

Fourier-Weierstrass Mode Analysis for Thermally Driven Turbulence

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The eddy size and temperature distribution in the inertial subrange of turbulent Rayleigh-Bénard flow at a Rayleigh number $Ra \approx 10^{11}$ is calculated from the Boussinesq equations. Velocity and temperature fields are decomposed into Fourier modes with geometrically scaling wave numbers. $\frac{1}{3}$ scaling for both velocity \mathbf{u} and temperature Θ is obtained in agreement with Shraiman and Siggia's finding, but there is no indication of a new scaling range as claimed recently. We resolve this discrepancy. The isotropization by eddy decay is demonstrated in terms of the $u_3\Theta$ scaling, which appreciably differs from $\frac{2}{3}$; testing this new quantity experimentally would be of great interest.

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Recent experiments by Libchaber and co-workers [1-3] offered new insight into high-Rayleigh-number Rayleigh-Bénard (RB) convection. Particularly the frequency power spectrum $P(\omega)$ of the temperature fluctuations $\Theta(\mathbf{x}, t)$ and its analog in r space, the temperature structure function $D_{\Theta\Theta}(r) \equiv \langle [\Theta(\mathbf{x} + \mathbf{r}) - \Theta(\mathbf{x})]^2 \rangle$, attracted attention. Procaccia and Zeitak [4,5] proposed a new large-eddy scaling range $D_{\Theta\Theta}(r) \propto r^{2/5}$, in distinction to classical $\frac{2}{3}$ scaling. Their argument is based on a dimensional analysis derived from a Lagrangian real-space mean-field theory [6]. At first glance, this new scaling range seems to be supported by the data of Wu *et al.* [3], but Castaing [7] showed how to account for the same data without demanding a new scaling range. Shraiman and Siggia [8] even raised serious doubts about the existence of the new scaling range, pointing out inconsistencies.

In this situation the questions that urgently arise are as follows: What is the correct temperature and velocity scaling in RB convection, how can the contradictions in theory be resolved, and how does the anisotropy by gravity in RB flow scale? The latter is responsible for the temperature-velocity coupling via gravitation. In this Letter we address these questions.

We solve the full underlying dynamical equations (Navier-Stokes and heat transfer equations) in the Boussinesq approximation, using a Fourier decomposition with a geometrically scaling selection of wave numbers introduced recently [9] to deal with high-Reynolds-number turbulence. Since the wave numbers are chosen as algebraically scaling and the mode amplitudes also turn out to behave algebraically, we denote this mode expansion the Fourier-Weierstrass analysis.

According to [9] we take twelve basic wave numbers, $\mathbf{p}_1^{(0)}, \dots, \mathbf{p}_{12}^{(0)}$, whose Fourier amplitudes describe the largest eddies. The amplitudes of the geometrically scaling wave numbers $\mathbf{p}_n^{(l)} = \pm \lambda^l \mathbf{p}_n^{(0)}$, $l = 1, 2, \dots, 10$, $\lambda = 2$, deal with successively smaller eddies. Thus eddies over 3 orders of magnitude are admitted. The interactions between the resulting 792 mode amplitudes for the velocity and temperature fluctuations via the Navier-Stokes non-

linearity, the nonlinearity in the heat equation, and the coupling of the temperature field to the velocity field by gravity are included exactly. Those interactions, on the other hand, which result in a wave number not belonging to the selected set are omitted. Within the restricted set of wave numbers, we solved the complete dynamical equations without any further approximation.

No new scaling range for $P(\omega)$ or $D_{\Theta\Theta}(r)$ could be found; instead, classical $\frac{2}{3}$ scaling for $D_{\Theta\Theta}(r)$ is confirmed, in agreement with [7] and [8]. For the velocity-temperature cross correlation we found an unexpected new scaling behavior.

The boundary conditions are crucial. As is common for plane-wave $\exp[i\mathbf{p} \cdot \mathbf{x}]$ mode analyses, their effect is condensed in the choice of an external forcing $f(\mathbf{x}, t)$. Procaccia and Zeitak [4,5] chose a forcing on the largest scale in the heat equation [see Eq. (5) in [4]]. This is the well agreed upon picture of how turbulence is driven and we adopt it as a reasonable representation of the larger-scale thermal winds. The effect of smaller-scale plumes from the boundary layers is not yet taken into account here; it will be dealt with separately.

The control parameters of the RB system are the Rayleigh number $Ra = \beta g L^3 \Delta / \nu \kappa$ and the Prandtl number $Pr = \nu / \kappa$, where L and Δ are the outer length and temperature scale, and β , g , ν , and κ are volume expansion coefficient, gravity, viscosity, and thermal conductivity, respectively. We numerically solve the Boussinesq equations with $Ra = 3.1 \times 10^{11}$, $Pr = 1$. According to [2], the Reynolds number behaves as $Re \propto Ra^{3/7}$, so $L/\eta \approx Re^{3/4} \approx Ra^{9/28} \approx 5 \times 10^3 \approx 2^{11}$. Therefore, eleven levels are suitable to represent the inertial subrange (ISR) for that Ra number. The resulting spectra are shown in Fig. 1. There is clearly no deviation in the spectra of velocity and temperature from classical $\frac{2}{3}$ scaling, whereas the temperature-velocity cross spectrum $\text{Real}[\langle u_3^*(p^{(l)}) \times \Theta(p^{(l)}) \rangle]$, which is apart from the coupling constant βg the average energy input $E_{in}^{(l)}$ into the velocity equation by buoyancy on the scale $r^{(l)} \approx (p^{(l)})^{-1} \approx L \lambda^{-l}$, is definitely steeper, and hence different from what one would naively expect from the scaling of $u(p^{(l)})$ and

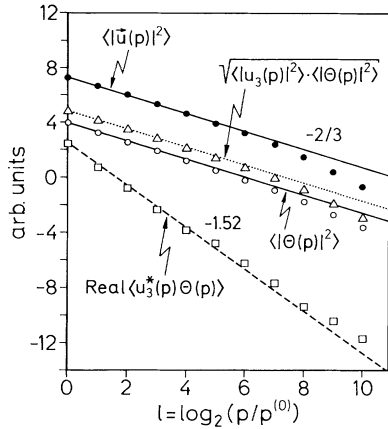


FIG. 1. \log_2 of the calculated time averages $\langle |\mathbf{u}(p^{(l)})|^2 \rangle$ (●), $\langle |\Theta(p^{(l)})|^2 \rangle$ (○), and $E_{in}^{(l)} = \beta g \text{Real}[\langle u_3^*(p^{(l)})\Theta(p^{(l)}) \rangle]$ (□) vs inverse scale $\log p^{(l)} \propto$ level l . The solid lines show the classical $\frac{2}{3}$ -scaling laws; the dashed one is 1.52 scaling. For comparison, $\beta g [\langle |u_3(p^{(l)})|^2 \rangle \langle |\Theta(p^{(l)})|^2 \rangle]^{1/2}$ is also shown (△ and dotted line). Beyond $l \approx 6$ the energy loss by viscosity becomes visible. The time average extends over 800 large-eddy turnover times.

$\Theta(p^{(l)})$. If we fit the scaling law $E_{in}^{(l)} \propto (p^{(l)})^{-\zeta}$ to our data, we get $\zeta \approx 1.52 \pm 0.07$.

The significantly faster decrease of the asymmetry, i.e., of the $u_3\Theta$ cross correlation, means a fast approach to isotropy with decreasing scale. This can also be demonstrated by looking at the ‘‘coherence,’’ which we define as the mean value of $\text{sgn}[u_3(\mathbf{x},t)\Theta(\mathbf{x},t)]$ in real space. The smaller the scales that contribute to $u_3(\mathbf{x},t)$ and $\Theta(\mathbf{x},t)$, the less coherence we find. Defining $u_3^{(l)}(0,t) \equiv \sum_{\mathbf{p} \in K_l} [u_3^{(l)}(\mathbf{p},t) + u_3^{(l)*}(\mathbf{p},t)]$ and analogously for $\Theta^{(l)}(0,t)$, we obtain about 12% coherence of $u_3^{(l)}(0,t)$ and $\Theta^{(l)}(0,t)$ at the input level $l=0$. This coherence decreases to about 4% at the first level $l=1$ and 1% after a few more eddy decays. There seems to remain a small but finite level of anisotropy even at very small scales. The Nusselt number $Nu \equiv \langle u_3(\mathbf{x})\Theta(\mathbf{x}) \rangle L / \kappa \Delta$ is of order 10^7 . Note that this describes the heat conductance of the bulk turbulence and does not include the heat resistance of the boundary layers determining the much smaller experimental Nusselt number [2].

The structure functions $D_{uu}(r)$, $D_{\Theta\Theta}(r)$, and $D_{u_3\Theta}(r)$ corresponding to the spectra of Fig. 1 are shown in Fig. 2. In the ISR, $\frac{2}{3}$ scaling for D_{uu} and $D_{\Theta\Theta}$ and the steeper scaling for $D_{u_3\Theta}$ are clearly visible, whereas, of course, in the viscous subrange (VSR) all structure functions behave like r^2 .

Now, to understand why no new scaling range for $\Theta(p)$ and $u(p)$ can result, we examine the stationary kinetic and thermal energy balance resulting from the Bousinesq equations:

$$T_u^{l \rightarrow l+1} - T_u^{l+1 \rightarrow l} + E_{in}^{(l)} - E_{diss}^{(l)} = 0, \quad (1)$$

$$T_\Theta^{l \rightarrow l+1} - T_\Theta^{l+1 \rightarrow l} - E_{diff}^{(l)} + \epsilon_\Theta \delta_{l,0} = 0. \quad (2)$$

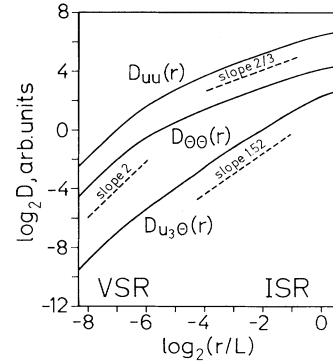


FIG. 2. log-log plot of the structure functions vs eddy size r . The scaling exponents of r and p only differ by the sign, since discrete p are used.

Here,

$$T_u^{l \rightarrow l+1} \propto p^{(l)} u(p^{(l+1)}) u(p^{(l+1)}) u(p^{(l)}) \quad (3)$$

and

$$T_\Theta^{l \rightarrow l+1} \propto p^{(l)} u(p^{(l+1)}) \Theta(p^{(l+1)}) \Theta(p^{(l)}) \quad (4)$$

denote the kinetic and thermal energy transport from level l to $l+1$, and $\epsilon_\Theta \delta_{l,0}$ is the (constant) thermal energy input into the largest temperature eddies, reflecting, as mentioned above, the thermal boundary conditions. $E_{diss}^{(l)}$ and $E_{diff}^{(l)}$ denote dissipation and thermal diffusion on scale $r^{(l)}$; they contribute negligibly little in the ISR (i.e., l not too large) for the large Rayleigh numbers we consider.

Figure 3 demonstrates the high degree to which the

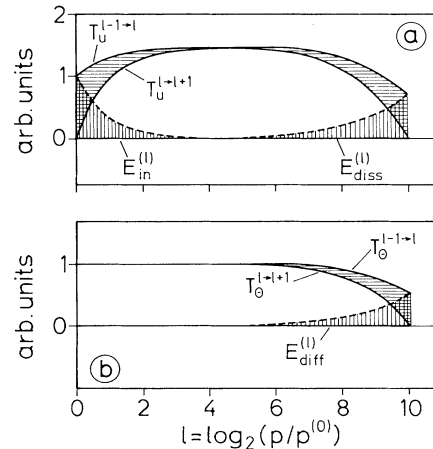


FIG. 3. (a) Stationary kinetic energy and (b) thermal intensity balance equations (1) and (2) vs inverse scale $\log p^{(l)} \propto$ level l . In the ISR, $T_u^{l \rightarrow l+1}$ and $T_\Theta^{l \rightarrow l+1}$ are constant, resulting in classical scaling. $E_{in}^{(l)}$ behaves like a geometric sequence and does not play a significant role for $l > 2$. For $l \geq 8$, dissipation $E_{diss}^{(l)}$ and thermal diffusion $E_{diff}^{(l)}$ become relevant. The shaded areas between the curves are equal, proving the high degree to which the balance equations (1) and (2) are satisfied.

balance equations (1) and (2) are satisfied. Already for small l , i.e., the largest scales, $T_u^{l \rightarrow l+1}$ becomes constant. Define as usual the scaling exponent α of the velocity by $u(p^{(l)}) \propto \lambda^{-l\alpha}$. To keep consistency with incompressibility, all components u_i scale identically. If, now, $T_u^{l \rightarrow l+1}$ is independent of l , then (3) implies $\alpha = \frac{1}{3}$.

To get better insight into why we do *not* find a $\frac{3}{5}$ scaling, despite the use of the same equations and the same forcing as in [4] and [5], we perform the l sum of the energy balance equation (1) from the largest (outer) scale $l=0$ down to a variable inertial range scale l . The quantities one obtains correspond to the super- r -scale fields of the variable-range decomposition [6], employed in [4] and [5]. Equation (1) (omitting $E_{\text{diss}}^{(l)}$) yields

$$T_u^{l \rightarrow l+1} = \sum_{l'=0}^l E_{\text{in}}^{(l')} = E_{\text{in}}^{(0)} \sum_{l'=0}^l \lambda^{-\zeta l'} \propto \frac{1 - \lambda^{-(l+1)\zeta}}{1 - \lambda^{-\zeta}}. \quad (5)$$

For the second equality we used the buoyancy input $E_{\text{in}}^{(l)} \propto \lambda^{-l\zeta}$. From the Boussinesq equation we calculated $\zeta \approx 1.52$ as explained above and in Fig. 1, but any other (positive) scaling exponent ζ would lead to the same result: $T_u^{l \rightarrow l+1}$ rapidly becomes practically l independent, as α rapidly converges to $\frac{1}{3}$. Particularly, this is valid also for $\zeta = \frac{4}{5}$ as proposed by Procaccia and Zeitak [4,5].

Analyzing analogously the thermal balance equation (2), it is physically clear that $T_\Theta^{l \rightarrow l+1}$ also has to be constant in the ISR because thermal energy is fed only into the largest eddies. This is confirmed by the numerical solution, cf. Fig. 3(b). Thus, the temperature scales like $\Theta^{(l)} \propto \lambda^{-\beta l}$ with $\beta = (1 - \alpha)/2$, which converges to $\beta = \frac{1}{3}$ since α does, i.e., classical Kolmogorov scaling again.

These p -space arguments have their complete analog in r space. The velocity-temperature coupling [4,5], $G^{(r)} \equiv \beta g \langle u_3^{(r)} \Theta^{(r)} \rangle$, is by definition the *total input* on *all* scales from the outer scale L down to r , since in variable-range decomposition [6] the fields $u_3^{(r)}$ and $\Theta^{(r)}$ contain *all* scales $\geq r$. There is no freedom to make an assumption on this well-defined quantity [10]. The only freedom is the choice of the forcing of the *heat* equation. $G^{(r)}$ corresponds to the geometrical sum $\sum_{l'=0}^l E_{\text{in}}^{(l')}$ in Eq. (5) and behaves as

$$\beta g (\langle u_3^{(r=0)} \Theta^{(r=0)} \rangle - \text{const} \times r^\zeta).$$

This quickly becomes r independent for decreasing r . It is the *differential* input $E_{\text{in}}^{(l)}$ and its r -space analog which obey a power law. The total inputs by buoyancy [Eq. (5) as well as $G^{(r)}$] are geometrical sums and do *not* scale algebraically as long as there is only large-eddy forcing [10]. So the real-space analysis with variable-range decomposition [6] is in perfect agreement with the Fourier-Weierstrass mode analysis, if properly treated.

Finally, we demonstrate that a properly done dimensional analysis also leads to correct scaling. The $\frac{6}{5}$ and $\frac{2}{3}$ scaling laws [11,12] for the velocity and temperature structure functions, $D_{uu}(r)$ and $D_{\Theta\Theta}(r)$, respectively, result from the misleading assumption that the essential pa-

rameter in the ISR is βg , besides, of course, ϵ_Θ . Then $D_{uu}(r) \sim \epsilon^{2/5} (\beta g)^{4/5} r^{6/5}$ and $D_{\Theta\Theta}(r) \sim \epsilon^{4/5} (\beta g)^{-2/5} r^{2/5}$, which we just proved by analytical argument as well as by numerical simulation to be self-contradictory. The physical reason is that, although βg determines the *way* in which kinetic energy is fed into the Navier-Stokes equation (namely, by gravity), the *relevant parameter* nevertheless is the energy flux ϵ as in the classical dimensional analysis for homogeneous, isotropic turbulence.

If for dimensional analysis we take the constant fluxes ϵ and ϵ_Θ as the relevant parameters, this leads to $D_{uu}(r) \sim (\epsilon r)^{2/3}$, $D_{\Theta\Theta}(r) \sim \epsilon r^{2/3} \epsilon_\Theta^{-1/3}$, in complete agreement with our numerical solution.

To fully explain the observed properties of RB flow, the thermal and kinetic energy supply caused by the plumes detaching from the boundary layers has to be taken into account. We mimicked thermal plumes by an additional thermal energy input on $l=4$, corresponding to a scale about 1 order of magnitude below the external one to cope with typical plume diameters. Preliminary numerical solutions of the Boussinesq equations indicate a substantial decrease in the temperature scaling exponent between external and plume scale [13], while the $\frac{2}{3}$ scaling of the velocity spectrum is hardly changed.

The additional thermal input is also seen in the scaling of the temperature-velocity cross spectrum; the resulting exponent decreases from 1.52 ± 0.07 to 1.25 ± 0.07 (see [13]). An experimental measurement of this exponent would provide an interesting test of this theory.

A further extension of the Fourier-Weierstrass cascade model has to allow for resolving the spatial spottiness of the fluctuations. Intermittency corrections and the measured exponential-type probability distribution functions for velocity differences can be accounted for in this way [14].

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i.e., a physically unreasonable average heat flow $\text{Re}(\langle u^* T^{(l)} \rangle) = E_{\text{in}}^{(l)} / \beta g$ from the cold top to the hot bottom of the Rayleigh-Bénard cell.

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