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Equations of motion for Hamiltonian systems with constraints

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Abstract. In this paper the problem of obtaining the equations of motion for Hamiltonian systems with constraints is considered. Conditions are given which ensure that the phase space points satisfying the primary and secondary constraints form a symplectic manifold, on which the resulting equations of motion are Hamiltonian and uniquely determined.

1. Introduction

In a classic paper ([1], see also [2]) Dirac considered the problem of converting the equations of motion given in Lagrangian form into equations in Hamiltonian form, in the case where the Lagrangian is degenerate. Recall that from a Lagrangian function $L(q, \dot{q})$, with $q=(q_1, \ldots, q_n)$ and $\dot{q}=(\dot{q}_1, \ldots, \dot{q}_n)$, we obtain the dynamical equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=\frac{\partial L}{\partial q_i} \qquad i=1, \ldots, n. \tag{1}$$

To pass over to the Hamiltonian formalism the momentum variables $p=(p_1, \ldots, p_n)$ are introduced by

$$p_i=\partial L/\partial \dot{q}_i \qquad i=1, \ldots, n. \tag{2}$$

Usually it is assumed that the mapping from $\dot{q}$ to p defined by (2) (the Legendre transformation) is a local diffeomorphism, or equivalently that the $n \times n$ matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \qquad i, j=1, \ldots, n \tag{3}$$

has everywhere full rank. In the case where this assumption is not satisfied L is called degenerate, and the definition of the momenta in (2) does not yield n independent variables $p_1, \ldots, p_n$. Dirac considered the case that the rank of the matrix (3) is everywhere $n-m$, and so there exist m (smooth) relations connecting the momentum variables

$$\phi_s(q, p)=0 \qquad s=1, \ldots, m. \tag{4}$$

Equations (4) are called the primary constraints. As in the non-degenerate case the Hamiltonian is defined as

$$H=\sum_{i=1}^{n} p_i \dot{q}_i-L(q, \dot{q}) \tag{5}$$

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and it can be easily seen (e.g. by making variations in q and $\dot{q}$ and using (2)) that H is only a function of q and p. However *a priori* the Hamiltonian is not uniquely defined since we may add to H any linear combination of the functions ϕ_s (which have to be zero):

$$H^*(q, p) = H(q, p) + \sum_{j=1}^{m} u_j \phi_j(q, p) \tag{6}$$

where u_j are coefficients which *a priori* can be any function of q and p. As Dirac argues, H^* is as good as H and the theory cannot distinguish between H and H^*.

From variational calculus [1, 2] it follows that the dynamical equations of motion corresponding to H^* are given as

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} + \sum_{j=1}^{m} u_j \frac{\partial \phi_j}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} - \sum_{j=1}^{m} u_j \frac{\partial \phi_j}{\partial q_i} \end{aligned}\right. \qquad i = 1, \ldots, n. \tag{7}$$

As a consequence of (7) the functions u_j are in general not completely arbitrary but have to satisfy some consistency conditions. These are obtained by noting that the primary constraints (4) have to be satisfied for any time, and hence all the (repeated) time derivatives of the functions ϕ_s along the equations of motion (7) have to be zero. This gives rise to the following algorithm given by Dirac [1, 2]. The first time derivative $\dot{\phi}_s$ of a function ϕ_s along (7) is given as

$$\dot{\phi}_s = \{H, \phi_s\} + \sum_{j=1}^{m} u_j \{\phi_j, \phi_s\} \tag{8}$$

where { } denotes the usual Poisson bracket of functions of q and p. This results in m consistency conditions $\dot{\phi}_s = 0$. If for a certain s the expressions $\{\phi_j, \phi_s\}$ are identically zero for all $j = 1, \ldots, m$, then the condition $\dot{\phi}_s = 0$ reduces to the condition $\{H, \phi_s\}\ (q, p) = 0$ only involving the q and p variables. Such a condition is called a secondary constraint, since it is of the same type as the primary constraints. In general, if l_s is the smallest positive integer such that all the kth time derivatives $\phi_s^{(k)}$ of ϕ_s for $k = 1, \ldots, l_s$ do not involve the functions u_j, then we obtain (see § 2) the secondary constraints

$$\phi_s^{(k)} = \{H, \phi_s^{(k-1)}\} = 0 \qquad k = 1, 2, \ldots, l_s \tag{9}$$

together with consistency conditions involving u_j variables:

$$\phi_s^{(l_s+1)} = \{H, \phi_s^{(l)}\} + \sum_{j=1}^{m} u_j \{\phi_j, \phi_s^{(l_s)}\}. \tag{10}$$

By doing this for any $s = 1, \ldots, m$, we finally obtain a sequence of secondary constraints (9), together with remaining equations of the form (10), which have to be solved for the unknown functions u_j yielding a (not necessarily unique) solution $\bar{u}_1(q, p), \ldots, \bar{u}_m(q, p)$. We end up with a 'phase space' defined by the primary and secondary constraints (4) and (9), and a (not necessarily unique) Hamiltonian H^* as in (6) defined by a solution $\bar{u}_1, \ldots, \bar{u}_m$. This algorithm was cast into a more abstract algebraic and geometric form by Gotay *et al* [3, 4], thereby elucidating, among other things, the global geometric aspects of the algorithm and making it more applicable to infinite-dimensional Hamiltonian systems. (For a geometrical treatment of the concepts of Dirac constraint theory we also refer to [5-7].)

As was already noted by Dirac, the algorithm sketched above does not always yield a well defined set of resulting equations, since the set of primary and secondary constraints may define an inconsistent system of equations and there may not always exist a solution $\bar{u}_1, \ldots, \bar{u}_m$. Furthermore, in general it is not clear what the structure of the resulting 'phase space' will be. The purpose of this paper is to give conditions on H and the functions ϕ_s which immediately ensure the existence of a well defined resulting 'phase space' and a Hamiltonian H^*. Furthermore, these conditions will imply that the space defined by the primary and secondary constraints is a symplectic manifold (and so really can be interpreted as a phase space) and that $\bar{u}_1, \ldots, \bar{u}_m$, and therefore H^*, are uniquely defined on this space.

Let us note that the above problem considered by Dirac is an example of the more general problem of Hamiltonian dynamics with constraints. In this case the Hamiltonian H as well as the constraint functions ϕ_s are arbitrary functions of q and p. The functions u_j are interpreted as constraint forces (see for example [8]) or, from the point of view of variational calculus, as Lagrange multipliers. Indeed, the theorem of the next section applies to this more general case.

Finally we mention that we were actually led to this theorem by looking at the equations of motion (7) as a Hamiltonian control system (cf [9, 10]). In this interpretation the functions u_j, $j=1, \ldots, m$, are viewed as (arbitrary) control time functions. The resulting solution functions $\bar{u}_j$, which depend through q and p on time, are obtained as the feedback which makes the subset defined by the primary and secondary constraints invariant [11]. (In the control literature such a subset is called controlled invariant [12, 13]).

2. Main theorem

Consider a $2n$-dimensional symplectic manifold M with symplectic form ω and canonical coordinates (q, p), i.e. locally $\omega = \Sigma_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i$. Let $H(q, p)$ and $\phi_s(q, p)$, $s=1, \ldots, m$, be arbitrary smooth functions on M. Define as usual for any two functions their Poisson bracket (locally) as

$$\{F(q,p), G(q,p)\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right)(q,p). \tag{11}$$

Define inductively

$$\begin{aligned} &\mathrm{ad}_H^0\, \phi_s = \phi_s \\ &\mathrm{ad}_H^k\, \phi_s = \{H, \mathrm{ad}_H^{k-1}\, \phi_s\} \qquad k=1,2,\ldots. \end{aligned} \tag{12}$$

In addition, for any $s=1, \ldots, m$, define ρ_s as the smallest non-negative integer such that for some $(q,p) \in M$ there exists a $j \in \{1, \ldots, m\}$ for which $\{\phi_j, \mathrm{ad}_H^{\rho_s}\, \phi_s\}(q,p) \neq 0$. (Of course, ρ_s may not exist, i.e. ρ_s may equal ∞.) A submanifold N of M will be called a symplectic submanifold if the symplectic form ω restricted to N, denoted $\omega|_N$, defines a symplectic form on N. In particular N is then a symplectic manifold in its own right.

Theorem. Assume that $\rho_s < \infty$, $s=1, \ldots, m$. Define the $m \times m$ matrix $A(q,p)$ with (i,j)th element

$$A_{ij}(q,p) = \{\phi_j, \mathrm{ad}_H^{\rho_i}\, \phi_i\}(q,p). \tag{13}$$

Assume that the rank of $A(q,p)$ is equal to m for all (q,p) satisfying the primary constraints (4). Then the secondary constraints are given as

$$\mathrm{ad}_H^{k_s}\,\phi_s(q,p)=0 \qquad k_s=1,\ldots,\rho_s,\ s=1,\ldots,m. \tag{14}$$

Furthermore, the primary and secondary constraints are independent functions for any (q,p) satisfying the primary constraints, defining a resulting phase space

$$N=\{(q,p)\in M \mid \mathrm{ad}_H^{k_s}\,\phi_s(q,p)=0,\ k_s=0,1,\ldots,\rho_s,\ s=1,\ldots,m\} \tag{15}$$

which is a symplectic submanifold of the original phase space M. The resulting equations of motion on N are uniquely determined and are given as the Hamiltonian equations of motion corresponding to the Hamiltonian $H|_N$ and symplectic form $\omega|_N$ on N, where $H|_N$ denotes restriction of H to N.

Proof. (See also [11].) If $\rho_s=0$ then $\dot\phi_s=\{H,\phi_s\}+\Sigma_{j=1}^m u_j\{\phi_j,\phi_s\}$. Now suppose $\rho_s\geq 1$. We prove by induction that, for $k_s=1,\ldots,\rho_s$, the time derivatives $\phi_s^{(k_s)}$ along (7) are equal to $\mathrm{ad}_H^{k_s}\,\phi_s$. Indeed for $k_s=1$ we have

$$\dot\phi_s=\{H,\phi_s\}+\sum_{j=1}^m u_j\{\phi_j,\phi_s\}=\{H,\phi_s\} \tag{16}$$

since $\rho_s\geq 1$. Now assume that $\phi_s^{(k_s-1)}=\mathrm{ad}_H^{(k_s-1)}\,\phi_s$; then

$$\begin{aligned}\phi_s^{(k_s)}&=\{H,\phi_s^{(k_s-1)}\}+\sum_{j=1}^m u_j\{\phi_j,\phi_s^{(k_s-1)}\}\\ &=\{H,\mathrm{ad}_H^{(k_s-1)}\,\phi_s\}+\sum_{j=1}^m u_j\{\phi_j,\mathrm{ad}_H^{(k_s-1)}\,\phi_s\}\\ &=\{H,\mathrm{ad}_H^{(k_s-1)}\,\phi_s\}=\mathrm{ad}_H^{k_s}\,\phi_s\end{aligned} \tag{17}$$

since $k_s-1<\rho_s$.

Furthermore, the (ρ_s+1)th time derivatives are given as

$$\begin{pmatrix}\phi_1^{(\rho_1+1)}\\ \cdots\\ \phi_m^{(\rho_m+1)}\end{pmatrix}(q,p)=\begin{pmatrix}\{H,\mathrm{ad}_H^{\rho_1}\,\phi_1\}\\ \cdots\\ \{H,\mathrm{ad}_H^{\rho_m}\,\phi_m\}\end{pmatrix}(q,p)+A(q,p)\begin{pmatrix}u_1\\ \cdots\\ u_m\end{pmatrix} \tag{18}$$

where $A(q,p)$ is the $m\times m$ matrix introduced in (13). Since rank $A(q,p)=m$ for (q,p) satisfying (4) the equations $\phi_1^{(\rho_1+1)}(q,p)=\ldots=\phi_m^{(\rho_m+1)}(q,p)=0$ have a unique solution $\bar u_1(q,p),\ldots,\bar u_m(q,p)$ for these (q,p) and the secondary constraints are given as in (14).

To prove the independence of the primary and secondary constraints we use the following argument given in [13]. Take an arbitrary point (q,p) satisfying (4). Consider the differentials $\mathrm{d}(\mathrm{ad}_H^{k_s}\,\phi_s)$, $k_s=0,1,\ldots,\rho_s$, $s=1,\ldots,m$, at (q,p). Suppose there exist constants c_{sk} such that

$$\sum_{s=1}^m\sum_{k=0}^{\rho_s} c_{sk}\,\mathrm{d}(\mathrm{ad}_H^k\,\phi_s)(q,p)=0. \tag{19}$$

We shall prove that all these constants c_{sk} are zero. Consider, as in [13], the function

$$\lambda(q,p)=\sum_{s=1}^m\sum_{k=0}^{\rho_s} c_{sk}\,\mathrm{ad}_H^k\,\phi_s(q,p). \tag{20}$$

According to the definition of $\rho_1, \ldots, \rho_m$ we have

$$\{\phi_j, \lambda\} = \sum_{s=1}^{m} \sum_{k=0}^{\rho_s} c_{sk}\{\phi_j, \mathrm{ad}_H^k \phi_s\}$$

$$= \sum_{s=1}^{m} c_{s\rho_s}\{\phi_j, \mathrm{ad}_H^{\rho_s} \phi_s\} = \sum_{s=1}^{m} c_{s\rho_s} A_{sj}. \tag{21}$$

Since $\mathrm{d}\lambda(q, p) = 0$ by (19) it follows that $\{\phi_j, \lambda\}(q, p) = 0$ and so

$$\sum_{s=1}^{m} c_{s\rho_s} A_{sj}(q, p) = 0 \qquad j = 1, \ldots, m. \tag{22}$$

Hence, because rank $A(q, p) = m$, we must have

$$c_{1\rho_1} = c_{2\rho_2} = \ldots = c_{m\rho_m} = 0 \tag{23}$$

and so $\lambda(q, p) = \Sigma_{s=1}^{m} \Sigma_{k=0}^{\rho_s - 1} c_{sk}\, \mathrm{ad}_H^k\, \phi_s(q, p)$. Consequently, from the definition of ρ_i it follows that $\{\phi_j, \lambda\}(q, p) = 0$ for all $(q, p) \in M$ and $j = 1, \ldots, m$. By the Jacobi identity we have

$$\{\{\phi_j, H\}, \lambda\} = \{\phi_j, \{H, \lambda\}\} + \{\{\phi_j, \lambda\}, H\}. \tag{24}$$

Hence, since $\{\phi_j, \lambda\} = 0$,

$$\{\{\phi_j, H\}, \lambda\} = \{\phi_j, \{H, \lambda\}\} = \left\{\phi_j, \sum_{s=1}^{m} \sum_{k=0}^{\rho_s - 1} c_{sk}\, \mathrm{ad}_H^{k+1}\, \phi_s\right\}$$

$$= \sum_{s=1}^{m} c_{s,\rho_s - 1}\{\phi_j, \mathrm{ad}_H^{\rho_s}\, \phi_s\} = \sum_{s=1}^{m} c_{s,\rho_s - 1} A_{sj}. \tag{25}$$

However, since $\mathrm{d}\lambda(q, p) = 0$ it follows that $\{\{\phi_j, H\}, \lambda\}(q, p) = 0$ and so

$$\sum_{s=1}^{m} c_{s,\rho_s - 1} A_{sj}(q, p) = 0 \qquad j = 1, \ldots, m. \tag{26}$$

Because rank $A(q, p) = m$ we conclude that

$$c_{1,\rho_1 - 1} = \ldots = c_{m,\rho_m - 1} = 0. \tag{27}$$

By repeating the above procedure we obtain that all the constants c_{sk} are zero in (19), and hence the constraint functions $\mathrm{ad}_H^{k_s}\, \phi_s$, $k_s = 0, 1, \ldots, \rho_s$, $s = 1, \ldots, m$, are all independent in every point (q, p) satisfying (4). Consequently, the set N defined in (15) is a smooth submanifold, which is either empty or has dimension $2n - \Sigma_{s=1}^{m}(\rho_s + 1)$.

In the first case N is trivially symplectic. To prove that in the second case N is symplectic, we first note that by the Jacobi identity we have for any $i, j = 1, \ldots, m$

$$\{\phi_j, \mathrm{ad}_H^{\rho_i}\, \phi_i\} = -\{\{H, \phi_j\}, \mathrm{ad}_H^{\rho_i - 1}\, \phi_i\} + \{H, \{\phi_j, \mathrm{ad}_H^{\rho_i - 1}\, \phi_i\}\}. \tag{28}$$

By definition of ρ_i the last term is zero in any (q, p) and inductively we obtain

$$\{\phi_j, \mathrm{ad}_H^{\rho_i}\, \phi_i\}(q, p) = (-1)^k \{\mathrm{ad}_H^k\, \phi_j, \mathrm{ad}_H^{\rho_i - k}\, \phi_i\}(q, p) \qquad k = 0, 1, \ldots, \rho_i \tag{29}$$

for any (q, p). From now on we only consider points (q, p) satisfying (4). Apply a permutation such that $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_m$. First suppose that $\rho_1 > \rho_2 > \ldots > \rho_m$. Then it follows that

$$\{\phi_j, \mathrm{ad}_H^{\rho_i}\, \phi_i\} = (-1)^{\rho_i} \{\mathrm{ad}_H^{\rho_i}\, \phi_j, \phi_i\} = 0 \qquad \text{for } j < i. \tag{30}$$

Hence $A_{ij}(q,p)=0$ for $j<i$ and so $A(q,p)$ is an upper triangular matrix. Since by assumption $A(q,p)$ is non-singular for (q,p) satisfying (4) it follows that the diagonal elements $\{\phi_i, \mathrm{ad}_H^{\rho_i}\phi_i\}(q,p)$, $i=1,\ldots,m$, are non-zero. By (29) this implies that $\{\mathrm{ad}_H^k\phi_i, \mathrm{ad}_H^{\rho_i-k}\phi_i\}(q,p)\neq 0$ for $k=0,1,\ldots,\rho_i$. Hence for any constraint function $\mathrm{ad}_H^k\phi_i$ there exists another constraint function $\mathrm{ad}_H^{\rho_i-k}\phi_i$ such that the Poisson bracket is non-zero. Now suppose that $\rho_1=\rho_2>\rho_3>\ldots>\rho_m$. By the same argument $A(q,p)$ has the form

$$\begin{pmatrix} * & * & \ldots & \ldots & * \\ * & * & \ldots & \ldots & \ldots \\ 0 & 0 & * & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & * \end{pmatrix} \tag{31}$$

where the 2×2 submatrix $(\{\phi_j, \mathrm{ad}_H^{\rho_i}\phi_i\})_{i,j=1,2}$ has rank 2. Now take a fixed point (q,p) satisfying (4). If $\{\phi_i, \mathrm{ad}_H^{\rho_i}\phi_i\}(q,p)\neq 0$, $i=1,2$, we are in the same situation as above. If $\{\phi_1, \mathrm{ad}_H^{\rho_1}\phi_1\}(q,p)=0$ then necessarily $\{\phi_2, \mathrm{ad}_H^{\rho_1}\phi_1\}(q,p)\neq 0$. But since $\rho_2=\rho_1=\rho$ this implies by (29) that $\{\mathrm{ad}_H^k\phi_2, \mathrm{ad}_H^{\rho-k}\phi_1\}(q,p)\neq 0$ for $k=0,1,\ldots,\rho$. Hence, again, for any constraint function $\mathrm{ad}_H^k\phi_1$ or $\mathrm{ad}_H^k\phi_2$ there exists another constraint function such that the Poisson bracket in (q,p) is non-zero. If more integers ρ_i are equal then we proceed in the same way by looking at the corresponding non-singular submatrix of A.

Concluding, in any point (q,p) satisfying the primary constraints (4), there exists for any (primary or secondary) constraint function $\mathrm{ad}_H^k\phi_i$, $k=0,1,\ldots,\rho_i$, $i=1,\ldots,m$, another constraint function $\mathrm{ad}_H^l\phi_j$, $l\leqslant\rho_j$, $j\in\{1,\ldots,m\}$, such that $\{\mathrm{ad}_h^l\phi_j, \mathrm{ad}_H^k\phi_i\}(q,p)\neq 0$. This implies that N is a symplectic submanifold as follows.

Take a point $(q,p)\in N$ and an arbitrary tangent vector X to N in (q,p), i.e. $X\in T_{(q,p)}N$. Hence $X(\mathrm{ad}_H^{k_s}\phi_s)(q,p)=0$ for $k_s=0,1,\ldots,\rho_s$, $s=1,\ldots,m$. Denote the Hamiltonian vector fields on M corresponding to the constraint functions $\mathrm{ad}_H^k\phi_i$ by $X_{\mathrm{ad}_H^k\phi_i}$. Then

$$\omega(X, X_{\mathrm{ad}_H^{k_s}\phi_s})(q,p)=X(\mathrm{ad}_H^{k_s}\phi_s)(q,p)=0 \qquad k_s=0,1,\ldots,\rho_s,\ s=1,\ldots,m. \tag{32}$$

Hence the vectors $X_{\mathrm{ad}_H^{k_s}\phi_s}(q,p)$, $k_s=0,1,\ldots,\rho_s$, $s=1,\ldots,m$, are all elements of

$$(T_{(q,p)}N)^\perp := \{Y\in T_{(q,p)}M \mid \omega(X,Y)=0 \text{ for any } X\in T_{(q,p)}N\}.$$

Since the functions $\mathrm{ad}_H^{k_s}\phi_s$ are independent and $\dim(T_{(q,p)}N)^\perp=2n-\dim T_{(q,p)}N=\Sigma_{s=1}^m(\rho_s+1)$ it follows that the vectors $X_{\mathrm{ad}_H^{k_s}\phi_s}(q,p)$, $k_s=0,1,\ldots,\rho_s$, $s=1,\ldots,m$, form a basis of $(T_{(q,p)}N)^\perp$. We just proved above that for any constraint function $\mathrm{ad}_H^k\phi_i$ there exists another constraint function $\mathrm{ad}_H^l\phi_j$ such that the Poisson bracket is non-zero, and therefore

$$\omega(X_{\mathrm{ad}_H^k\phi_i}, X_{\mathrm{ad}_H^l\phi_j})(q,p)=\{\mathrm{ad}_H^k\phi_i, \mathrm{ad}_H^l\phi_j\}(q,p)\neq 0. \tag{33}$$

Using the fact that the vectors $X_{\mathrm{ad}_H^{k_s}\phi_s}(q,p)$ form a basis of $(T_{(q,p)}N)^\perp$ it immediately follows that for an arbitrary vector $X\in(T_{(q,p)}N)^\perp$ there exists a vector $Y\in(T_{(q,p)}N)^\perp$ such that $\omega(X,Y)\neq 0$. Hence the intersection of $(T_{(q,p)}N)^\perp$ with $((T_{(q,p)}N)^\perp)^\perp$ is zero, i.e. $T_{(q,p)}N\cap(T_{(q,p)}N)^\perp=0$. However this is also equivalent to the fact that for any $X\in T_{(q,p)}N$ there exists $Y\in T_{(q,p)}N$ such that $\omega(X,Y)\neq 0$. Therefore the symplectic form ω restricted to N, $\omega|_N$, is non-degenerate. Closedness immediately follows from closedness of ω, and so $\omega|_N$ defines a symplectic form on N. Finally note that the solution $\bar{u}_1, \ldots, \bar{u}_m$ of (18) on N is uniquely determined. Hence the equations of motion (7) on N are uniquely determined. Furthermore they are given as the Hamiltonian equations of motion corresponding to the Hamiltonian $H^* = H + \Sigma_{j=1}^m \bar{u}_j \phi_j$. Since restricted to N the Hamiltonian H^* is equal to H, it follows that the equations of motion on N are also given as the Hamiltonian equations of motion corresponding to the Hamiltonian $H|_N$ and symplectic form $\omega|_N$.

In conclusion, the above theorem states sufficient conditions under which a Hamiltonian system with constraints on a symplectic manifold M yields a uniquely determined Hamiltonian system without constraints on a lower-dimensional symplectic manifold N. For the case of Hamiltonian systems with constraints as arising from degenerate Lagrangians, it would be nice to state these conditions also directly in terms of the Lagrangian.

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