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On the existence and uniqueness of the boundary layer equations for a rotating conducting flow

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ABSTRACT

In this paper we have considered the question of the existence of the solution of a pair of coupled ordinary differential equations depending on a parameter s. By using a corollary of the Schauder-Tychonoff fixed point theorem, and some hard analysis – needed to cover essentially the range of s near $s=0$ – we were able to show that all the conditions for the theorem could be fulfilled. Hence existence for all values of $s>0$ was established.

We also considered the question of uniqueness. There we used a number of rough estimates. In this way the calculations are largely simplified, but it is then not possible to prove uniqueness for all values of s. We only proved it for $s>2.33$.

1. INTRODUCTION

The study of the behaviour of rotating flows has always been an important branch of fluid dynamics. This of course is so, because of the many physical phenomenae and technical problems where rotating flows occur.

In certain cases the full Navier-Stokes equations or a simplified boundary layer version of these equations can be reduced to systems of ordinary differential equations by using similarity transformations. For examples we refer to refs. [1] and [2]. From these papers it will be clear that the question of uniqueness of solutions is an important one.

In the present paper we will consider the existence and uniqueness of the solution of a pair of ordinary differential equations, which describe the boundary layer flow over a fixed disk as induced by a vortex flow far from the disk. The fluid is considered to be conducting and forces induced by an applied magnetic field are taken into account.

The equations describe a special case of a more general problem considered by King and Lewellen in ref. [3].

To be more specific we introduce a coordinatesystem, where the z-axis is perpendicular to the disk and where r is the radial distance as measured from this axis.

Following ref. [3] the radial velocity u, the tangential velocity v and the axial velocity w can be expressed as follows

(1.1)a $\quad u = r^{-1} f'(\eta)$

(1.1)b $\quad v = r^{-1} g(\eta)$

(1.1)c $\quad w = \sqrt{\nu}\, r^{-1}\{f(\eta) + \eta f'(\eta)\}$

with

(1.2) $\quad \eta = z(\sqrt{\nu})^{-1} r^{-1}$ and where ν is the kinematic viscosity.

As is clear from eq. (1.2) $\eta = 0$ for $z = 0$ and $\eta \to \infty$ for $z \to \infty$. In fact η is a conical coordinate.

The required vortex-like behaviour of the flow can be obtained by demanding that $g(\eta) \to 1$ and $f'(\eta) \to 0$ for $\eta \to \infty$. At the disk the velocities should vanish. This leads to the following boundary conditions.

(1.3) $\quad f(0) = f'(0) = g(0) = 0,\ f'(\infty) = 0,\ g(\infty) = 1.$

When $s > 0$ is a measure for the applied magnetic force, the governing equations can be written as

(1.4)a $\quad f''' + ff'' + f'^2 - sf' = 1 - g^2$

(1.4)b $\quad g'' + fg' + s(1 - g) = 0.$

As has been remarked by Stewartson and Troesch in ref. [4] and by Troesch in refs. [5] and [6] this system is very hard to solve for small values of s. The crux to this lies in the anomalous behaviour of f for these small values. In ref. [4] an asymptotic estimate is given for $f(\infty)$ for small s

(1.5) $$f(\infty) = -0.1481\ (s/2\pi)^{\frac{1}{4}} \exp\left[\left(\frac{\pi}{2} + 1\right)/s\right].$$

Due to the stiff behaviour which is the result hereof, it becomes very hard to solve this problem. A number of exploratory studies was made on the difficulties involved in this problem. A survey is given in ref. [7], which also contains a sketch of an existence proof, without considering uniqueness.

Questions of existence and uniqueness are so important however, that they deserve a more detailed treatment. This is the main purpose of this paper.

The proofs will rely heavily on the property of positiveness of the operators. For existence use will be made of the fixed point theorem of Tychonoff. In order to apply this theorem we need a number of a priori estimates. These will be given first.

Uniqueness will be proved for $s>2.33$. The reason for this we feel is not a fundamental one, but for $s<2.33$., better estimates are needed. Although these estimates in principle may be obtained they require the use of elaborate numerical calculations, and we have refrained from those. In the sequel we shall write x in stead of η.

2. A NUMBER OF A PRIORI ESTIMATES

In this section we will show that the solution f of equations (1.4) has to be negative. This can be proved by first showing that g is an increasing function from the value 0 to the value 1.

PROPOSITION 2.1.

(2.1) $\quad g'>0.$

PROOF. There can be no maximum of g with $g>1$ for then $g'=0$, $s(1-g)<0$ and $g''\leq 0$ which leads to a contradiction. For the same reason there can be no minimum with $g<1$. But this means that due to the boundary condition there can also be no maximum with $g<1$. Hence g has to increase monotonously.

PROPOSITION 2.2.

(2.2) $\quad f'\leq 0.$

PROOF. By integrating eq. (1.4) we obtain

(2.3) $$f''+ff'-sf=\int_0 (1-g^2)+f''(0).$$

The proof will consist of two parts. First we establish that $f''(0)<0$. Suppose that $f''(0)\geq 0$. Then the right hand side of eq. (2.3) is >0 and it follows that f can have no positive maximum. But this means that $f'(x)\geq 0$ for $x\neq 0$ and hence also $f\geq 0$. Moreover f' cannot decrease if $f'<s$. For then $f''\leq 0$ and $f(f'-s)<0$ and thus we have a contradiction. Since this means that we cannot fulfill $f'(+\infty)=0$, we conclude

$$f''(0)<0.$$

From this fact together with $f(0)=f'(0)=0$ it follows that $f'(x)$ starts to decrease. Because $f'(+\infty)=0$, the function f' has to obtain a minimum value and then has to increase again. If we suppose that f' increases to a positive value, it has to decrease again because of the boundary condition at infinity. It then decreases to zero at $x=x_2$ where x_2 may be $+\infty$. (See fig. 1).

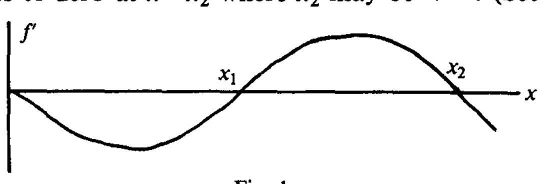

Fig. 1.

We now apply eq. (2.3) at x_1 and x_2. Since due to the fact that $1-g^2>0$, the right hand side of eq. (2.3) is increasing with x. Hence we find

$$f''(x_2)-f''(x_1)-s\{f(x_2)-f(x_1)\}>0.$$

Now $f''(x_2)-f''(x_1)\leq 0$ and also $-s\{f(x_2)-f(x_1)\}<0$. Thus we have again a contradiction. This proves the proposition.

We are now ready for the existence proof which we will give in the next section.

3. EXISTENCE OF SOLUTIONS

In this section we shall prove that the system (1.3), (1.4) has a continuously differentiable solution f, g defined on $[0, +\infty[$ with

$$0\geq f\geq -m(s) \text{ and } 0\geq f'\geq -\frac{1}{s}$$

where $m(s)$ is a strictly positive number, only dependent on s, to be determined in the course of our exposition.

To prove existence we use a corollary of the Schauder-Tychonoff theorem which states:

Let E be a Banach space, A a non empty closed and convex subset of E, $\Phi: A\to E$ a continuous mapping with $\Phi(A)\subset A$ and $\Phi(A)$ relatively compact. Then Φ has a fixed point. (See ref. 8).

For E we take

$$BC^1=\{f:[0,+\infty]\to\mathbb{R}\,|\,f \text{ continuous and } f' \text{ regular}\}$$

with norm $\|f\|_{1,\infty}=\|f\|_\infty+\|f'\|_\infty$ where $\|\cdot\|_\infty$ is the usual supnorm.

"f' regular" means that $f'(x+0)$ exists for $x\in[0,+\infty[$ and $f'(x-0)$ for $x\in]0,+\infty]$ (see ref. 9).

$$A=\left\{f\in BC^1 | 0\geq f\geq -m(s),\ 0\geq f'\geq -\frac{1}{s},\ f(0)=0\right\}.$$

It is clear that A is convex and closed.

To define Φ we rewrite the system (1.3), (1.4) as follows

$$(3.1)\qquad -\tilde{g}''-f\tilde{g}'+s\tilde{g}=0,\ \tilde{g}(0)=1$$

$$(3.2)\qquad -f''-ff'+sf=\int^{\infty}(2\tilde{g}-\tilde{g}^2)+sf(+\infty)$$

where $\tilde{g}=1-g$.

Introducing $\bar{f}=f-f(+\infty)$, eq. (3.2) becomes

$$(3.3)\qquad -\bar{f}''-f\bar{f}'+s\bar{f}=\int^{\infty}(2\tilde{g}-\tilde{g}^2),\ \bar{f}'(0)=0.$$

We now take an $f \in A$, then solve $\tilde{g}$ from (3.1), put $\tilde{g}$ in the right hand side of (3.3), solve $\tilde{f}$ and finally put

$$(3.4) \qquad \Phi(f) = \tilde{f} - \tilde{f}(0).$$

In the sequel we write g instead of $\tilde{g}$ for convenience of notation.

The left hand side of (3.1) and (3.3) shows an operator of the form

$$-(\ )'' - f(\ )' + s(\ )$$

so a closer look at this operator will be worthwhile. First we remark that there are functions ϕ_1 and ϕ_2 such that

$$(3.5) \qquad -g'' - fg' + sg = -(D - \phi_2)(D - \phi_1)g$$

where D is the differentiation operator. We find for ϕ_1 and ϕ_2

$$(3.6) \qquad \begin{cases} \phi_1 + \phi_2 = -f \text{ and } \phi_1' - \phi_1\phi_2 = s \text{ or} \\ \phi_1' + f\phi_1 + \phi_1^2 = s \end{cases}$$

Let $f \in A$ and let μ_1 be the (strictly) positive and μ_2 be the (strictly) negative root of

$$\mu^2 + f(+\infty)\mu - s = 0, \text{ thus}$$

$$\mu_1 = -\tfrac{1}{2}f(+\infty) + \tfrac{1}{2}\sqrt{f^2(+\infty) + 4s}, \; \mu_2 = -\tfrac{1}{2}f(+\infty) - \tfrac{1}{2}\sqrt{f^2(+\infty) + 4s}.$$

PROPOSITION 3.1. *The Riccati equation* (3.6) *has one solution* ϕ_1 *with* $\phi_1(0) = 0$. *The domain of definition is* $[0, +\infty[$. ϕ_1 *is an increasing function and* $\phi_1(+\infty) = \mu_1$.

PROOF. There is a maximum interval $J = [0, \beta[$ on which there exists an unique solution (ref. (10)). ϕ_1 and ϕ_1' are continuous on J. (3.6) can be written in the form

$$(3.7) \qquad \phi_1' + (\phi_1^2 + f(+\infty)\phi_1 - s) + (f - f(+\infty))\phi_1 = 0.$$

$\phi_1'(0) = s > 0$ so that ϕ_1 starts increasing. If $\phi_1(x) > \mu_1$ for some $x \in J$ then it follows from (3.7) that $\phi_1'(x) < 0$. It turns thus out that $0 \le \phi_1(x) \le \mu_1$ for all $x \in J$. But then $\beta = +\infty$ (ref. (10)).

Let $\mu_1(a)$ be the positive root of $\mu^2 + a\mu - s = 0$ $(a \le 0)$ then the function $a \mapsto \mu_1(a)$ is strictly decreasing.

Now suppose there is a point on $[0, +\infty[$ where $\phi_1' < 0$ then there exists an $x \in]0, +\infty[$ such that $\phi_1' \ge 0$ on $[0, x]$, $\phi_1'(x) = 0$ and a nonempty interval $]x, y]$ where $\phi_1' < 0$ everywhere. Now $\phi_1'(x) = 0$ so that $\phi_1(x) = \mu_1(f(x))$. $\phi_1'(y) < 0$ so that $\phi_1^2(y) + f(y)\phi_1(y) - s > 0$ and that means that $\mu_1(f(y)) < \phi_1(y) < \phi_1(x) = \mu_1(f(x))$ which implies that $f(y) > f(x)$. But an f with this property $\notin A$. $\phi_1(+\infty)$ must exist and $\phi_1'(+\infty) = 0$ so that $\phi_1(+\infty) = \mu_1$.

REMARK. It is clear that $\phi_2(+\infty) = \mu_2$.

It is known that for $u \in BC^0 (= \{f: [0, +\infty] \to \mathbb{R} | f \text{ continuous}\})$, the set of solutions of

$$(3.8) \qquad -g'' - fg' + sg = u$$

is a two parameter family defined on $[0, +\infty[$ (see ref. 10) which is asymptotically equal to that of

$$-g'' - f(+\infty)g' + sg = u(+\infty) \text{ (see ref. (11))}.$$

Replacing the left hand side of eq. (3.8) as in (3.5) and defining $h = (D - \phi_1)g$ we immediately find

$$(3.9) \qquad h(x) = \frac{c_1}{p_2(x)} - \frac{1}{p_2(x)} \int_0^x p_2 u, \; g(x) = \frac{c_2}{p_1(x)} - \frac{1}{p_1(x)} \int_x^\infty p_1 h$$

where c_1 and c_2 are constants and

$$p_1(x) = \exp\left(-\int_0^x \phi_1\right) \text{ and } p_2(x) = \exp\left(-\int_0^x \phi_2\right).$$

It will be clear from the properties of ϕ_1 and ϕ_2 that h is bounded, and that g will be bounded if and only if

$$(3.10) \qquad c_2 = 0.$$

Assuming that (3.10) holds, also h' is bounded. This immediately will make clear that also g' and g'' are bounded and continuous.

Let $BC^2 = \{g: [0, +\infty] \to \mathbb{R} | g, g' \text{ and } g'' \text{ continuous}\}$ with norm

$$\|g\|_{2,\infty} = \|g\|_\infty + \|g'\|_\infty + \tfrac{1}{2}\|g''\|_\infty$$

and

$$BC_0^2 = \{g \in BC^2 | g(0) = 0\}.$$

PROPOSITION 3.2. Let $L: BC_0^2 \to BC^0$ be the mapping

$$g \mapsto -g'' - fg' + sg$$

for a fixed $f \in A$ then L is linear and a homeomorphism.

PROOF. The linearity of L is obvious. We have seen above that L is onto. L is one to one because $g \equiv 0$ is the only bounded solution of $-g'' - fg' + sg = 0$ with $g(0) = 0$ as follows from (3.9).

$$\|Lg\|_\infty \le \|g''\|_\infty + \|f\|_\infty \|g'\|_\infty + s\|g\|_\infty \le c\|g\|_{2,\infty}$$

for a suitable constant c. So L is continuous. The open mapping theorem yields the continuity of L^{-1}.

PROPOSITION 3.3. The mapping $f \mapsto g$ ($f \in A, g \in BC^2$) where g is the bounded solution of

$$-g'' - fg' + sg = 0 \text{ with } g(0) = 1$$

is continuous.

PROOF. We shall show that $f \mapsto g$ is continuous in every $f_0 \in A$.

Let $g_0 = g(f_0)$ and $g = g(f)$ then $g_0(0) = g(0) = 1$ and $-g'' - fg' + sg = 0$ and $-g_0'' - f_0 g_0' + s g_0 = 0$. Substraction gives

$$-(g-g_0)'' - f_0(g-g_0)' + s(g-g_0) = (f-f_0)(g'-g_0') + (f-f_0)g_0'.$$

According to proposition 3.2

$$\|g-g_0\|_{2,\infty} \leq c\|f-f_0\|_\infty \|g'-g_0'\|_\infty + c\|f-f_0\|_\infty \|g_0'\|_\infty \leq$$

$$\leq c\|f-f_0\|_\infty \|g-g_0\|_{2,\infty} + c\|f-f_0\|_\infty \|g_0'\|_\infty$$

for a suitable c depending on f_0.

Choose f such that

$$\|f-f_0\|_{1,\infty} \leq \frac{1}{2c}$$

then

$\|g-g_0\|_{2,\infty} \leq 2c\|f-f_0\|_{1,\infty}\|g_0'\|_\infty$ since $\|f-f_0\|_{1,\infty} \geq \|f-f_0\|_\infty$.

So $\|g-g_0\|_{2,\infty} \leq \varepsilon$ if

$$\|f-f_0\|_{1,\infty} \leq \min\left(\frac{1}{2c}, \frac{\varepsilon}{2c\|g_0'\|_\infty}\right).$$

REMARK. We have used in this proof that BC^0 is a Banach *algebra*. In fact for every $n \in \mathbb{N}$, BC^n with norm

$$\|f\|_{n,\infty} = \sum_{k=0}^{n} \frac{1}{k!} \|f^{(k)}\|_\infty$$

is a Banach algebra (see ref. (12)).

This is the first step to show that Φ is continuous. To go further we need an ordering principle.

We say that $f \leq g$ for f and $g \in F([0, +\infty], \mathbb{R})$ if $f(x) \leq g(x)$ for each $x \in [0, +\infty]$.

PROPOSITION 3.4. L^{-1} (the inverse of the L of prop. 2) *is a positive operator, i.e. if* $Lg \geq 0$, $g(0) \geq 0$ *and* $g(+\infty) = 0$ *then* $g \geq 0$. *Moreover* g *cannot have a strictly positive maximum if* $Lg = 0$.

The proof follows the same line as that of prop. 2.1.

PROPOSITION 3.5. *The mapping* $f \mapsto g$ *is decreasing.*

PROOF. Let $f_1 \leq f_2$, $g(f_1) = g_1$ and $g(f_2) = g_2$, $g_1(0) = g_2(0) = 1$ then substracting the equations for g_1 and g_2 gives

$$-(g_1-g_2)'' - f_1(g_1-g_2)' + s(g_1-g_2) = (f_1-f_2)g_2'.$$

It follows from (3.9) that $g_1(+\infty)=g_2(+\infty)=0$; $g_2'\le 0$, according to prop. 3.4. It follows now from prop. 3.4 that $g_1-g_2\ge 0$.

COROLLARY. If $-g''-fg'+sg=0$, $g(0)=1$ then

$$g(f\equiv 0)=e^{-x\sqrt{s}}\le g(x)\le e^{\mu_2 x}=g(f\equiv f(+\infty)).$$

PROPOSITION 3.6. *The mapping* $f\mapsto\int^{\infty}(2g-g^2)$ *is continuous.*

PROOF. Let

$$u(x)=\int_x^{\infty}(2g-g^2),\ g(f)=g,\ g(f_0)=g_0 \text{ and } u_0(x)=\int_x^{\infty}(2g_0-g_0^2)$$

where f_0 is some element of A, the mapping $f\mapsto g$ being that of prop. 3.3. Let $a>0$ then

$$\|u-u_0\|_\infty\le\int_0^a|2g-g^2-2g_0+g_0^2|+\int_a^{\infty}|(2g-g^2)-(2g_0-g_0^2)|\le$$

$$\le\|g-g_0\|_\infty\int_0^a(2-g-g_0)+2\int_a^{\infty}g+2\int_a^{\infty}g_0$$

because $0\le g\le 1$ according to the corollary. Using this again we see that

$$\|u-u_0\|_\infty\le\|g-g_0\|_\infty\int_0^a(2-g_0-g)-\frac{2}{\mu_2(f_0)}\,e^{\mu_2(f_0)a}-\frac{2}{\mu_2(f)}\,e^{\mu_2(f)a}.$$

The functional $f\mapsto\mu_2(f)$ is continuous so that one can choose a in such a way that the terms with the exponents are small. Afterwards one can choose g properly in view of prop. 3.3.

PROPOSITION 3.7. *Let* $L^1:\{g\in BC^2|g'(0)=0\}\to BC^0$ *be the mapping* $g\mapsto -g''-fg'+sg$ *for a fixed* $f\in A$. *Then* L^1 *is linear and a homeomorphism.* ${L^1}^{-1}$ *is positive.*

PROOF. Take a look at the formulae (3.9). Because $\phi_1(0)=0$ there results that $h(0)=0$ so that $c_1=0$. From (3.9) follows then the bijectivity of L^1. That L^1 is linear and a homeomorphism is seen by the same reasoning as in the proof of prop. 3.2.

That ${L^1}^{-1}$ is positive follows directly from (3.9), (3.10) for $c_1=0$.

PROPOSITION 3.8. The mapping $(f,u)\mapsto\bar f$ ($f\in A, u\in BC^0$ and $\bar f\in BC^2$) where $\bar f$ is the solution of

$$-\bar f''-f\bar f'+s\bar f=u \text{ with } \bar f'(0)=0$$

is continuous.

The proof follows the same lines as that of prop. 3.3.

THEOREM 1. Φ is continuous.

PROOF. This follows from prop. 3.3, 3.6 and 3.8.

We now will prove that $\Phi(A) \subset A$.

PROPOSITION 3.9 *If $\bar{f}$ is the bounded solution of* (3.3) *then* $0 > \bar{f}' \geq -(1/s)$.

PROOF. Because $\bar{f}'(0) = 0$ it follows that in (3.9) $c_1 = 0$. According to ref. (13)

$$\int_0^x p_2 u \sim \frac{p_2^2(x) u^2(x)}{|p_2'(x) u(x) + p_2(x) u'(x)|} \text{ for } x \to \infty.$$

Now $p_2'(x) = -\phi_2(x) p_2(x)$ so that

$$-h(x) \sim \frac{u^2(x)}{|-\phi_2(x) u(x) + u'(x)|}$$

and that means that $h(+\infty) = 0$, for

$$u = \int_x^\infty (2g - g^2).$$

In the same way as for h we derive that

$$\bar{f}(+\infty) = \bar{f}'(+\infty) = 0.$$

Now suppose that $\bar{f}' > 0$ somewhere, then there is an x where $\bar{f}'$ has a strict positive maximum.

Let $k > 0$ then

$$-\bar{f}''(x+k) - f(x+k)\bar{f}'(x+k) + s\bar{f}(x+k) = \int_{x+k}^\infty (2g - g^2).$$

Taking $k = 0$, where $\bar{f}''(x) = 0$ and substracting gives

$$(3.11) \quad \begin{cases} -\bar{f}''(x+k) - f(x+k)\bar{f}'(x+k) + f(x)\bar{f}'(x) + s(\bar{f}(x+k) - \bar{f}(x)) = \\ = -\int_x^{x+k} (2g - g^2). \end{cases}$$

Now

$$-\int_x^{x+k} (2g - g^2) \leq 0 \text{ and } -\bar{f}''(x+k) \geq 0.$$

Furthermore

$$\bar{f}(x+k) - \bar{f}(x) = k\bar{f}'(x) + o_1(k),\ \bar{f}'(x+k) - \bar{f}'(x) = o_2(k)\ (\bar{f}''(x) = 0).$$

Using this we find

$$-\bar{f}''(x+k) + (-f(x+k) + f(x))\bar{f}'(x+k) + sk\bar{f}'(x) + o_3(k) =$$

$$= -\int_x^{x+k} (2g - g^2).$$

All the terms at the left hand are positive, whereas $o_3(k)$ can be made as small as necessary. Hence a contradiction. So $\bar{f}' \leq 0$.

Now let y be the point where $\bar{f}'$ reaches its global minimum. For $k>0$ we again can write down eq. (3.11) with x replaced by y. Since $g^2-2g \geq -1$ and $-f''(y+k) \leq 0$ we find

$$-f(y+k)\bar{f}'(y+k)+f(y)\bar{f}'(y)+s\{\bar{f}(y+k)-\bar{f}(y)\} \geq -k.$$

Using Taylor series and the fact that $\bar{f}'(y)(f(y)-f(y+k)) \leq 0$ we derive

$$ks\bar{f}'(y)+o(k) \geq -k \text{ or } \bar{f}'(y) \geq -\frac{1}{s}.$$

It follows that everywhere $\bar{f}' \geq -(1/s)$.

COROLLARY. $\Phi(f)=\bar{f}-\bar{f}(0) \leq 0$.

PROPOSITION 3.10. *The mapping $f \mapsto \bar{f}$ is decreasing.*

PROOF. Let $f_1, f_2 \in A$ with $f_1 \leq f_2$. If $g_1=g(f_1)$ and $g_2=g(f_2)$ (prop. 3.3) then $g_1 \geq g_2$. If

$$u_i = \int^{\infty} (2g_i - g_i^2) \text{ then } u_1 \geq u_2, \text{ for } \frac{d}{dg}(2g-g^2)=2-2g \geq 0.$$

Now subtracting equations (3.3) for $i=1,2$ yields

$$-(\bar{f}_1-\bar{f}_2)''-f_1(\bar{f}_1-\bar{f}_2)'+s(\bar{f}_1-\bar{f}_2)=(f_1-f_2)\bar{f}_2'+u_1-u_2.$$

The result follows from prop. 3.9 and 3.7.

THEOREM 2. $\Phi(A) \subset A$.

PROOF. According to prop. 3.9 the inequality $0 \geq \Phi(f)' \geq -(1/s)$ holds if $f \in A$. The cor. of prop. 3.9 yields $\Phi(f) \leq 0$. One needs some hard analysis to show that $\Phi(f) \geq -m(s)$. We will do this in the lemma below. Now obviously $\Phi(f)(0)=0$ and $\Phi(f)'$ is continuous. The theorem follows.

LEMMA. For every $s>0$ there is a number $m(s)>0$ such that the function f_s defined as

$$f_s(x) = -\frac{x}{s} \text{ for } x \in [0, +sm(s)]$$

$$f_s(x) = -m(s) \text{ for } x \in [+sm(s), +\infty[$$

is the minimum of A, i.e. if $f \in A$, then f and $\Phi(f) \geq f_s$.

PROOF. The only thing we need to prove is that $\bar{f}_s(0) \leq m(s)$ for then $\Phi(f) \geq f_s$. This follows from the fact that $\Phi(f)' \geq -(1/s)$ and hence $\Phi(f) \geq -(x/s)$ for $x \in [0, sm(s)]$. On the other hand we know that $\Phi(f)' \leq 0$. Applying proposition

(3.10) and knowing that $\Phi(f_s)(+\infty) \geq -m(s)$ gives the result that $\Phi(f) > -m(s)$ for $x \in [sm(s), +\infty[$. These two facts together imply $\Phi(f) \geq f_s$.

We now will prove that $\bar{f}_s(0) \leq m(s)$ for a certain $m(s)$ which has to be found. From eq. (3.3) we derive

$$s\bar{f}_s(0) = \int_0^\infty (2g_s - g_s^2) + \bar{f}_s''(0).$$

From the corollary of prop. 3.9 together with $\Phi(f)(0) = \Phi(f)'(0) = 0$ it follows that $\Phi(f_s)''(0) = \bar{f}_s''(0) \leq 0$. Hence we find

$$\text{(3.12)} \quad \bar{f}_s(0) < \frac{2}{s} \int_0^\infty g_s.$$

Using eq. (3.1) and the actual form of f_s we find

$$\bar{f}_s(0) < \frac{2}{s^2} \left[\int_0^{sm(s)} \left(g_s'' - \frac{x}{s} g_s' \right) + \int_{sm(s)}^\infty (g_s'' - m(s) g_s') \right].$$

By partial integration and again using eq. (3.1) there is obtained for $s \neq 1$

$$\text{(3.13)} \quad \bar{f}_s(0) < \frac{2}{s^2(s^2-1)} \{ -s^2 g_s'(0) + g_s'(sm(s)) - m(s) g_s(sm(s)) \}.$$

In order to evaluate eq. (3.13) we have to find g_s. Now

$$\text{(3.14)a} \quad g_s'' - \frac{x}{s} g_s' - s g_s = 0 \text{ for } x \in [0, sm(s)]$$

$$\text{(3.14)b} \quad g_s'' - m(s) g_s' - s g_s = 0 \text{ for } x \in [sm(s), +\infty[.$$

Remark that for $s=1$, there holds $m(1)\, g(1) = g'(m(1)) - g'(0)$ as is found by integration eq. (3.14)a. Hence the denominator of eq. (3.13) vanishes. It is not difficult to find the relevant expression for $s=1$ for eq. (3.13) but we will not give it here.

The conditions to imply are $g_s(0) = 1$, g_s and g_s' continuous at $sm(s)$ and $g_s(+\infty) = 0$.

Eq. (3.14)a can be brought into a standard form by introducing $z = (x^2/2s)$. We find $zg''(z) + (\tfrac{1}{2} - z)g'(z) - (s^2/2)g(z) = 0$.

The general solution of this equation is (ref. 14)

$$\text{(3.15)} \quad g = \alpha_1 M\left(\frac{s^2}{2}, \frac{1}{2}, \frac{x^2}{2s}\right) + \alpha_2 U\left(\frac{s^2}{2}, \frac{1}{2}, \frac{x^2}{2s}\right).$$

The solution of eq. (3.14)b is immediately found to be

$$\text{(3.16)} \quad g = a e^{-\frac{1}{2}\{\sqrt{m^2(s)+4s} - m(s)\}x}.$$

By using the relevant proporties of the hypergeometric functions the continuity equations for g and g' at $x = sm(s)$ can be written as

$$\alpha_1 M\left(\frac{s^2}{2}, \frac{1}{2}, \frac{sm^2(s)}{2}\right) + \alpha_2 U\left(\frac{s^2}{2}, \frac{1}{2}, \frac{sm^2(s)}{2}\right) = a e^{-(sm(s)/2)\{\sqrt{m^2(s)+4s} - m(s)\}}$$

$$\alpha_1(1-s^2)M\left(\frac{s^2}{2},\frac{3}{2},\frac{sm^2(s)}{2}\right)+\alpha_2 U\left(\frac{s^2}{2},\frac{3}{2},\frac{sm^2(s)}{2}\right)=$$
$$=a\chi e^{-(sm(s)/2)\{\sqrt{m^2(s)+4s}-m(s)\}}$$

where

$$\chi=\tfrac{1}{2}\left\{1+\sqrt{1+\frac{4s}{m^2(s)}}\right\}.$$

For $x=0$ we have

$$\alpha_1+2^{s^2-1}\left(\Gamma\left(\frac{s^2}{2}\right)/\Gamma(s^2)\right)\alpha_2=1.$$

From these equations we can solve for the quantities α_1, α_2 and a. We find

(3.17)a $$\alpha_2=\frac{1}{2^{s^2-1}\Gamma\left(\frac{s^2}{2}\right)/\Gamma(s^2)-\Psi}.$$

where

(3.17)b $$\Psi=\frac{U\left(\frac{s^2}{2},\frac{3}{2},\frac{sm^2(s)}{2}\right)-\chi U\left(\frac{s^2}{2},\frac{1}{2},\frac{sm^2(s)}{2}\right)}{(1-s^2)M\left(\frac{s^2}{2},\frac{3}{2},\frac{sm^2(s)}{2}\right)-\chi M\left(\frac{s^2}{2},\frac{1}{2},\frac{sm^2(s)}{2}\right)}.$$

$\alpha_1=-\alpha_2\Psi.$

(3.17)c $$a=e^{sm(s)/2\{\sqrt{m^2(s)+4s}-m(s)\}}\alpha_2\left\{-\Psi M\left(\frac{s^2}{2},\frac{1}{2},\frac{sm^2(s)}{2}\right)+U\left(\frac{s^2}{2},\frac{1}{2},\frac{sm^2(s)}{2}\right)\right\}$$

From eq. (3.15) we derive

(3.17)d $$g_s'(0)=-\sqrt{\frac{2\pi}{s}}\,\frac{1}{\Gamma\left(\frac{s^2}{2}\right)}\,\alpha_2.$$

Evaluating the right hand side of eq. (3.13) with the above obtained we finally have

(3.18) $$\tilde{f}_s(0)<\frac{2m(s)}{s^2(1-s^2)}\,\chi a e^{-\frac{1}{2}sm(s)\{\sqrt{m^2(s)+4s}-m(s)\}}-\frac{2}{1-s^2}\sqrt{\frac{2\pi}{s}}\,\frac{\alpha_2}{\Gamma\left(\frac{s^2}{2}\right)}.$$

If the right hand side of eq. (3.18) is equal to $m(s)$ we have fulfilled the original requirement. Now eq. (3.18) is a very involved function of $m(s)$. It is possible to conduct asymptotic analysis for s small and s large. We will omit the details of the analysis and only give the results. For s large we find

(3.19) $$m(s)\sim 2s^{-3/2}.$$

For s small we have

$$(3.20)\qquad m(s) \sim \sqrt{\frac{2}{s}}\, \exp\left\{-\frac{1}{s^2}\ln\left(s^2(1-s^2)2^{s^2-2}\,\frac{\Gamma\left(\frac{s^2}{2}\right)}{\Gamma(s^2)}\right)\right\}.$$

For a number of values of s we have computed the values of $m(s)$. They are compared with the asymptotic values below.

	$m(s)$ computed	asymptotic value	
$s=$ 0.1	0.58385 10^{231}	0.58385 10^{231}	
0.2	0.10036 10^{44}	0.10036 10^{44}	
0.3	0.26788 10^{16}	0.26788 10^{16}	eq. (3.20)
0.4	0.19563 10^{8}	0.19563 10^{8}	
0.5	0.11105 10^{5}	0.11109 10^{5}	
0.6	0.32920 10^{3}	0.33332 10^{3}	
0.7	0.51522 10^{2}	0.56115 10^{2}	
0.8	0.17397 10^{2}		
0.9	0.86377 10^{1}		
1.1	0.36815 10^{1}		
2.0	0.84223		
3.0	0.41388		
4.0	0.26017	0.25000	
5.0	0.18346	0.17888	
6.0	0.13848	0.13608	
7.0	0.10938	0.10799	eq. (3.19)
8.0	0.08926	0.08839	
9.0	0.07465	0.07407	
10.0	0.06364	0.06325	

THEOREM 3. $\Phi(A)$ is relatively compact.

PROOF. Let $f \in A$, then $\|f\| \le m(s)$, $\|\tilde{f}\| \le m(s)$, $\|\tilde{f}'\| \le 1/s$. According to the proof of prop. 3.10 and according to the cor. of prop. 3.5.

$$0 \le \int_x^\infty (2g-g^2) \le \int_x^\infty (2e^{\mu_2 \xi})d\xi = \frac{-2}{\mu_2}\, e^{\mu_2 x}.$$

So

$$\left\| \int^\infty (2g-g^2)\right\| \le -\frac{2}{\mu_2}.$$

Now $\mu_2 = -\frac{1}{2}f(+\infty) - \frac{1}{2}\sqrt{f^2(+\infty)+4s}$. This function is decreasing for $f(+\infty) < 0$. It follows that

$$-\frac{2}{\mu_2} \le -\frac{2}{\mu_2(-m(s))}$$

where $\mu_2(-m(s))=\frac{1}{2}m(s)-\frac{1}{2}\sqrt{m(s)^2+4s}$. With these inequalities (3.3) yields

$$\|\tilde{f}''\| \leq M(s)$$

where $M(s)$ is a number >0 only dependent on s.

Ascoli's theorem gives us the theorem.

We can now apply the cor. of the Schauder-Tychonoff-theorem in order to find a solution of (1.3) and (1.4).

4. ON THE UNIQUENESS OF SOLUTIONS

In order to prove uniqueness we suppose that there are two solutions f_1 and f_2 of the system (1.4), with the maximum norm of the difference of $\tilde{f}_1$ and $\tilde{f}_2$ denoted by ε. We shall have proven uniqueness if the maximum norm of the difference of the images of $\tilde{f}_1$ and $\tilde{f}_2$ is smaller than ε.

Remark, that any solution lies in a subclass of A as defined in section 3.

We first need a number of preliminaries and lemmas. We consider two arbitrary functions f_1 and f_2 belonging to A. We denote $g(f_i)$ by g_i and define

$$F=f_2-f_1,\ G=g_2-g_1,\ \tilde{F}=\tilde{f}_2-\tilde{f}_1.$$

REMARK. g is here the original g and not $\tilde{g}$.

It is not difficult to derive from eqs. (3.1) that (see prop. 3.5)

(4.1)a $\quad -G''-f_2G'+sG=Fg_1'$

with the boundary conditions

(4.1)b $\quad G(0)=0$

and

(4.2)a $\quad -\tilde{F}''-f_2\tilde{F}'+s\tilde{F}=\tilde{F}f_1'-\int^{\infty} G(g_2+g_1)$

with

(4.2)b $\quad \tilde{F}'(0)=0$ and $\tilde{F}(+\infty)=0$.

First we consider the following equation

(4.3) $\quad -G''-f_2G'+sG=\beta>0.$

According to proposition 3.4 and the boundary condition (4.1)b the solution has to be positive. We now consider an arbitrary α such that $\beta>|\alpha|$. It will be immediately clear that $G_{-\alpha}=-G_\alpha$. Now $\beta\pm\alpha>0$ and thus the differences $G_\beta\pm G_\alpha\geq 0$. This leads to

$$|G_\beta|\geq|G_\alpha|.$$

We have thus proved the following lemma

LEMMA 4.1. If $\beta>|\alpha|$ then $|G_\beta|\geq|G_\alpha|$.

We now consider the equation

$$-\tilde{F}''-f_2\tilde{F}'+s\tilde{F}=\beta>0$$

with the boundary conditions (4.2)b. According to proposition 3.7 we have $\tilde{F}\geq 0$. By an analogous proof as for G we immediately find

LEMMA 4.2. If $\beta>|\alpha|$ then $|\tilde{F}_\beta|\geq|\tilde{F}_\alpha|$.

Since for an arbitrary f_2 belonging to A it may be very hard to find a solution of these equations we need further approximations. To derive these approximations we proceed as follows. As an example we take the equation for G. From the forgoing discussion we know that the same applies for $\tilde{F}$.

We start from eq. (4.3) and apply the transformation

$$H=G\exp\left(\tfrac{1}{2}\int_0 f_2\right). \tag{4.4}$$

Then we have

$$-H''+\left(s+\tfrac{1}{2}f_2'+\tfrac{1}{4}f_2^2\right)H=\beta\exp\left(\tfrac{1}{2}\int_0 f_2\right). \tag{4.5}$$

To find an estimate for H we remark the following.

Consider the two equations

$$-H_1''+\gamma_1 H_1=X>0$$

$$-H_2''+\gamma_2 H_2=X$$

with $\gamma_1>\gamma_2>0$, then $H_1\geq 0$ (see prop. 2.1).

Subtracting the two equations gives

$$-(H_2-H_1)''+\gamma_2(H_2-H_1)=(\gamma_2-\gamma_1)H_1. \tag{4.6}$$

The right hand side is negative and since $\gamma_2>0$, this means that $H_2-H_1\leq 0$.

This leads immediately to the following lemma.

LEMMA 4.3. If $\gamma_1>\gamma_2>0$, then $|H_1|\leq|H_2|$.

For $\tilde{F}$ the same reasoning applies.

Now we can apply the foregoing lemmas to construct estimates for $\|\tilde{F}\|$.

First we remark that for every member of A there holds $f'\geq-(1/s)$ and $f^2\geq 0$ and hence

$$s+\tfrac{1}{2}f'+\tfrac{1}{4}f^2\geq s-\frac{1}{2s}=s^*. \tag{4.7}$$

It will be immediately clear that by doing this, we restrict ourselves to cases where $s>\frac{1}{2}\sqrt{2}$ and that we should need better estimates for $s+\frac{1}{2}f'+\frac{1}{4}f^2$ to treat smaller values of s. However this we will not consider.

REMARK. It should be observed that the idea for the construction of the function H stems from an asymptotical analysis. In that case of course $s^*\sim s$ and the uniqueness is readily proved.

We start from eq. (4.1)a and write

$$(4.8)\qquad -G''-f_2G'+sG=Fg_1'\le 2\varepsilon\sqrt{s}(1-g_1)\le 2\varepsilon\sqrt{s}e^{\mu_2 x}$$

where use has been made of the inequalities

$$\|F\|=\|f_2-f_1\|=\|\tilde{f}_2-\tilde{f}_1-(\tilde{f}_2(0)-\tilde{f}_1(0))\|\le 2\varepsilon$$

and $g'\le\sqrt{s}(1-g)$ which has been derived in ref. (7) eq. (2.8)a. and the corollary of prop. 3.5. Observe that μ_2 denotes $\mu_2(f_1(+\infty))$.

We now write

$$(4.9)\qquad -H''+s^*H=2\varepsilon\sqrt{s}e^{\mu_2 x}\cdot\exp\cdot(\tfrac{1}{2}\int_0^x f_2)\le 2\varepsilon\sqrt{s}e^{\mu_2 x}=u.$$

According to lemmas 4.1 and 4.3 we know that

$$|H|\ge|G|\ \exp\ (\tfrac{1}{2}\int_0 f_2).$$

By using eq. (3.6), (3.9) and (3.10) and realizing that $\phi_1=\sqrt{s^*}$ in the case of eq. (4.9) we find

$$(4.10)\qquad H=e^{-\sqrt{s^*}x}\int_0^x e^{2\sqrt{s^*}\lambda}\int_\lambda^\infty e^{-\sqrt{s^*}v}u\,dv\,d\lambda.$$

Performing the integration we get

$$(4.11)\qquad H=\frac{2\varepsilon\sqrt{s}}{(\mu_2+\sqrt{s^*})(\sqrt{s^*}-\mu_2)}\,(e^{\mu_2 x}-e^{-\sqrt{s^*}x}).$$

This is indeed a positive quantity for all μ_2 and $\sqrt{s^*}$.

Since as will be shown in the appendix for the s values considered $\mu_2>-\sqrt{s^*}$ we apply the inequality $1-e^{(-\mu_2-\sqrt{s^*})x}\le x(\mu_2+\sqrt{s^*})$ to obtain

$$(4.12)\qquad |H|\le\frac{2\varepsilon\sqrt{s}}{\sqrt{s^*}-\mu_2}\,xe^{+\mu_2 x}\ \text{and hence}\ |G|=|H|e^{-\frac{1}{2}\int_0 f_2}\le\frac{2\varepsilon\sqrt{s}}{\sqrt{s^*}-\mu_2}\,xe^{\{\mu_2+\frac{1}{2}|f_\infty|\}x}$$

where $f_2(+\infty)$ is denoted by f_∞.

We can now proceed to obtain an estimate for $\|\tilde{F}\|$ from eqs. (4.2)a–b. Using the fact that $\|F\|\le 2\varepsilon$ and $(g_1+g_2)<2$ we have to consider

$$(4.13)\qquad -\tilde{F}''-f_2\tilde{F}'+s\tilde{F}=2\varepsilon|\tilde{f}_1'|+2\int^\infty|G|.$$

As will be clear, we need an estimate for $|\tilde{f}_1'|$. To obtain one we differentiate eq. (3.3). The result is

$$(4.14)\qquad -\tilde{f}'''-f\tilde{f}''+(s-f')\tilde{f}'=-2\tilde{g}+\tilde{g}^2\ge-2\tilde{g}.$$

We know that $\tilde{f}'(0)=0$, $s-f'\ge s$ and $-2\tilde{g}+\tilde{g}^2\le 0$.

From the lemmas 4.1 and 4.3 it then follows

$$|-\tilde{f}'|\le|H_1|e^{\frac{1}{2}|f_\infty|x}$$

where H_1 is the solution of

$$-H_1''+s^*H_1=2e^{\mu_2 x}.$$

This is completely analogous to eq. (4.9) and leads immediately to

(4.15) $$|-\tilde{f}'| < \frac{2}{\sqrt{s^*}-\mu_2}\, x e^{\{\mu_2+\frac{1}{2}|f_\infty|\}x}.$$

By using eq. (4.12), (4.15) and inserting in (4.13) and by applying that $\mu_2 > -\sqrt{s^*} > -\sqrt{s}$, we find after integrating and using again lemma's 4.2 and 4.3 that

$$|F| < |H_2| e^{\frac{1}{2}|f_\infty|x}$$

where H_2 is the solution of

(4.16) $$-H_2'' + s^* H_2 = \frac{2\varepsilon\sqrt{s}}{\mu_2(\mu_2+\frac{1}{2}|f_\infty|)}\, e^{(\mu_2+\frac{1}{2}|f_\infty|)x}\left\{2x - \frac{1}{\mu_2+\frac{1}{2}|f_\infty|}\right\}$$

where the right hand side is positive; at least for $s>1$ as follows from the estimates in the appendix.

We have to apply the boundary condition

$$H_2'(0)=0.$$

It is a simple matter to calculate from eqs. (3.9) and (3.10) that if the right hand side of eq. (4.16) is denoted by R, the function H_2 is given by

$$H_2 = \frac{e^{\sqrt{s^*}x}}{2\sqrt{s^*}}\int_x^\infty Re^{-\sqrt{s^*}\lambda}d\lambda + \frac{e^{-\sqrt{s^*}x}}{2\sqrt{s^*}}\int_0^\infty Re^{-\sqrt{s^*}\lambda}d\lambda + \frac{e^{-\sqrt{s^*}x}}{2\sqrt{s^*}}\int_0^x Re^{\sqrt{s^*}\lambda}d\lambda.$$

We need an estimate for $|H_2|e^{\frac{1}{2}|f_\infty|x}$.

Denoting the maximum of $Re^{\frac{1}{2}|f_\infty|x}$ by M we find that

(4.17) $$\max_{\mathbb{R}+} |\tilde{F}| < \frac{1}{\sqrt{s^*}}\,\frac{M}{\sqrt{s^*}-\frac{1}{2}|f_\infty|}$$

where M can be given by

(4.18) $$M = \frac{-4\varepsilon\sqrt{s}}{\mu_2(\mu_2+\frac{1}{2}|f_\infty|)(\mu_2+|f_\infty|)}\, e^{-1/2}.$$

It will be clear that as long as

(4.19) $$-4\sqrt{\frac{s}{s^*}}\,\frac{e^{-1/2}}{\mu_2(\mu_2+\frac{1}{2}|f_\infty|)(\mu_2+|f_\infty|)(\sqrt{s^*}-\frac{1}{2}|f_\infty|)} < 1$$

then there holds

(4.20) $$\|\tilde{F}\| = \max_{\mathbb{R}+} |\tilde{F}| < \varepsilon.$$

But in that case the original functions f_1 and f_2 can be no solutions of our problem, for then there should hold

$$\|\tilde{F}\| = \varepsilon.$$

So for all values for which (4.19) is valid we have thus proved uniqueness of the solution.

It is easy to deduct that the left hand side of eq. (4.19) is a function which increases with $|f_\infty|=|f_2(+\infty)|$ and with $|f_1(\infty)|$ occurring in μ_2. Hence if we know an upper bound for $|f(+\infty)|$ we can calculate a maximum value of this left hand side as a function of s. An upper bound for $|f(+\infty)|$ has been given in the appendix. It turns out that as long as $s>2.33$ unequality (4.19) is fulfilled. This means that the solution has been proven to be unique for

$$s>2.33.$$

It should be clear that we have used a number of rather rough estimates. So a more carefull analysis would yield a lower value of s. But to extend the range to $s=0$ we need other techniques.

APPENDIX

In section 4, we have used the fact that $\mu_2>-\sqrt{s^*}$. This should at least be true for $s>2.16$.

Now

$$-\mu_2=\frac{s}{\frac{1}{2}\sqrt{f_\infty^2+4s}+\frac{1}{2}|f_\infty|}<\frac{s}{\frac{1}{2}\sqrt{\frac{25}{36s^3}+4s}+\frac{5}{12}\frac{1}{s\sqrt{s}}}=R_2$$

where we have used the fact that

$$|f_\infty|>\frac{5}{6s\sqrt{s}} \text{ as proved in ref. 7.}$$

There should hold that

A(1) $$\sqrt{s-\frac{1}{25}}>R_2.$$

A simple calculation learns that this certainly is true for $s>1$.

The second thing we should consider is securing an upper bound for $|f_\infty|$. Such bounds also have been given in ref. 7. However we will give here another one which is asymptotically correct for $s\to\infty$, but which breaks down for *sm* 1,2.

Starting from eq. (3.3), using the fact that $-f_\infty\tilde{f}'<-f\tilde{f}'$ and the cor. of prop. 3.5 for $\tilde{g}$ we find

$$-\tilde{f}''-f_\infty\tilde{f}'+s\tilde{f}<\int_x^\infty(2e^{\mu_2\lambda}-e^{2\mu_2\lambda})d\lambda.$$

Multiplying left- and righthand side with $e^{-\varrho x}$ where $\varrho=\mu-f_\infty$ it is readily found that

$$-\mu_2|f_\infty|<\int_0^\infty e^{-\varrho x}\int_x^\infty(2e^{\mu_2\lambda}-e^{2\mu_2\lambda})d\lambda.$$

Performing the integration leads to the equation.

A(2) $$|f_\infty| < \frac{1}{\mu_2^2} \frac{2f_\infty + 5\sqrt{f_\infty^2 + 4s}}{\sqrt{f_\infty^2 + 4s}(f_\infty + 3\sqrt{f_\infty^2 + 4s})}.$$

From this inequality an upper bound for $|f_\infty|$ can be calculated.

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