

Unidirectional wave propagation in one-dimensional first order Hamiltonian systems

E. W. C. van Groesen

Mathematical Institute, Catholic University, Toernooiveld, Nijmegen, The Netherlands

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Defining the velocity of a conserved density to be the velocity of the center of gravity of this density, it is shown that for linear equations this velocity equals the weighted group velocity (with the density as weight function). For nonlinear equations, expressions for the centrovelocity of several conserved densities are derived. In particular, for a class of nonlocal equations, the centrovelocity of the energy density turns out to be some weighted average of the group velocity of the corresponding linearized equation. For a specific equation of this type, viz, the BBM equation, it is shown that upon restricting it to solutions whose initial form represents a long, low wave, the centrovelocity of the energy density is positive for all positive time.

1. INTRODUCTION

In this paper we shall consider equations of the form

$$\partial_t u = -\partial_x h'(u), \quad (1.1)$$

wherein u is a scalar function of one space variable $x \in R^1$ and the time $t \in R^1$. Equation (1.1) may be envisaged as an evolution equation in a function space U consisting of sufficiently smooth functions which tend, together with their derivatives, to zero for $|x| \rightarrow \infty$. Given a functional h on this space, $h'(u)$ denotes the functional derivative of h at the point u (i.e., the formal Gateaux derivative with respect to the L_2 inner product $(d/d\epsilon)h(u + \epsilon v)|_{\epsilon=0} = \langle h'(u), v \rangle$ for $u, v \in U$, where \langle, \rangle is the L_2 inner product).

Equation (1.1) is called a first order Hamiltonian system, with the functional h as the Hamiltonian. Equations of this type often occur in mathematical physics. Probably the oldest example is the KdV equation

$$\partial_t u = -\partial_x (u + u^2 + u_{xx}), \quad (1.2)$$

which is of the form (1.1) with Hamiltonian

$$h(u) = \int \left\{ \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{2}u_x^2 \right\} dx \quad (1.3)$$

(integration is over the whole real line). Korteweg and de Vries¹ derived this equation in 1895 as an approximate description of the evolution of surface waves on a layer of fluid, under the assumption that the waves were mainly running in one direction. Due to this last assumption, they obtained an equation which is of first order in time, whereas Boussinesq had already obtained an equation of second order in time for the evolution of surface waves which admits solutions representing waves running in both directions. It turned out that this is just a particular instance of a much more general situation: Starting with a Hamiltonian system with canonically conjugate variables q and p , for which q enters as a potential function (in which form Boussinesq's equation can be rephrased), the restriction to solutions which mainly run in one direction leads, within some approximation, to an equation of the form (1.1) (cf. Whitham², Benjamin³, and in particular Broer⁴, Broer *et al.*⁵, and van Groesen⁶).

Initiated by the pioneering work of Gardner *et al.* (cf.

Ref. 7 for references) and Lax⁸, the interest in the KdV and related equation(s) has considerably increased during the past ten years because it turned out that this equation is a completely integrable system (Zacharov and Faddeev⁹) which is soluble with the aid of inverse scattering theory.

In this paper we shall not deal with these aspects of (some) equations of the form (1.1). Our aim is to investigate what sense can be given to such statements as "Eq. (1.1) (approximately) describes unidirectionally propagating waves" and to derive conditions under which an equation of this type deserves such a qualification. Although several notions of propagativity are mentioned in the literature (such as phase, group, front, and signal velocity amongst many others), especially for nonlinear equations this point seems to have been overlooked somewhat.

Therefore, in Sec. 2 we shall define and comment on a notion of propagativity which can be used for any (nonlinear) system which has a conserved density and which is essentially nothing but the velocity of the center of gravity (centrovelocity) of this conserved density. Adopting this definition, it is shown in Sec. 3 that for *linear* equations of the form (1.1), the centrovelocity of any definite quadratic density is equal to the weighted group velocity, with the density as the weight function. This result shows that for linear equations the group velocity plays a fundamental role in the propagation of arbitrary wave forms (not necessarily "small wave packets" or domains of only "weak dispersion"), in contradistinction to a statement by Krotscheck and Kundt¹⁰ who pose that "... the group velocity cannot be used for a rigorous discussion of propagation speeds: it has only approximate character, and loses its meaning in domains of strong dispersion".

In Sec. 4 we shall derive expressions for the centrovelocity of several conserved densities of Eq. (1.1). In Sec. 5 we shall specialize these results to a class of equations for which Eq. (1.1) is a local equation (i.e., h' is a local operator), and give sufficient conditions which assure that such an equation is unidirectionally propagative for all solutions. In Sec. 6 we consider nonlocal equations. In particular, we shall show that for a restricted class of such equations the centrovelocity of the conserved energy density [i.e., the density corre-

sponding to the invariant integral $h(u)$ is given by a weighted average of the group velocity of the corresponding linearized equation. An important example of a nonlocal equation of the form (1.1) is the BBM equation

$$(1 - \partial_x^2) \partial_t u = - \partial_x (u + u^2), \quad (1.4)$$

which was proposed by Benjamin, Bona, and Mahony¹¹ as an alternative to the KdV equation to describe long, low waves.

In the final section we shall rigorously prove that the centroid velocity of the energy density of Eq. (1.4) is positive for all $t > 0$ for any solution of Eq. (1.4) whose initial value belongs to a specified class of functions, where this class can be interpreted as the class of long, low wave forms.

2. PRELIMINARIES AND DEFINITIONS

In order to assure that with every evolution equation of the form

$$\partial_t u = - \partial_x h'(u) \quad (2.1)$$

there corresponds a unique Hamiltonian $h \in C^1(U, \mathbb{R})$, we require

$$h(0) = h'(0) = 0.$$

Moreover, we only consider translational invariant functionals h (i.e., h' does not depend explicitly on $x \in \mathbb{R}$).

In the following we shall call Eq. (2.1) a local equation if h' is a local operator, i.e., if $h'(u)(x)$ depends on the function u and its derivatives at the place x only. Equation (2.1) is said to be nonlocal if h' is not a local operator. Clearly, the KdV equation (1.2) is a local equation. The BBM equation (1.4) is not of the form (2.1) but it can be brought to this form by a simple linear transformation.

More generally, we shall consider equations of the form

$$\partial_t Dv = - \partial_x k'(v) \quad [k(0) = k'(0) = 0], \quad (2.2)$$

wherein D is some symmetric, regular operator on U , commuting with ∂_x . Transformation properties between Eqs. (2.1) and (2.2) can easily be described.

Proposition 2.1: (i) Let L be a regular operator on U , commuting with ∂_x . Under the transformation $v = L^{-1}u$, Eq. (2.1) transforms into Eq. (2.2) with

$$D = L * L \quad \text{and} \quad k(v) = h(Lv).$$

(ii) If the operator D admits the representation $D = L * L$ for some regular operator L , then under the transformation $u = Lv$, Eq. (2.2) transforms into Eq. (2.1) with

$$h(u) = k(L^{-1}u).$$

Proof: If $k(v) = h(Lv)$, then on differentiating with respect to v there results

$$\langle k'(v), w \rangle = \langle h'(Lv), Lw \rangle, \quad \forall w \in U,$$

or

$$k'(v) = L * h'(Lv).$$

Hence, if $u = Lv$,

$$L * [\partial_t u + \partial_x h'(u)] = \partial_t L * Lv + \partial_x k'(v),$$

which proves (i). Part (ii) is proved analogously. \square

The BBM equation (1.4) is of the form (2.2) with

$$D = I - \partial_x^2, \quad k(v) = \int \left(\frac{1}{2} v^2 + \frac{1}{3} v^3 \right) dx. \quad (2.3)$$

As follows from proposition 2.1, it can be brought to the form (2.1) by the linear transformation $u = D^{1/2}v$, where $D^{1/2}$ is the symmetric, positive square root of the operator D . Then it is seen that the BBM equation is a nonlocal equation.

However, for many purposes it is simpler to deal directly with equations in the form (2.2) than with the corresponding transformed form of the equations. When dealing with nonlocal operators, the following definition is not completely standard.

Definition 2.2: A functional $e \in C^1(U, \mathbb{R})$ is said to be an invariant integral for Eq. (2.1) if

$$\partial_t e(u) = 0$$

for every solution of Eq. (2.1). Any operator E on U for which

$$e(u) = \int E(u) dx$$

is an invariant integral will be called a conserved density for Eq. (2.1).

Remark 2.3: From this definition it follows that if E is a conserved density, then there exists a flux density T such that

$$\begin{aligned} \partial_t E(u) + \partial_x T(u) &= 0, \\ T(u) &\rightarrow 0 \quad \text{for } |x| \rightarrow \infty \end{aligned} \quad (2.4)$$

for every solution of Eq. (2.1). The expressions (2.4) are of the form of a local conservation law. However, only if T is a local operator do we obtain the usual result that for arbitrary interval $(a, b) \subset \mathbb{R}$, $\partial_t \int_a^b E(u)$ depends only on the value of u and its derivatives with respect to x at the points $x = a$ and $x = b$.

Lemma 2.4: The functionals $\int u dx$, h , and m , with

$$m(u) = \int u^2 dx,$$

are invariant integrals for Eq. (2.1), whereas the functionals $\int v dx$, k , and

$$m(v) = \int v Dv dx$$

are invariant integrals for Eq. (2.2).

Proof: Integrating the equations over the whole real axis, the statement for the linear densities follows immediately. Furthermore, for solutions of Eq. (2.1) we have

$$\partial_t h(u) = \langle h'(u), \partial_t u \rangle = \langle h'(u), - \partial_x h'(u) \rangle = 0$$

and

$$\begin{aligned} \partial_t m(u) &= 2 \langle u, \partial_t u \rangle \\ &= - 2 \langle u, \partial_x h'(u) \rangle = 2 \langle h'(u), u_x \rangle = 0 \end{aligned}$$

(the last equality because h is a translational invariant functional). The corresponding results for Eq. (2.2) now follow because of proposition 2.1. \square

Remark 2.5: If Eq. (2.1) is meant to describe a specific physical system (such as the water wave problem), invariant integrals often admit a physical interpretation (such as conservation of mass, energy, and momentum).

We now come to the main ideas concerning the propagation of conserved densities.

Definition 2.6: Let E be any conserved density of (2.1) with $e(u) = \int E(u) dx$ the corresponding invariant integral. For $u \in U$, the center of gravity of $E(u)$ will be denoted by $X_c(u)$ and is defined if $e(u) \neq 0$ by

$$\int [x - X_E(u)] \cdot E(u) dx = 0. \quad (2.5)$$

For a solution u of Eq. (2.1), the *centrovelocity* of E is defined to be the velocity of the center of gravity of $E(u)$:

$$V_E(u)(t) := \partial_t X_E[u(t)]. \quad (2.6)$$

The density E is said to be propagated to the right by a solution u of Eq. (2.1) at time t if $V_E(u)(t) > 0$, and Eq. (2.1) is said to be unidirectionally propagative (to the right) with respect to E if $V_E(u)(t) > 0$ for every nontrivial solution of Eq. (2.1) and all $t \in \mathbb{R}$.

With respect to this definition some remarks have to be made.

Remark 2.7: If T is the flux density corresponding to the conserved density E [i.e., E and T satisfy Eq. (2.4) for every solution of Eq. (2.1)], the V_E is easily seen to be given by

$$V_E(u) = \frac{I}{e(u)} \cdot \int T(u) dx. \quad (2.7)$$

Remark 2.8: Although the centrovelocity of a conserved density as defined above has some physical significance, it is of course by no means the only possible way to describe propagation phenomena. However, some advantages of the proposed definition may be noticed: (i) it is not necessary to restrict to a special class of solutions (such as monochromatic solutions in linear equations); (ii) the definition is the same for nonlinear and for linear equations; (iii) it is possible (as will become clear in the next sections) to formulate, with relative ease, general conditions on the functional h (or on the operator D and the functional k) which assure that Eq. (2.1) [Eq. (2.2)] is unidirectionally propagative with respect to some density; (iv) if the equation admits a solution which travels undisturbed in shape with constant velocity c , say $u(x,t) = \varphi(x - ct)$, then $V_E(\varphi)(t) = c$ for all time and every conserved density E .

Remark 2.9: As a consequence of the proposed definition, with every conserved density there is associated a velocity for every solution. Suppose E_1 and E_2 are two different conserved densities [possibly with the same invariant functional (!); cf. Remark 2.10]. Then, if V_1 and V_2 denote the corresponding velocities, the velocity V_{12} of the conserved density $E_{12} = E_1 + E_2$ is given by

$$V_{12} = \frac{e_1 \cdot V_1 + e_2 \cdot V_2}{e_1 + e_2}$$

for every solution.

Furthermore, in general there is no evidence at all that if the density E_1 is being propagated to the right by some solution u , the same is true for the density E_2 . However, for linear equations with constant coefficients it will be shown in the next section that if Eq. (2.1) is unidirectionally propagative with respect to some definite, quadratic conserved density, then the same is true for every quadratic density. For

nonlinear equations no such strong relationship between the propagativities of different conserved densities has been found (nor can be expected to hold).

Remark 2.10: Closely related with the foregoing remark is the following observation: If E is a conserved density, with T the corresponding flux density, then E^* , defined by

$$E^*(u) := E(u) + \partial_x F(u),$$

where $F(u)$ is any expression in u satisfying $F(u) \rightarrow 0$ for $|x| \rightarrow \infty$ if $u \in U$, is also a conserved density with the same invariant integral

$$e(u) = \int E(u) dx = \int E^*(u) dx.$$

The flux density T^* corresponding to E^* is given by

$$T^*(u) = T(u) - \partial_x F(u),$$

and if X and X^* denote the centers of gravity of E and E^* , respectively, with corresponding velocities V and V^* , we have

$$X(u) - X^*(u) = \frac{1}{e(u)} \cdot \int F(u) dx$$

and

$$V(u) - V^*(u) = \frac{1}{e(u)} \cdot \int [T(u) - T^*(u)] dx.$$

From this it follows that, upon adding a term $\partial_x F$ to the density, the corresponding velocity will change in general: Only if the total flux is not altered will the velocity remain the same.

Remark 2.11: In general, the velocity functional V_E is not an invariant integral. However, if the conserved density E has a conserved flux, i.e., if the total flux itself is an invariant integral

$$\partial_t \int T(u) dx = 0,$$

then V_E as given by Eq. (2.7) is an invariant integral. In that case, the center of gravity is a linear function of t :

$$X(u)(t) = t \cdot V(u) + X_0(u), \quad (2.8)$$

where X_0 is an invariant integral (the position of the center of gravity at $t = 0$). Inserting Eq. (2.8) into (2.5) gives

$$\int [xE(u) - tT(u)] dx = X_0(u) \cdot \int E(u) dx,$$

which leads to the following invariant functional which contains the x and t variable explicitly:

$$\partial_t \int [xE(u) - tT(u)] dx = 0. \quad (2.9)$$

3. PROPAGATION IN LINEAR SYSTEMS

Linear first order Hamiltonian systems are described by an equation of the form

$$\partial_t u = -\partial_x Lu, \quad (3.1)$$

where L is some symmetric operator. The Hamiltonian for such equations is the quadratic functional

$$h(u) = \frac{1}{2} \langle u, Lu \rangle. \quad (3.2)$$

In the following we shall restrict ourselves to the simplest class of operators, viz., the class of pseudodifferential operators (with constant coefficients). If M is a pseudodifferential operator, we shall denote the symbol of M by \hat{M} as follows:

$$\widehat{Mu}(k) = \hat{M}(k) \cdot \hat{u}(k),$$

wherein \hat{u} denotes the Fourier transform of the function u .

Theorem 3.1: Any linear density $E(u) = Pu$, where P is a pseudodifferential operator, is conserved. The center of gravity and its velocity can be defined for solutions for which $\int Pu \, dx \neq 0$ and we have

$$V_E(u) = \hat{L}(0). \quad (3.3)$$

Hence, all linear densities are being propagated with the same constant speed $\hat{L}(0)$, independent of the particular solution.

Proof: Since P commutes with ∂_x , we have

$$\partial_x Pu + \partial_x PLu = 0.$$

From this it follows with Eq. (2.7) and Fourier transformation that

$$V_E(u)(t) = \frac{\int PLu \, dx}{\int Pu \, dx} = \frac{\hat{P}(0) \cdot \hat{L}(0) \cdot \hat{u}(0)}{\hat{P}(0) \cdot \hat{u}(0)} = \hat{L}(0). \quad \square$$

Quadratic conserved densities are more interesting and have been studied in great detail. It is at this point that the concept of group velocity enters the discussion of propagation. The dispersion relation for Eq. (3.1) is

$$\omega = k \cdot \hat{L}(k), \quad (3.4)$$

and corresponding to the group velocity $d\omega/dk$, we define an operator G .

Definition 3.2: The group-velocity operator G is defined to be the pseudodifferential operator with symbol

$$\hat{G}(k) := \frac{d\omega}{dk}(k) = \frac{d}{dk} [k \cdot \hat{L}(k)]. \quad (3.5)$$

Lemma 3.3: As L is symmetric, the operator G is symmetric.

Proof: The proof is immediate from the fact that a pseudodifferential operator M is symmetric if and only if its symbol satisfies $\hat{M}(k) \in Rl$ for $k \in Rl$. \square

Lemma 3.4: Any quadratic density $E(u) = Pu \cdot Qu$, where P and Q are pseudodifferential operators, is a conserved density for Eq. (3.1).

Proof: Generally, $E(u) = Pu \cdot Qu$ is a conserved density of Eq. (3.1) if P and Q satisfy

$$P^* Q \partial_x L = L \partial_x P^* Q.$$

Clearly, for pseudodifferential operators P and Q , this condition is satisfied. \square

As a typical result concerning the relation between group velocity and the propagation of conserved densities by monochromatic solution of Eq. (3.1), we quote the following result:

Theorem 3.5: Let E be a quadratic conserved density, with corresponding flux density T . Then E is being propagated with the group velocity in the following sense: For monochromatic solutions $\phi(x,t) = \phi_0 \exp(i(k_0 x - \omega_0 t))$, where ϕ_0 is a constant and $\omega_0 = \omega(k_0)$, the following relation holds:

$$\frac{T(\phi)}{E(\phi)} = \frac{d\omega}{dk}(k_0). \quad (3.6)$$

This theorem is well known and can be found, for example, in de Graaf and Broer¹². With the proposed definition 2.6, it turns out that the group velocity plays an equally important role in the propagation of quadratic densities by nonperiodic solutions.

Theorem 3.6: Consider a definite (conserved) quadratic density $E(u) = Au \cdot Au$, where A is some pseudodifferential operator. Then, the centrovelocity of this density is (an invariant integral and is) given by

$$V_E(u) = \langle Au, Au \rangle^{-1} \cdot \langle Au, GAu \rangle, \quad (3.7)$$

wherein G is the group-velocity operator. Hence, $V_E(u)$ equals the weighted group velocity, with weight function $|\hat{Au}|^2$.

Proof: Inserting Eq. (3.1) directly into definition (2.6), it follows that

$$V_E(u) = 2 \cdot \langle \hat{Au}, \hat{Au} \rangle^{-1} \cdot \langle \hat{Au}, -x \partial_x L \hat{Au} \rangle.$$

With Parseval's theorem and Eq. (3.4) we find

$$V_E(u) = 2 \cdot \langle \hat{Au}, \hat{Au} \rangle^{-1} \cdot \langle \hat{Au}, \partial_k [\omega \cdot \hat{Au}] \rangle,$$

and after some straightforward manipulations

$$\begin{aligned} V_E(u) &= \langle \hat{Au}, \hat{Au} \rangle^{-1} \cdot \left\langle \hat{Au}, \frac{d\omega}{dk} \cdot \hat{Au} \right\rangle \\ &= \langle Au, Au \rangle^{-1} \cdot \langle Au, GAu \rangle. \end{aligned} \quad (3.8)$$

This proves Eq. (3.7), and as V_E is a quadratic functional, it is an invariant integral according to Lemma 3.4. \square

An immediate consequence of this theorem is the following:

Corollary 3.7: Equation (3.1) is unidirectionally propagative (to the right) with respect to any definite, quadratic density if and only if the group velocity is nonnegative for all wave numbers:

$$\hat{G}(k) \geq 0, \quad \text{for all } k \in Rl. \quad (3.9)$$

Remark 3.8: For more general quadratic densities of the form $E(u) = Au \cdot Bu$, with A and B pseudodifferential operators, the corresponding centrovelocity is easily found to be the invariant integral

$$V_E(u) = \langle Au, Bu \rangle^{-1} \cdot \langle Au, GBu \rangle \quad (3.10)$$

for solutions for which $e(u) = \langle Au, Bu \rangle \neq 0$.

Remark 3.9: For two specific densities, viz., the momentum density [$A = I$ in Eq. (3.7)] and the energy density [$A^2 = L$ in Eq. (3.7)], the result (3.7) can already be found in Wehausen and Laitone (Ref. 13, Sec. 15).

Our more general formulation makes their results somewhat more transparent.

4. PROPAGATION IN NONLINEAR SYSTEMS

For the first order Hamiltonian equations under consideration

$$\partial_t u = -\partial_x h'(u), \quad (4.1)$$

the Hamiltonian h can be written as

$$h(u) = \int H(u) dx, \quad (4.2)$$

where H is some (nonlinear) operator on U . Only differentia-

ble operators H will be considered, i.e., operators H for which, for every $u \in U$, there exists a linear operator, depending on u , and denoted by $H'(u)$, the derivative of H at the point u , such that

$$\frac{d}{d\epsilon} H(u + \epsilon v) \Big|_{\epsilon=0} = H'(u)v,$$

for every function $v \in U$. As we also have

$$\frac{d}{d\epsilon} h(u + \epsilon v) \Big|_{\epsilon=0} = \int h'(u) \cdot v dx,$$

it follows that

$$\int h'(u) \cdot v dx = \int H'(u)v dx$$

for every $v \in U$. Hence, there exists an operator B such that

$$h'(u) \cdot v = H'(u)v - \partial_x B(u, v). \quad (4.3)$$

Note that $B(u, v)$ is linear in v , nonlinear in u (except when H is quadratic), and that $B(u, v) \rightarrow 0$ for $|x| \rightarrow \infty$ on the considered class of functions. As h is assumed to be a translational invariant functional, it follows that

$$h'(u) \cdot u_x = \partial_x [H(u) - B(u, u_x)] \quad (4.4)$$

and, as h does not depend explicitly on t , we also have

$$\partial_t H(u) = h'(u) \cdot u_t + \partial_x B(u, u_t). \quad (4.5)$$

With these results it is not difficult to find the local conservation laws for the conserved densities of Eq. (4.1). In fact, as it stands, Eq. (4.1) is of the form of a local conservation law for the density u . For the conserved density u^2 , the corresponding flux density follows with the aid of Eq. (4.4):

$$\begin{aligned} \partial_t u^2 &= 2u \cdot u_t = -2u \cdot \partial_x h'(u) \\ &= -2\partial_x [u h'(u) - H(u) + B(u, u_x)]. \end{aligned}$$

In the same way, using Eq. (4.5), we find the following for the conserved density $H(u)$:

$$\begin{aligned} \partial_t H(u) &= -h'(u) \cdot \partial_x h'(u) - \partial_x B(u, \partial_x h'(u)) \\ &= -\partial_x \left\{ \frac{1}{2} [h'(u)]^2 + B(u, \partial_x h'(u)) \right\}. \end{aligned}$$

Having found the flux densities corresponding to the conserved densities of Eq. (4.1), the centrovocities of these densities follow immediately with Eq. (2.7) and we have obtained theorem 4.1.

Theorem 4.1: For Eq. (4.1) we have the following: (i) The centrovelocity of the conserved density u is given by

$$V(u) = \left(\int u dx \right)^{-1} \cdot \int h'(u) dx. \quad (4.6)$$

(ii) The centrovelocity of the conserved quadratic density u^2 is given by

$$V(u) = 2 \cdot \|u\|^{-2} \cdot \int \{u \cdot h'(u) - H(u) + B(u, u_x)\} dx. \quad (4.7)$$

(iii) The centrovelocity of the (energy) density $H(u)$ is given by

$$V(u) = h(u)^{-1} \cdot \int \left\{ \frac{1}{2} [h'(u)]^2 + B(u, \partial_x h'(u)) \right\} dx. \quad (4.8)$$

With these general results it is possible, at least in principle, to investigate whether one of the conserved densities is being propagated to the right by a particular solution. However, the possibility to find conditions on the functional h which assure that some of these velocity functionals are positive depends very much on the "nature" of the operator h' . In fact, for local equations it is possible to investigate this matter completely. By way of example we shall consider a class of local equations in the next section. This class is relatively simple but has nonetheless all the peculiarities of more general local equations. In contradistinction, even the most simple nonlocal equations are difficult to deal with. This is due to the mathematical difficulties encountered in the investigation of the positivity of functionals with nonquadratic nonlocal integrands. Nevertheless, for a special class of nonlocal equations we shall derive some remarkable results concerning the propagation of the energy density in Secs. 6 and 7.

5. A CLASS OF LOCAL EQUATIONS

We consider local equations of the form (4.1) for which the Hamiltonian $h(u)$ is given by

$$h(u) = \int \{N(u) + S(u_x)\} dx, \quad (5.1)$$

where N and S are smooth (C^2) functions of their arguments with derivatives n and s , respectively:

$$n(y) := N'(y) = \frac{dN}{dy}(y), \quad y \in R^1,$$

$$s(z) := S'(z) = \frac{dS}{dz}(z), \quad z \in R^1$$

(primes denote differentiations with respect to the arguments). We assume that

$$N(0) = S(0) = n(0) = s(0) = 0,$$

$0 = N(0) = S(0) = n(0)$ to assure that $h(0) = h'(0) = 0$ and $s(0) = 0$ is no restriction. Equation (4.1) with h as in Eq. (5.1) then reads

$$\partial_t u = -\partial_x \{n(u) - \partial_x s(u_x)\}. \quad (5.2)$$

In a straightforward way, the following results are obtained from theorem 4.1.

Theorem 5.1: The centrovocities of the conserved densities

$$u, \quad u \cdot u, \quad N(u) + S(u_x) \quad (5.3)$$

for Eq. (5.2) are given by

$$V(u) = \left(\int u dx \right)^{-1} \cdot \int n(u) dx, \quad (5.4)$$

$$\begin{aligned} V(u) &= 2 \cdot \|u\|^{-2} \cdot \int \{u \cdot n(u) - N(u) + 2u_x \cdot s(u_x) \\ &\quad - S(u_x)\} dx, \end{aligned} \quad (5.5)$$

$$\begin{aligned} V(u) &= h(u)^{-1} \cdot \int \left\{ \frac{1}{2} n^2(u) + 2u_x \cdot s(u_x) \cdot n'(u) \right. \\ &\quad \left. + \frac{3}{2} [\partial_x s(u_x)]^2 \right\} dx. \end{aligned} \quad (5.6)$$

To assure that Eq. (5.2) is unidirectionally propagative with respect to the densities u^2 and/or $N(u) + S(u_x)$ it suffices to

state conditions for the functions N and S which assure that the velocity functionals (5.5) and/or (5.6) are nonnegative on the considered class of functions U . In that way we get Theorem 5.2.

Theorem 5.2: Equation (5.1) is unidirectionally propagative to the right with respect to (i) the density u^2 if

$$y \cdot n(y) - N(y) \geq 0, \quad \forall y \in \mathbb{R}I,$$

$$2z \cdot s(z) - S(z) \geq 0, \quad \forall z \in \mathbb{R}I;$$

(ii) the (energy) density $N(u) + S(u_x)$ (required to be positive) if

$$N(y) \geq 0, \quad n'(y) \geq 0, \quad \forall y \in \mathbb{R}I,$$

$$S(z) \geq 0, \quad 2z \cdot s(z) \geq 0, \quad \forall z \in \mathbb{R}I;$$

(iii) both the density u^2 and the energy density if

$$N(y) \geq 0, \quad y \cdot n(y) - N(y) \geq 0, \quad n'(y) \geq 0 \quad \forall y \in \mathbb{R}I,$$

$$S(z) \geq 0, \quad 2z \cdot s(z) - S(z) \geq 0, \quad \forall z \in \mathbb{R}I.$$

Remark 5.3: Linearizing Eq. (5.2) gives

$$\partial_t u = -\partial_x \{n'(0) \cdot u - s'(0) \cdot \partial_x u\}, \quad (5.7)$$

which has the dispersion relation

$$\omega(k) = n'(0) \cdot k + s'(0) \cdot k^3$$

and group velocity

$$\frac{d\omega}{dk}(k) = n'(0) + 3s'(0) \cdot k^2.$$

The velocity functionals of the density u^2 and the linearized energy density $\frac{1}{2}n'(0) \cdot u^2 + \frac{1}{2}s'(0) \cdot u_x^2$ of this linear equation (5.7) [as given by Eq. (3.7)] are easily seen to be the quadratic terms in a Taylor expansion of the integrands of Eqs. (5.5) and (5.6), respectively.

Remark 5.4: As u and u^2 are conserved densities for Eq. (5.2), the velocity (5.4) is an invariant functional if

$$n(u) = \alpha u + \beta u^2, \quad \alpha, \beta \in \mathbb{R}I.$$

In that case, u is a conserved density with conserved flux and according to remark 2.11, there exists an invariant functional which depends on x and t explicitly; in this case

$$\partial_t \int \{xu - tn(u)\} dx = 0. \quad (5.8)$$

Remark 5.5: The KdV equation (1.2) belongs to the considered class of equations with

$$N(y) = \frac{1}{2}y^2 + \frac{1}{6}y^3, \quad S(z) = \frac{1}{2}z^2.$$

This equation is neither unidirectionally propagative with respect to the density u^2 nor with respect to its energy density, as may be confirmed from the expressions (5.5) and (5.6). The velocity functional (5.4) is an invariant functional, and Eq. (5.8) reads

$$\partial_t \int \{xu - t(u + \frac{1}{2}u^2)\} dx = 0. \quad (5.9)$$

Apart from a trivial transformation, this functional was already given by Miura *et al*¹⁴.

6. A CLASS OF NONLOCAL EQUATIONS

Here we shall examine equations of the form

$$\partial_t Du = -\partial_x e'(u), \quad (6.1)$$

where the functional e is given by

$$e(u) = \int N(u) dx, \quad (6.2)$$

with N a smooth (C^2) function of its argument, $n(u) = (dN/du)(u)$, $N(0) = n(0) = 0$, and where $D = A^2$ with A some positive symmetric pseudodifferential operator. From Sec. 2 it follows that, via a simple linear transformation, Eq. (6.1) can be brought into an equation of the form (4.1), but except when A^{-1} is an ordinary differential operator, this equation will be of nonlocal type. In these cases it is somewhat simpler to deal directly with the form (6.1).

For Eq. (6.1) we have three invariant integrals:

$$\int u dx, \quad \int \langle u, Du \rangle, \quad \int e(u)$$

(cf. lemma 2.4). The velocity of the linear density u is again given by Eq. (5.2) and remark 5.4 applies as well: If $\int n(u) dx$ is an invariant functional, then

$$\partial_t \int \{x Du - tn(u)\} dx = 0.$$

For the following we define the symmetric operator G as the pseudodifferential operator with symbol \hat{G} , where

$$\hat{G}(k) = \partial_k [k \cdot \hat{D}^{-1}(k)]$$

$$= \hat{D}^{-1}(k) \cdot [1 - k \cdot \hat{D}^{-1}(k) \cdot \partial_k \hat{D}(k)]. \quad (6.3)$$

Remark 6.1: Note that if u has a linear term, say $n'(0) = 1$, then $\omega(k) = k \cdot \hat{D}^{-1}(k)$ is the dispersion relation of the linearized equation (6.1), and then $\hat{G}(k)$ is the corresponding group velocity. However, if Eq. (6.1) does not admit a formal linearization, i.e., if $n'(0) = 0$, this interpretation of ω and \hat{G} no longer makes sense, but the results to be derived remain valid!

Theorem 6.2: The centrovLOCITY of the energy density $N(u)$ is given by

$$V(u) = \frac{1}{2}e(u)^{-1} \cdot \langle n(u), Gn(u) \rangle. \quad (6.4)$$

Consequently, Eqs. (6.1) and (6.2) are unidirectionally propagative to the right with respect to the energy density (assumed to be positive) if and only if

$$\hat{G}(k) \geq 0, \quad \forall k \in \mathbb{R}I. \quad (6.5)$$

Proof: The simplest way to derive this result is analogous to the proof of Theorem 3.6: Using Fourier-transform techniques and writing \hat{n} for the Fourier transform of the expression $n(u)$, we find

$$\partial_t \int xN(u) dx = \langle x \cdot n(u), \partial_t u \rangle$$

$$= \langle n(u), -x \partial_x D^{-1}n(u) \rangle$$

$$= \frac{1}{2} \langle \hat{n}(k), \hat{G}(k) \cdot \hat{n}(k) \rangle.$$

Hence,

$$V(u) = \frac{1}{2}e(u)^{-1} \cdot \langle \hat{n}, \hat{G}\hat{n} \rangle$$

$$= \frac{1}{2}e(u)^{-1} \cdot \langle n(u), Gn(u) \rangle$$

and the theorem follows. \square

Remark 6.3: The BBM equation (1.4) belongs to the considered class of equations with

$$D = 1 - \partial_x^2, \quad N(u) = \frac{1}{2}u^2 + \frac{1}{6}u^3. \quad (6.6)$$

For this operator D , the function \hat{G} is not positive for all $k \in \mathbb{R}I$:

$$\hat{G}(k) = (1 + k^2)^{-2} \cdot (1 - k^2). \quad (6.7)$$

Hence, the BBM equation is not unidirectionally propagative with respect to its (nondefinite) energy density. (See however the results of the next section for a restricted set of solutions.)

Remark 6.4: More generally, if D can be written as

$$\hat{D} = (1 + k^2)^\sigma, \quad \sigma \in \mathbb{R}I, \quad (6.8)$$

then \hat{G} is given by

$$\hat{G}(k) = (1 + k^2)^{-\sigma-1} \cdot [1 + (1 - 2\sigma) \cdot k^2]. \quad (6.9)$$

Hence, for such operators, condition (6.5) is satisfied if and only if

$$\sigma \leq \frac{1}{2}. \quad (6.10)$$

The energy velocity as given by Eq. (6.4) is remarkably simple. However, matters are much more complicated for the velocity of the (positive) quadratic density $Au \cdot Au$. Using Fourier-transform techniques in intermediate steps, it is not difficult to derive the following result in a direct way.

Theorem 6.5: The centrovlocity of the positive density $Au \cdot Au$ can be expressed with the operator G as

$$V(u) = \langle Au, Au \rangle^{-1} \cdot \int \{u \cdot n(u) - 2N(u) + n(u) \cdot DGu\} dx. \quad (6.11)$$

Note that for linear equations $N(u) = \frac{1}{2}u^2$, Eq. (6.11) agrees with Eq. (3.7). However, in the more interesting case on nonlinear equations it seems to be impossible to derive conditions on N and D such that Eq. (6.11) is a positive functional.

7. UNIDIRECTIONAL PROPAGATION IN LONG, LOW WAVE MODELS

In this section we shall once again examine equations of the form

$$\partial_t Du = -\partial_x n(u), \quad (7.1)$$

where D is a pseudodifferential operator and $n(u)$ a smooth function of its argument. The energy density $N(u)$, with $n(u) = (dN/du)(u)$, $N(0) = n(0) = 0$, is no longer required to be positive. In view of the results of the foregoing section we shall only consider the centrovlocity of the energy density. This velocity is given by Eq. (6.4):

$$V(u) = \frac{1}{2} e(u)^{-1} \cdot \langle n(u), Gn(u) \rangle, \quad (7.2)$$

where G is the pseudodifferential operator with symbol given by Eq. (6.3). We shall suppose that

$$n'(0) = 1, \quad (7.3)$$

such that G can be interpreted as the group-velocity operator of the linearized problem. In the foregoing section it was shown that the BBM equation

$$(1 - \partial_x^2) \partial_t u = -\partial_x (u + \frac{1}{2}u^2) \quad (7.4)$$

is not unidirectionally propagative with respect to the energy density. However, the BBM equation (as many other equations of this type) is derived as an approximate equation for the description of "fairly long, fairly low" waves (cf. Broer,⁴

Broer *et al.*,⁵ and Benjamin *et al.*¹¹ for details about this approximate character of the equation).

Therefore, it is reasonable to investigate the positivity of the functional (7.2) on the restricted class of functions which can be described as long, low waves. To make this idea more concrete, let us suppose that we can define two functionals ϵ and λ whose values $\epsilon(u)$ and $\lambda(u)$ are a measure of the height and of the "length" of the function u , respectively. Then the class of long, low waves can be described as the set of functions satisfying

$$\begin{aligned} \epsilon(u) &< \epsilon_0, \\ \lambda(u) &> \lambda_0, \end{aligned} \quad (7.5)$$

where ϵ_0 and λ_0^{-1} are small positive numbers. Now suppose that numbers ϵ_0 and λ_0 can be found such that $V_E(u)$ is of the same sign (positive say) for every function u which satisfies Eqs. (7.5). Then, if u is a solution of Eq. (7.1) which satisfies Eqs. (7.5) at some instant t_0 , $V_E(u)(t)$ will be positive at $t = t_0$ and for times $t > t_0$ as long as $u(t)$ satisfies conditions (7.5). *A priori*, it is by no means clear that solutions corresponding to initial data which satisfy Eqs. (7.5) satisfy this condition for all $t > 0$.

Especially for nonlinear equations this is a critical point. To demonstrate this for the long wavelength condition for instance, consider the solution of Eq. (7.1) corresponding to an initial value $g(x)$ whose Fourier transform \hat{g} satisfies $\hat{g}(k) = 0$ for $|k| \geq k_0$, $k \in \mathbb{R}I$ (i.e., g consists of long wave components only). A Fourier transformation of Eq. (7.1) shows that if the equation is *linear*, then $\hat{u}(k, t) = 0$ for $|k| \geq k_0$ for all $t \geq 0$, but if the equation is *nonlinear*, then $\hat{u}(k, t) \neq 0$ for almost all $k \in \mathbb{R}I$, no matter how small $t > 0$: *Initial long wave components generate short wave components instantly*. From these remarks and observations the following definition will be acceptable.

Definition 7.1: Let there be given two positive functionals ϵ and λ [for which $\epsilon(u)$ and $\lambda(u)$ are a measure for the height and the length of a function $u \in U$, respectively]. Then, Eq. (7.1) is said to be unidirectionally propagative with respect to the energy density $N(u)$ for (λ) long and (ϵ) low waves if positive numbers ϵ_0 and λ_0 can be found such that $V(u)(t) \geq 0$ for all $t > 0$ for every solution u whose initial value u_0 satisfies

$$\begin{aligned} \epsilon(u_0) &< \epsilon_0, \\ \lambda(u_0) &> \lambda_0. \end{aligned} \quad (7.6)$$

For the following we suppose that the symbol of the group-velocity operator G can be estimated as

$$\hat{G}(k) \geq \hat{G}(0) \cdot [1 - \ell^2 k^2], \quad k \in \mathbb{R}I, \quad (7.7)$$

where $\hat{G}(0)$ and ℓ are positive numbers. For long wave models such an estimate is generally possible: The long wave components propagate with the largest, positive speed (the group velocity has a positive maximum at $k = 0$). With Eq. (7.7) the velocity functional (7.2) can be estimated as

$$V(u) \geq \frac{1}{2} \hat{G}(0) \cdot \frac{\|n(u)\|^2}{e(u)} \cdot \left\{ 1 - \ell^2 \frac{\|\partial_x n(u)\|^2}{\|n(u)\|^2} \right\}. \quad (7.8)$$

From this it immediately follows that $V(u) \geq 0$ if

$$\lambda(u) \geq \ell, \quad (7.9)$$

if the functional λ is defined by

$$\lambda(u)^{-1} = \frac{\|\partial_x n(u)\|}{\|n(u)\|} \quad (7.10)$$

Remark 7.2: The functional λ defined by Eq. (7.10) can indeed be interpreted as an averaged wave length: $\lambda(u)^{-2}$ is the weighted average of k^2 with weight function $|\widehat{n(u)}|^2$. Another way to interpret $\lambda(u)$ as a measure of the "length" of the function u follows from the observation

$$\lambda(u_\mu)^{-2} = \mu^2 \cdot \lambda(v)^2, \quad \text{for } u_\mu(x) := v(\mu x); \quad (7.11)$$

hence, $\lambda(u_\mu) \rightarrow \infty$ for $\mu \rightarrow 0$.

In the following we shall show that it is sometimes possible to find conditions of the form (7.6), i.e., conditions imposed on the initial data only, which assure that the resulting solutions satisfy condition (7.9) for all $t \geq 0$. For simplicity we shall restrict ourselves in the first instance to a specific equation, viz., the BBM equation (7.4). Note that this equation satisfies Eq. (7.7) with

$$\hat{G}(0) = 1, \quad \ell^2 = 3. \quad (7.12)$$

For what follows we have to recall that $H^1(RI)$ denotes the first Sobolev space of functions $u \in L_2(RI)$ which have (generalized) derivatives $u_x \in L_2(RI)$. Supplied with the norm $\|\cdot\|_1$ it is a Banach space which is continuously embedded in $C^0(RI)$:

$$\|u\|_1^2 \leq \frac{1}{2} \|u\|_1^2 + \|u_x\|_1^2, \quad u \in H^1(RI), \quad (7.13)$$

where

$$\|u\|_\infty := \sup_{x \in RI} |u(x)|, \\ \|u\|_1^2 := \|u\|^2 + \|u_x\|^2.$$

Concerning the existence of a classical solution of the initial value problem for the BBM equation, we quote the following result:

Lemma 7.3: Let $u_0 \in C^2(RI) \cap H^1(RI)$. Then there exists a unique (classical) solution u of Eq. (7.4) with

$$u(x,0) = u_0(x)$$

and

$$u(\cdot, t), \partial_t u(\cdot, t) \in C^2(RI) \cap H^1(RI), \quad \text{for all } t \geq 0.$$

Consequently, the (momentum) functional

$$m(u) := \frac{1}{2} \langle u, Du \rangle$$

and the energy functional $e(u)$ are neatly defined and are invariant integrals:

$$m(u(t)) = m(u_0), \quad e(u(t)) = e(u_0), \quad \forall t \geq 0.$$

Proof: The proof of this result can be found in Benjamin et al.¹¹ \square

We are now in a position to formulate the main result.

Theorem 7.4: The BBM equation (7.4) is unidirectionally propagative to the right with respect to the energy density for the class of long, low waves, which is characterized as the solutions whose initial value u_0 satisfy

$$u_0 \in C^2(RI) \cap H^1(RI), \\ \lambda(u_0) > \lambda_0, \\ \epsilon(u_0) < \epsilon_0, \quad (7.14)$$

for sufficiently small positive numbers ϵ_0 and λ_0^{-1} . Here, λ is

the functional defined by Eq. (7.10) and

$$\epsilon(u) := \|u\|_1.$$

Proof: In view of the estimate (7.8) and result (7.12) we have to show that λ_0 and ϵ_0 can be found such that

$$\lambda(u)^{-2} \leq \frac{1}{3} \quad (7.15)$$

for every $t \geq 0$ and every solution with initial data satisfying Eq. (7.14). Let u_0 denote the initial value and define $\delta > 0$ by

$$\|u_0\|_1^2 = \frac{1}{2} \delta^2.$$

As

$$m(u) = \frac{1}{2} \langle u, (1 - \partial_x^2)u \rangle = \frac{1}{2} \|u\|_1^2 \quad (7.16)$$

is an invariant functional, it follows that

$$\|u\|_1^2 = \frac{1}{2} \delta^2, \quad \forall t \geq 0$$

and with Eq. (7.13), that

$$\|u\|_\infty < \delta, \quad \forall t \geq 0. \quad (7.17)$$

Then we can derive the following useful estimates for the functionals e and λ :

$$\frac{1}{2} (1 - \frac{1}{3} \delta) \|u\|^2 \leq e(u) \leq \frac{1}{2} (1 + \frac{1}{3} \delta) \|u\|^2, \quad \forall t \geq 0 \quad (7.18)$$

and, provided $\delta < 2$,

$$\left(\frac{1 - \delta}{1 + \frac{1}{2} \delta} \right)^2 \left(\frac{\|u\|_1^2}{\|u\|^2} - 1 \right) \leq \lambda(u)^{-2} \\ \leq \left(\frac{1 + \delta}{1 - \frac{1}{2} \delta} \right)^2 \left(\frac{\|u\|_1^2}{\|u\|^2} - 1 \right), \quad \forall t \geq 0. \quad (7.19)$$

As e is an invariant functional, it follows from Eq. (7.18) that

$$(1 - \frac{1}{3} \delta) \|u_0\|^2 \leq (1 + \frac{1}{3} \delta) \|u\|^2, \quad \forall t \geq 0. \quad (7.20)$$

Writing $\lambda_0 = \lambda(u(x,0))$, it follows from Eq. (7.19) that

$$\left(\frac{1 - \delta}{1 + \frac{1}{2} \delta} \right)^2 \left(\frac{\|u_0\|_1^2}{\|u_0\|^2} - 1 \right) \leq \lambda_0^{-2},$$

and, as $\|u\|_1$ is invariant, we obtain, provided $\delta < 1$,

$$\frac{\|u\|_1^2}{\|u_0\|_2} \leq 1 + \lambda_0^{-2} \left(\frac{1 + \frac{1}{2} \delta^2}{1 - \delta} \right), \quad \forall t \geq 0. \quad (7.21)$$

With Eqs. (7.20) and (7.21) we can majorize the right hand side of Eq. (7.19) and obtain

$$\lambda(u)^{-2} \leq \left(\frac{1 + \delta}{1 - \frac{1}{2} \delta} \right)^2 \cdot \left\{ \left(\frac{1 + \frac{1}{3} \delta}{1 - \frac{1}{3} \delta} \right) \right. \\ \left. \cdot \left[1 + \lambda_0^{-2} \left(\frac{1 + \frac{1}{2} \delta^2}{1 - \frac{1}{2} \delta} \right)^2 \right] - 1 \right\}, \quad \forall t \geq 0. \quad (7.22)$$

This result shows that $\lambda(u)^{-2}$ can be majorized uniformly with respect to t in terms of initial value δ and λ_0 . Moreover, it is easily seen that the right hand side of Eq. (7.22) can be bounded by $\frac{1}{3}$ if δ and λ_0^{-1} are taken sufficiently small. This shows that condition (7.15) is satisfied for δ (and hence ϵ_0) and λ_0 sufficiently small. With the extra observation that $e(u)$ is positive if $\delta < 3$, as follows from Eq. (7.18), this proves the theorem. \square

Remark 7.5: From a physical point of view the foregoing theorem is satisfactory because the requirements define the functions to be low waves, as follows from the estimate

(7.13), and to be long waves in the sense of remark 7.2. However, it is possible to show that the velocity functional (7.2) is positive on a larger class of functions. Therefore, define the functional A by

$$A(u) := \frac{m(u)}{e(u)}. \quad (7.23)$$

Then it can be shown that

$$V(u) > 0, \quad \text{for every } u \in S_\gamma, \quad (7.24)$$

where S_γ is the set of functions for which

$$\epsilon(u) := \|u\|_1 < \gamma,$$

for some γ , $0 < \gamma < 2$, where the function $\Gamma(\gamma)$ is given by

$$\Gamma(\gamma) = \left(1 + \frac{1}{3}\gamma\right)^{-1} \cdot \left(1 + \frac{1}{3} \frac{1 - \frac{1}{2}\gamma}{1 + \gamma}\right).$$

[Note that $\Gamma(0) = \frac{4}{3}$, $\Gamma(2) = \frac{3}{5}$, and

$$\Gamma(\gamma) > 1, \quad \text{for } 0 < \gamma < \gamma_0 = \frac{1}{2} \left(\sqrt{\frac{11}{2}} - \frac{3}{2} \right). \quad (7.26)$$

As the functionals ϵ and A are invariant functionals for the BBM equation, it follows that $V(u)(t) > 0$ for all $t \geq 0$ for every solution whose initial value satisfies conditions (7.25). Although the functional A has the advantage of being an invariant functional, its relevance as a measure of the "length" of a function is less clear. Nevertheless, for functions

$$u_{\delta,\mu}(x) := \delta v(\mu x),$$

we have

$$A(u_{\delta,\mu}) = \left[\|v\|^2 + \mu^2 \|\partial_x v\|^2 \right] \cdot \left[\|v\|^2 + \frac{1}{6} \delta \int_{-\infty}^{\infty} dy v^3(y) \right]^{-1},$$

such that

$$A(u_{\delta,\mu}) \rightarrow 1 \quad \text{for } \delta, \mu > 0.$$

From this it follows that for the class of long, low waves $A \approx 1$, and hence, because of Eq. (7.26), this class is included in the set S_γ for $\gamma < \gamma_0$. This shows that the result stated above includes the contents of Theorem 7.5.

Remark 7.6: It is illustrative to apply the above described method to more general equations of the form (7.1), where D is given by Eq. (6.8). The first problem to be considered concerns the existence of global solutions of the initial value problem for such equations. For $\sigma > 0$, let H^σ denote the fractional order Sobolev space, defined as the set

$$H^\sigma := \{u \in L_2 \mid \hat{u} \cdot (1 + k^2)^{\sigma/2} \in L_2\},$$

supplied with the norm

$$\|u\|_{H^\sigma}^2 = \int_{-\infty}^{\infty} (1 + k^2)^\sigma \cdot |\hat{u}|^2 dk.$$

For arbitrary $\sigma > 0$, H^σ is a Banach space, but only if $\sigma > \frac{1}{2}$ is H^σ continuously embedded in C^0 and have the property that multiplication of elements from H^σ is a continuous operation in H^σ .

Along the same lines as in the proof of Lemma 7.3, it is possible (using a contraction mapping principle, now in a

space based on H^σ) to prove the existence of a unique solution, over some time interval $[0, T]$ of the integral equation corresponding to Eq. (7.1):

$$u(x, t) = u_0(x) - \int_0^t \partial_x D^{-1} n(u(\tau)) d\tau, \quad (7.27)$$

where T depends on the H^σ norm of the initial value u_0 only:

$$T = T(\|u_0\|_{H^\sigma}). \quad (7.28)$$

Assuming $u_0 \in H^s$, for $s > \sigma$, it can be shown, using well-known bootstrap arguments, that the solution of Eq. (7.27) is also in H^s . This implies that if s is sufficiently large, u is also a solution of the original equation over the time interval $[0, T]$. Moreover, in that case the functional $m(u) = \frac{1}{2} \langle u, Du \rangle = \frac{1}{2} \|u\|_{H^\sigma}^2$ is in fact an invariant integral, from which it follows that because of Eq. (7.28) the local solution can be continued over an arbitrary time interval, which gives us the desired existence result.

To arrive at a useful estimate for the functional V , note that the group velocity (6.9) can be estimated as

$$\hat{G}(k) \geq 1 - 3[(1 + k^2)^\sigma - 1]. \quad (7.29)$$

Then V as given by Eq. (7.2) can be estimated as

$$V_\sigma(u) \geq \frac{1}{2} e(u)^{-1} \cdot \{ \|n(u)\|^2 - 3[\|n(u)\|_{H^\sigma}^2 - \|n(u)\|^2] \} \\ = \frac{1}{2} \frac{\|n(u)\|^2}{e(u)} \cdot [1 - 3\lambda_\sigma^{-2}(u)],$$

if the "averaged wavelength" functional λ_σ is defined by

$$\lambda_\sigma(u) = \left(\frac{\|n(u)\|_{H^\sigma}^2}{\|n(u)\|^2} - 1 \right)^{-1/2}$$

In much the same way as was done in the proof of Theorem 7.4, it is then possible to show that $V_\sigma(u) \geq 0$ for all time $t > 0$ if u is a solution with initial data satisfying $\lambda_\sigma(u_0) > \lambda^0$ and $m(u) = \frac{1}{2} \|u\|_{H^\sigma}^2 < \epsilon_0$, for λ^0 and ϵ_0 sufficiently small. [It is intriguing to observe that no value of σ for which these results can be obtained along the lines indicated above ($\sigma > \frac{1}{2}$) corresponds to an operator D which leads to a group velocity $\hat{G}(k)$ which is positive for all real k ; cf. Eq. (6.10)].

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