = 0; \tag{6}$$

and, from, e.g., [4] we know

$$ac_m(x, a) - ac_{m-1}(x, a) + xc_{m-1}(x-1, a) = 0 \tag{7}$$

and

$$ac_m(x, a) - ac_m(x-1, a) + mc_{m-1}(x-1, a) = 0. \tag{8}$$

(Note that the functions $G_m(x, a)$ in [4, 6] satisfy $G_m(x, a) = (-1)^m\, c_m(x, a)$.)

Our first theorem gives recurrence relations for $q_m(x, a)$. The theorem is known (see [3] for (9) and [6, 10] for (10)), but our proof is simpler than previous ones.

THEOREM 2. *For $x \geqq 0$, $a > 0$, and $m = 0, 1, ...,$ one has*

$$q_m(x+1, a) = 1 + \frac{1}{x+1}\left(m - a + \frac{a}{q_m(x, a)}\right) \tag{9}$$

*and*

$$q_{m+1}(x, a) = 1 + \frac{(m+1)\, q_m(x, a)}{a + xq_m(x, a)}. \tag{10}$$

*Proof.* The recurrence relation (6) immediately yields (9). To prove (10) observe that

$$1+\frac{(m+1)\,q_m(x,a)}{a+xq_m(x,a)}=1+\frac{(m+1)\;c_m(-x-1,a)}{ac_m(-x,a)+xc_m(-x-1,a)}$$
$$=1+\frac{(m+1)\,c_m(-x-1,a)}{ac_{m+1}(-x,a)}=\frac{ac_{m+1}(-x-1,a)}{ac_{m+1}(-x,a)}=q_{m+1}(x,a),$$

by (7) and (8), respectively. ∎

We next give a series of monotonicity results for $q_m(x, a)$. The first shows monotonicity as a function of $m$.

THEOREM 3. *For fixed $x \geqq 0$ and $a > 0$, $q_m(x, a)$ is strictly increasing with $m$ for $m = 0, 1, \ldots$.*

*Proof.* If $x = 0$ the statement clearly follows from Lemma 1 (iii), so suppose $x > 0$. Then, according to Karlin [8, p. 18 and (12.16)], the kernel $c_{n+m}(-x, a)$, $n$, $m = 0, 1, \ldots$, is strictly totally positive, implying in particular that

$$\frac{c_{m+1}(-x,a)}{c_m(-x,a)}>\frac{c_m(-x,a)}{c_{m-1}(-x,a)}. \tag{11}$$

With (7) we conclude

$$\frac{ac_m(-x,a)+xc_m(-x-1,a)}{c_m(-x,a)}>\frac{ac_{m-1}(-x,a)+xc_{m-1}(-x-1,a)}{c_{m-1}(-x,a)},$$

whence the theorem follows. ∎

COROLLARY 4. *For fixed $x \geqq 0$ and $a > 0$, the function $(q_m(x, a) - 1)/m$ is strictly increasing with $m$ for $m \in \mathbb{N}$.*

*Proof.* Follows immediately from (10) and Theorem 3. ∎

THEOREM 5. *For fixed $m \in \mathbb{N}$ and $a > 0$, $q_m(x, a)$ is strictly decreasing with $x$ for $x \geqq 0$.*

*Proof.* Observe that $c_m(x, a)$ is a polynomial in $x$ of degree $m$. Denoting the zeros of $c_m(x, a)$ by $x_{mi}(a)$, $i = 1, 2, \ldots, m$, we have

$$q_m(x,a)=\prod_{i=1}^{m}\frac{x+1+x_{mi}(a)}{x+x_{mi}(a)}. \tag{12}$$

Since $x_{mi}(a)$, 1, 2, ..., $m$, is real and positive (see, e.g., [8, p. 446]) the theorem follows. ∎

We remark that alternative proofs of this theorem can be based on sign regularity properties of $c_m(-x-y, a)$ for $x, y > 0$ (see [8, pp. 12, 446]) or on an induction argument involving (10).

COROLLARY 6. *For fixed* $m \in \mathbb{N}$ *and* $a > 0$, *the function* $x(q_m(x, a) - 1)$ *is strictly increasing with* $x$ *for* $x \geqq 1$.

*Proof.* Follows immediately from (9) and Theorem 5. ∎

THEOREM 7. *For fixed* $m \in \mathbb{N}$ *and* $x \geqq 0$, $q_m(x, a)$ *is strictly decreasing with* $a$ *for* $a > 0$.

*Proof.* Writing $q_m \equiv q_m(x, a)$ and $\tilde{q}_m \equiv (\partial/\partial a)\, q_m(x, a)$ we obtain from (10)

$$\frac{\tilde{q}_{m+1}}{m+1} = \frac{a\tilde{q}_m - q_m}{(a + xq_m)^2}.$$

Hence, by Lemma 1 (i), $\tilde{q}_{m+1} < 0$ if $\tilde{q}_m \leqq 0$. Since $\tilde{q}_0 = 0$, the theorem follows by an induction argument. ∎

We will finally characterize the behaviour of $q_m(x,a)$ along a line $x + a = \text{constant}$.

THEOREM 8. *For fixed* $m \in \mathbb{N}$ *and* $c > 0$ *the function* $q_m(x, c-x)$, $0 \leqq x \leqq c$, *is completely monotone, that is, for all* $j = 0, 1, ...,$

$$\left(-\frac{d}{dx}\right)^j q_m(x, c-x) \geqq 0. \tag{13}$$

*In addition strict inequality prevails in* (13) *for all* $j$ *if and only if* $m > 1$.

The proof of this theorem has been relegated to the Appendix because of its length.

Taking $j = 1$ and $j = 2$ in Theorem 8 it follows that $q_m(x, a)$ is decreasing and convex (both in the strict sense if $m > 1$) on a line $x + a = \text{constant}$. Hence,

$$q_m(x+a, 0) \leqq q_m(x, a) \leqq \frac{x}{x+a}\, q_m(x+a,\ 0) + \frac{a}{x+a}\, q_m(0, x+a),$$

with strict inequalities if $m > 1$. Upon substitution of the results of Lemma 1 (iii) and (iv) the next corollary is obtained.

COROLLARY 9. *For $x > 0$, $a > 0$, and $m \in \mathbb{N}$ one has*

$$1 + \frac{m}{x+a} \leqq q_m(x, a) \leqq \frac{x}{x+a}\left(1 + \frac{m}{x+a}\right) + \frac{a}{x+a} B^{-1}(m, x+a), \quad (14)$$

*with strict inequalities if $m > 1$.*

It is easy to see that the bounds for $q_m(x, a)$ given in Corollary 9 are at least as good as the bounds implied by Lemma 1 and the monotonicity theorems 3, 5, and 7.

Other bounds can be obtained by a type of argument first employed by Fredericks [3]. Throughout we assume $x > 0$, $a > 0$, and $m \in \{2, 3, ...\}$. First, the recurrence relation (9) together with Theorem 5 imply both

$$q_m(x, a) > 1 + \frac{1}{x+1} (m - a + aq_m^{-1}(x, a))$$

and

$$q_m(x, a) < 1 + \frac{1}{x} (m - a + aq_m^{-1}(x, a)).$$

Solution of these quadratic inequalities gives

$$f_m(x+1, a) < q_m(x, a) < f_m(x, a), \quad (15)$$

where

$$f_m(x, a) \equiv \frac{1}{2x} (m + x - a + \sqrt{(m + x - a)^2 + 4ax}). \quad (16)$$

(The upperbound in (15) is Fredericks' result [3].) The same type of argument involving the recurrence relation (10) and Theorem 3 also gives the bounds (15). Analogously, one can use the recurrence relation (9) together with Corollary 6. An upper bound resulting from this approach is $f_m(x, a)$ again, but as a lower bound one gets

$$q_m(x, a) > g_m(x, a) \equiv \frac{1}{2x} (m + x - a + 1 + \sqrt{(m + x - a - 1)^2 + 4a(x - 1)}). \quad (17)$$

Finally, using the same type of argument for the fourth time, one can combine (10) and Corollary 4. This approach yields two upper bounds; one is $f_m(x, a)$ again and the other one is

$$q_m(x, a) < h_m(x, a) \equiv \frac{m}{2x(m-1)} \times \left( m + x - a - 1 - \frac{2x}{m} + \sqrt{(m + x - a - 1)^2 + 4ax} \right). \quad (18)$$

Straightforward but tedious calculations reveal that

$$h_m(x, a) < f_m(x, a). \quad (19)$$

We can summarize our findings as follows.

THEOREM 10. *For* $x > 0$, $a > 0$ *and* $m \in \{2, 3, ...\}$ *one has*

$$\max\{g_m(x, a), f_m(x+1, a)\} < q_m(x, a) < h_m(x, a) < f_m(x, a). \quad (20)$$

## 3. BOUNDS FOR TIME CONGESTION

In view of (1) and (4) the time congestion of the overflow group in a Brockmeyer system satisfies

$$T(\lambda, \mu, M, N) = q_M(N, \lambda/\mu)\, B(M+N, \lambda/\mu). \quad (21)$$

In what follows we assume $M > 1$, $N > 0$, and $\lambda/\mu > 0$. As observed in [6, 10] explicit evaluation of $q_M(N, \lambda/\mu)$ should proceed via the recurrence relation (10) with initial value $q_0(N, \lambda/\mu) = 1$, which provides a stable and efficient scheme. The value of $B(M+N, \lambda/\mu)$ can be obtained from a table or via one of the existing, efficient schemes for computing the Erlang loss function, see, e.g., [5].

For two reasons approximations and bounds for $T$ are valuable. First, quoting Jagerman [6], "it is useful to have simple upper and lower bounds showing simple and explicit dependence on the arguments." Second, the overflow traffic in a Brockmeyer system (see Fig. 1) is often used as a prototype for traffic which is incompletely characterized (the *equivalent random method*, see, e.g., [5]). This technique, however, involves a formal generalization of the Brockmeyer system in which the size of the first group need not be an integer, so that the right-hand side of (21) is not always defined. Although continuation of $q_M(N, \lambda/\mu)$ to nonintegral values of $M$ is possible in principle, efficient computation of $q_M(N, \lambda/\mu)$ becomes problematical, since (10) requires $M$ to be integral, while a scheme based on (9) is violently unstable. Moreover, it is, in the context of the equivalent random method, clearly senseless to look for very accurate calculation schemes. Instead, simple approximations for $T(\lambda, \mu, M, N)$ that allow $M$ to be nonintegral are called for. Since continuation of the Erlang loss function to nonintegral values of the first argument is conceptually and computationally easy ([5], cf. also [7] and references there), the essential problem is then to find approximations for $q_M(N, \lambda/\mu)$ that allow $M$ to be nonintegral. (See [11] for a further discussion and elaboration of these issues.)

The results of the previous section immediately yield bounds for $T$. First, Corollary 9 and (21) show (recall our assumption $M > 1$)

$$T_{L1}(\lambda, \mu, M, N) < T(\lambda, \mu, M, N) < T_{U1}(\lambda, \mu, M, N),$$

where

$$T_{L1}(\lambda, \mu, M, N) \equiv \left(1 + \frac{M}{N + \lambda/\mu}\right) B(M + N, \lambda/\mu) \tag{22}$$

and

$$T_{U1}(\lambda, \mu, M, N) \equiv \left(N\left(1 + \frac{M}{N + \lambda/\mu}\right) + \frac{\lambda/\mu}{B(M, N + \lambda/\mu)}\right) \frac{B(M + N, \lambda/\mu)}{N + \lambda/\mu}. \tag{23}$$

Both bounds seem to be new. The upper bound improves upon the bound $B(M + N, \lambda/\mu)/B(M, N + \lambda/\mu)$ found by Le Gall and Bernussou [9] via cumbersome calculations.

Second, Theorem 10 and (21) imply

$$T_{L2}(\lambda, \mu, M, N) < T(\lambda, \mu, M, N) < T_{U2}(\lambda, \mu, M, N),$$

where

$$T_{L2}(\lambda, \mu, M, N) \equiv \max\{g_M(N, \lambda/\mu), f_M(N + 1, \lambda/\mu)\} B(M + N, \lambda/\mu) \tag{24}$$

and

$$T_{U2}(\lambda, \mu, M, N) \equiv h_M(N, \lambda/\mu) B(M + N, \lambda/\mu), \tag{25}$$

and $f$, $g$, and $h$ are given by (16), (17), and (18), respectively. The upper bound (25) seems to be new and is better (see Theorem 10) than the bound $f_M(N, \lambda/\mu) B(M + N, \lambda/\mu)$ found by Fredericks [3]. The lower bound (24) combines the results of Jagerman [6], who found $g_M(N, \lambda/\mu)$ $B(M + N, \lambda/\mu)$, and Lindberg [10] and Sanders and Van Doorn [11], who came up independently with $f_M(N + 1, \lambda/\mu) B(M + N, \lambda/\mu)$.

Some numerical results are displayed in Table I. These and other

TABLE I

Numerical Results

| $\lambda/\mu$ | $M$ | $N$ | $T$ | $T_{L1}$ | $L_{L2}$ | $T_{U1}$ | $T_{U2}$ | $T_{APP}$ |
|---|---|---|---|---|---|---|---|---|
| 17 | 5 | 25 | 0.0014 | 0.0014 | 0.0014 | 0.0014 | 0.0014 | 0.0014 |
| 25 | | | 0.0581 | 0.0579 | 0.0580 | 0.0581 | 0.0581 | 0.0581 |
| 33 | | | 0.2020 | 0.2014 | 0.2018 | 0.2020 | 0.2020 | 0.2020 |
| 41 | | | 0.3385 | 0.3376 | 0.3383 | 0.3385 | 0.3385 | 0.3385 |
| 17 | 15 | 15 | 0.0020 | 0.0019 | 0.0020 | 0.0021 | 0.0020 | 0.0020 |
| 25 | | | 0.0768 | 0.0723 | 0.0765 | 0.0786 | 0.0769 | 0.0768 |
| 33 | | | 0.2559 | 0.2433 | 0.2549 | 0.2590 | 0.2561 | 0.2559 |
| 41 | | | 0.4148 | 0.3979 | 0.4138 | 0.4178 | 0.4150 | 0.4161 |
| 17 | 25 | 5 | 0.0044 | 0.0027 | 0.0044 | 0.0124 | 0.0045 | 0.0043 |
| 25 | | | 0.1424 | 0.0964 | 0.1389 | 0.1948 | 0.1442 | 0.1415 |
| 33 | | | 0.4127 | 0.3073 | 0.4049 | 0.4642 | 0.4171 | 0.4139 |
| 41 | | | 0.6051 | 0.4844 | 0.5995 | 0.6373 | 0.6096 | 0.6081 |

experimental results indicate that for ranges of the parameter values that are of practical interest $T_{L2}$ and $T_{U2}$ are generally better than $T_{L1}$ and $T_{U1}$, respectively. Moreover, $T_{L2}$ and $T_{U2}$ perform excellently, with relative errors mostly far below 5%. For small values of $N$ the relative error of $T_{L2}$ tends to become greater than that of $T_{U2}$, so as an approximation to $T$ which is to be used in the whole range of parameter values we would suggest $T_{U2}$, which is simpler than $T_{L2}$ anyway. Alternatively, one might consider Lindberg's [10] suggestion to use

$$T_{\text{APP}}(\lambda, \mu, M, N) \equiv f_M(N+\tfrac{1}{2}, \lambda/\mu)\, B(M+N, \lambda/\mu) \tag{26}$$

(cf. Theorem 10). Numerical values of $T_{\text{APP}}$ are also included in Table I and show very satisfactory performance of $T_{\text{APP}}$ as a simple approximation to $T$.

## APPENDIX: PROOF OF THEOREM 8

From (10) we have

$$q_m(x, c-x) = 1 + \frac{m q_{m-1}(x, c-x)}{c + x(q_{m-1}(x, c-x) - 1)}. \tag{A.1}$$

Together with the initial condition $q_0(x, c-x) \equiv 1$, $x \in \mathbb{R}$, (A.1) allows us to extend the domain of $q_m(x, c-x)$ beyond the interval $0 \leqslant x \leqslant c$. By induction it is easy to see that

$$q_m(x, c-x) > 1, \qquad m > 0, x \geqslant 0. \tag{A.2}$$

It is convenient to introduce the auxiliary functions

$$F_m(x) = 1 - 1/q_m(x, c-x), \tag{A.3}$$

in terms of which the recurrence relation (A.1) becomes

$$F_m(x) = m/(c + m + (x-c)\, F_{m-1}(x)), \tag{A.4}$$

with initial condition $F_0(x) = 0$. By induction it is straightforward to convince oneself that $F_m(x)$ is a ratio of two polynomials, the numerator of degree $[\frac{1}{2}(m-1)]$ and the denominator of degree $[\frac{1}{2}m]$, where $[p]$ denotes the largest integer smaller than or equal to $p$. In particular, one has

$$F_1(x) = \frac{1}{c+1}, \qquad F_2(x) = \frac{2(c+1)}{x + c^2 + 2c + 1}. \tag{A.5}$$

Furthermore, (A.2) and (A.3) imply

$$0 < F_m(x) < 1, \qquad m > 0, x \geqq 0. \tag{A.6}$$

We will show by induction that $F_m(x)$, $m > 1$, is a (strictly) completely monotone function of $x$, $x \geqq 0$. Since sums and products of completely monotone functions are again completely monotone, and

$$q_m(x, c-x) = (1 - F_m(x))^{-1} = \sum_{k=0}^{\infty} (F_n(x))^k, \tag{A.7}$$

it then follows that $q_m(x, c-x)$, $m > 1$, is a (strictly) completely monotone function of $x$, $x \geqq 0$. Evidently, $q_1(x, c-x) = 1 + 1/c$ and hence completely monotone.

As an induction hypothesis suppose that for some $m > 0$ we have

$$F_{2m}(x) = \sum_{i=1}^{m} \frac{a_i}{x + b_i} \tag{A.8}$$

with

$$0 < a_i \equiv a_i(m), \qquad b_i \equiv b_i(m), \qquad 0 < b_1 < b_2 < \cdots < b_m, \tag{A.9}$$

so that $F_{2m}(x)$ is monotonically decreasing from 0 to $-\infty$ in the interval $(-\infty, -b_m)$, from $+\infty$ to $-\infty$ in the intervals $(-b_i, -b_{i-1})$, and from

$+\infty$ to 0 in the interval $(-b_1, \infty)$. We now observe from (A.4) that the $m$ zeros of $F_{2m+1}(x)$ are precisely the poles $-b_i(m)$, $i=1, 2, ..., m$, of $F_{2m}(x)$, while the $m$ poles of $F_{2m+1}(x)$ are precisely the roots of the equation $c+2m+(x-c)\, F_{2m}(x)=0$. By virtue of (A.6) and the aforementioned behaviour of $F_{2m}(x)$, the latter equation has $m$ real roots $-B_i$, $i=1, 2, ..., m$, such that

$$0<B_1<b_1<B_2<b_2< \cdots <B_m<b_m \quad (b_i \equiv b_i(m)). \tag{A.10}$$

As a consequence

$$F_{2m+1}(x)=A_0+\sum_{i=1}^{m} \frac{A_i}{x+B_i} \tag{A.11}$$

with

$$0<A_i \equiv A_i(m), \qquad B_i \equiv B_i(m), \tag{A.12}$$

since, by (A.4) and the behaviour of $F_{2m}(x)$, $F_{2m+1}(x)$ is decreasing in a neighbourhood of each of its zeros $b_i$.

By a similar type of reasoning we can deduce the behaviour of $F_{2m+2}(x)$ from (A.4) and (A.10)–(A.12), culminating in the conclusion that

$$F_{2m+2}(x)=\sum_{i=1}^{m+1} \frac{a_i}{x+b_i} \tag{A.13}$$

with

$$\begin{aligned} &0<a_i \equiv a_i(m+1), \qquad b_i \equiv b_i(m+1) \\ &0<b_1<B_1<b_2<B_2< \cdots <B_m<b_{m+1} \qquad (B_i \equiv B_i(m)). \end{aligned} \tag{A.14}$$

Since, by (A.5), $F_2(x)$ satisfies the hypotheses (A.8) and (A.9), an induction argument shows that (A.8)–(A.12) is valid for all $m>0$. Finally, it is evident from (A.8)–(A.12) that $F_m(x), m>1$, is a (strictly) completely monotone function of $x$, $x \geqq 0$, which concludes the proof of Theorem 8. ∎

## Acknowledgments

The authors are thankful to R. Balian and A. A. Jagers for helpful discussions in connection with the proof of Theorem 8.

## References

1. E. Brockmeyer, Det simple overflowproblem i telefontrafikteorien, *Teleteknik* **5** (1954), 361–374.

2. A. Erdélyi (Ed.), "Higher Transcendental Functions, Vol. II," McGraw–Hill, New York, 1953.
3. A. A. Fredericks, Approximating parcel blocking via state dependent birth rates, *in* "Proceedings, 10th Int. Teletraffic Cong., Montreal, P.Q., Canada, 1983," paper 5.3.2.
4. D. L. Jagerman, Some properties of the Erlang loss function, *Bell Syst. Tech. J.* **53** (1974), 525–551.
5 D. L. Jagerman, Methods in traffic calculations, *AT & T Bell Labs. Tech. J.* **63** (1984), 1283–1310.
6. D. L. Jagerman, Laplace transform inequalities with application to queueing, *AT & T Tech. J.* **64** (1985), 1755–1763.
7. A. A. Jagers and E. A. van Doorn, On the continued Erlang loss function, *Oper. Res. Lett.* **5** (1986), 43–46.
8. S. Karlin, "Total Positivity," Stanford Univ. Press, Stanford, CA, 1968.
9. F. Le Gall and J. Bernussou, An estimate for the time congestion of overflow processes, *IEEE Trans. Commun.* **COM-31** (1983), 1202–1203.
10. P. Lindberg, "Simple Computation Methods for Erlang's Formula and Wilkinson's ERT," Report Nst 85008 6171, Swedish Telecommunications Administration, Stockholm, Sweden, 1985.
11. B. Sanders and E. A. van Doorn, Estimating time congestion from traffic parameters, *IEEE Trans. Commun.* **COM-35** (1987), 856–862.
12. "Telephone Traffic Theory: Tables and Charts, Part I," 2nd rev. ed., Siemens Aktiengesellschaft, Munich, 1974.
13. "Table of the Erlang Loss Formula," Telefonaktiebolaget L. M. Ericsson, Telephone Exchange Division, Stockholm, 1979.