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ORIGINAL RESEARCH

Parameter reduction for the Yld2004-18p yield criterion

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Abstract The Yld2004-18p yield criterion uses 18 parameters to define anisotropy for a full 3D stress state. It is demonstrated in this paper that dependencies between the parameters exist and for a given set of experimental data the parameters are not uniquely defined. Analysis of the yield function shows that two specific combinations of parameters do not contribute to the value of the yield function. Therefore, the number of parameters can be reduced to 16, without any loss of flexibility. Similarly, the number of parameters for the plane stress version of this yield criterion reduces from 14 to 12.

Keywords Yield function · Anisotropic material · Parameter reduction · Yld2004-18p

Introduction

The Yld2004-18p yield criterion as proposed in [1] is used by a growing number of researchers e.g. [2–9]. One of the advantages of the model is its flexibility in describing plastic deformation of orthotropic materials and the availability of a 3D and a plane stress version. The model as published has 18 parameters in the 3D version. In the plane stress version it reduces to 14 parameters. The parameters are commonly determined by fitting to uniaxial yield stresses and Lankford R-values, the equi-biaxial yield stress and the R_b value (ratio between strain in rolling and transverse direction in an equi-biaxial stress state). It is also suggested that parameters can be fit to 'virtual' experiments with crystal plasticity models.

A least squares estimation was performed in this work with a gradient based algorithm. To avoid local minima, several starting values were used for the parameter set. Surprisingly, many different sets achieved exactly the same minimised error value. This lead to investigating the sensitivity of the yield locus to the parameter set as described in Section "Sensitivity analysis". It was found that 2 particular combinations of parameter variations do not influence the yield function at all. In Section "Parameter reduction", the origin of this non-uniqueness is demonstrated and a reduction of the number of parameters for the model is proposed.

Description of the Yld2004-18p yield function

The Yld2004-18p yield criterion as proposed in [1] is defined as

$$f = \phi - 4\bar{\sigma}^a = 0 \text{ with } \phi = \sum_{i=1}^{3}\sum_{j=1}^{3}\left|\tilde{S}'_i - \tilde{S}''_j\right|^a \tag{1}$$

where $\tilde{S}'_i$ and $\tilde{S}''_j$ are the eigenvalues of the transformed stress tensors $\tilde{\mathbf{s}}'$ and $\tilde{\mathbf{s}}''$ respectively. The transformed stresses are functions of the deviatoric stress tensor $\mathbf{s}$.

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{1} \tag{2}$$

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These transformed stresses are commonly derived in vector format from a transformation through the matrices $\mathbf{C}'$ and $\mathbf{C}''$:

$$\tilde{\mathbf{s}}' = \mathbf{C}'\mathbf{s} \qquad \text{and} \qquad \tilde{\mathbf{s}}'' = \mathbf{C}''\mathbf{s} \tag{3}$$

fully defined by:

$$\begin{Bmatrix} \tilde{s}'_{xx} \\ \tilde{s}'_{yy} \\ \tilde{s}'_{zz} \\ \tilde{s}'_{yz} \\ \tilde{s}'_{zx} \\ \tilde{s}'_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & -c'_{12} & -c'_{13} & 0 & 0 & 0 \\ -c'_{21} & 0 & -c'_{23} & 0 & 0 & 0 \\ -c'_{31} & -c'_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c'_{66} \end{bmatrix} \begin{Bmatrix} s_{xx} \\ s_{yy} \\ s_{zz} \\ s_{yz} \\ s_{zx} \\ s_{xy} \end{Bmatrix} \tag{4}$$

and

$$\begin{Bmatrix} \tilde{s}''_{xx} \\ \tilde{s}''_{yy} \\ \tilde{s}''_{zz} \\ \tilde{s}''_{yz} \\ \tilde{s}''_{zx} \\ \tilde{s}''_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & -c''_{12} & -c''_{13} & 0 & 0 & 0 \\ -c''_{21} & 0 & -c''_{23} & 0 & 0 & 0 \\ -c''_{31} & -c''_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c''_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c''_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c''_{66} \end{bmatrix} \begin{Bmatrix} s_{xx} \\ s_{yy} \\ s_{zz} \\ s_{yz} \\ s_{zx} \\ s_{xy} \end{Bmatrix} \tag{5}$$

The matrices $\mathbf{C}'$ and $\mathbf{C}''$ are not necessarily symmetric and contain in total 18 parameters to describe anisotropy of the yield function.

Sensitivity analysis

The sensitivity analysis starts with determining the influence of the model parameters on the position of the yield surface in stress space. For the Yld2004-18p yield function the model parameters c_i represent the $n = 18$ parameters $c'_{12}, c'_{13}, ..., c''_{55}, c''_{66}$. Since we want to investigate the change in position of the yield surface as function of the parameters, we define the 'result' r_j as the value of the Von Mises equivalent stress at the yield surface in a given stress direction $\boldsymbol{\sigma}_j$. I.e. $\boldsymbol{\sigma}_j$ is scaled to $\boldsymbol{\sigma}^*_j = \alpha\boldsymbol{\sigma}_j$ such that $\phi(\boldsymbol{\sigma}^*_j) = 1$ and $r_j = \sigma_{VM}(\boldsymbol{\sigma}^*_j)$.

The result r_j is determined for $m = 1000$ randomly distributed stresses to get an impression of the influence of the parameters c_i all over the stress space. The effect of model parameters on the yield surface in several stress directions can be gathered in a derivative matrix $\mathbf{D}$:

$$\mathbf{D} = \begin{bmatrix} \frac{\partial r_1}{\partial c_1} & \cdots & \frac{\partial r_m}{\partial c_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_1}{\partial c_n} & \cdots & \frac{\partial r_m}{\partial c_n} \end{bmatrix} \tag{6}$$

which is obtained by numerical differentiation in this work. The variation in results $\mathrm{d}\mathbf{r}$ as function of a small perturbation in parameters $\mathrm{d}\mathbf{c}$ is then

$$\mathrm{d}\mathbf{r} = \mathbf{D}^{\mathrm{T}}\mathrm{d}\mathbf{c} \tag{7}$$

The length of $\mathrm{d}\mathbf{r}$ is a measure for the effect of a change of parameters $\mathrm{d}\mathbf{c}$ over the complete yield surface:

$$\|\mathrm{d}\mathbf{r}\|^2 = \mathrm{d}\mathbf{r}^{\mathrm{T}}\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{c}^{\mathrm{T}}\mathbf{D}\mathbf{D}^{\mathrm{T}}\mathrm{d}\mathbf{c} = \mathrm{d}\mathbf{c}^{\mathrm{T}}\mathbf{S}\,\mathrm{d}\mathbf{c} \tag{8}$$

By definition, $\mathbf{S}$ is a symmetric, semi-positive definite matrix with consequently real and non-negative eigenvalues λ_i and orthogonal eigenvectors $\mathbf{p}_i$ such that

$$\mathbf{S}\mathbf{p}_i = \lambda_i\mathbf{p}_i \qquad i = 1..n \tag{9}$$

All eigenvectors can be normalised and assembled in an orthogonal matrix $\mathbf{P}$ and all eigenvalues in a diagonal matrix $\boldsymbol{\Lambda}$ such that

$$\mathbf{S}\mathbf{P} = \boldsymbol{\Lambda}\mathbf{P} \quad \Rightarrow \quad \mathbf{P}^{\mathrm{T}}\mathbf{S}\mathbf{P} = \boldsymbol{\Lambda} \tag{10}$$

To find the direction in the parameter space with the smallest influence on the output parameters, the linear orthogonal transformation $\mathbf{P}$ is applied on the parameter space:

$$\mathbf{c} = \mathbf{P}\mathbf{c}^* \tag{11}$$

$$\mathbf{c}^* = \mathbf{P}^{\mathrm{T}}\mathbf{c} \tag{12}$$

Substitution of Eqs. 11 in Eq. 8 yields

$$\|\mathrm{d}\mathbf{r}\|^2 = \mathrm{d}\mathbf{c}^{*\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{S}\mathbf{P}\mathrm{d}\mathbf{c}^* = \mathrm{d}\mathbf{c}^{*\mathrm{T}}\boldsymbol{\Lambda}\mathrm{d}\mathbf{c}^* \tag{13}$$

This shows that a variation of the parameters in the direction of an eigenvector of $\mathbf{S}$ contributes to a change $\|\mathrm{d}\mathbf{r}\|^2$ that is proportional to the corresponding eigenvalue. Most importantly for the current analysis, the eigenvector corresponding to an eigenvalue equal to zero represents a direction in the parameter space with no effect on the yield surface.

It must be emphasised that the rotation $\mathbf{P}$ is defined locally in the parameter space. When evaluating the sensitivity of the model with different parameters the effect may be different.

As an example, the sensitivity of the yield criterion is determined for the aluminium alloys 2090-T3 and 6111-T4, fitted with the Yld2004-18p model and parameters obtained from [1]. From Fig. 1 it can be seen that all parameters have

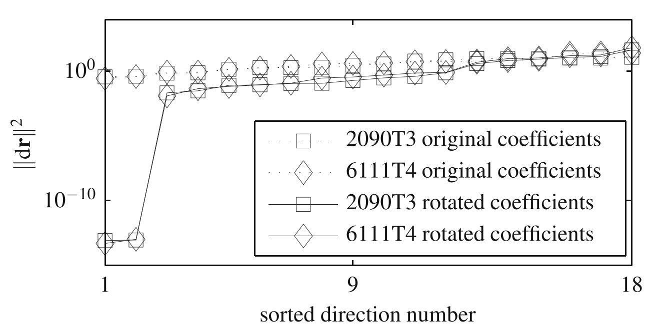

Fig. 1 Sorted logarithmic effect on the Yld2004-18p criterion for the original and the rotated parameters

influence on the yield surface when evaluated in the original orientation. However, after rotating the parameters to $\mathbf{c}^*$, two directions with a negligible influence on the yield surface are found for both materials.

The parameter subspace with zero influence can be represented as:

$$\begin{Bmatrix} c'_{12} \\ c'_{13} \\ c'_{21} \\ c'_{23} \\ c'_{31} \\ c'_{32} \\ c''_{12} \\ c''_{13} \\ c''_{21} \\ c''_{23} \\ c''_{31} \\ c''_{32} \end{Bmatrix} = \alpha \begin{Bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} + \beta \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{Bmatrix} \tag{14}$$

and this subspace is independent of the position in the parameter space as will be proven in the next section. As a result of this analysis it is observed that the 18 parameters of the Yld2004-18p model are not unique and a reduction by 2 parameters is possible.

Parameter reduction

The eigenvalues of a second order tensor $\mathbf{A}$ fulfil the condition

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{15}$$

where λ_i and $\mathbf{v}_i$ are the corresponding eigenvalue and eigenvector. Adding a scaled unit matrix $\mathbf{1}$ to $\mathbf{A}$ gives

$$(\mathbf{A} + p\mathbf{1})\,\mathbf{v}_i = (\lambda_i + p)\mathbf{v}_i \tag{16}$$

hence, the eigenvalues of $\mathbf{A} + p\mathbf{1}$ are $\lambda_i + p$. Using this property in Eq. 1 yields:

$$\begin{aligned} \phi(\tilde{\mathbf{s}}' + p\mathbf{1}, \tilde{\mathbf{s}}'' + p\mathbf{1}) &= \sum_{i=1}^{3}\sum_{j=1}^{3} \left| \tilde{S}'_i + p - \tilde{S}''_j - p \right|^a \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \left| \tilde{S}'_i - \tilde{S}''_j \right|^a = \phi(\tilde{\mathbf{s}}', \tilde{\mathbf{s}}'') \end{aligned} \tag{17}$$

So, if we can define a matrix $\mathbf{M}$ such that $\mathbf{Ms} = p\mathbf{1}$, while $\mathbf{M}$ has the same non-zero structure as $\mathbf{C}'$ and $\mathbf{C}''$ then ϕ is independent of the addition of $\mathbf{M}$ to the parameter matrices $\mathbf{C}'$ and $\mathbf{C}''$.

In index notation, the relation holds

$$M_{ijkl}s_{kl} = p\delta_{ij}, \quad \text{subject to} \quad s_{ij} = s_{ji}, \quad s_{ii} = 0 \tag{18}$$

Since this must hold for all s_{kl} it is required that $M_{ijkl} = \delta_{ij}B_{kl}$ and $B_{kl}s_{kl} = p$. In matrix format this leads to the condition

$$\mathbf{Ms} = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 & 0 & 0 \\ m_{21} & 0 & m_{23} & 0 & 0 & 0 \\ m_{31} & m_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} s_{xx} \\ s_{yy} \\ -s_{xx} - s_{yy} \\ s_{yz} \\ s_{zx} \\ s_{xy} \end{Bmatrix} = \begin{Bmatrix} p \\ p \\ p \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{19}$$

or, since p is arbitrary

$$\begin{aligned} m_{12}s_{yy} - m_{13}(s_{xx} + s_{yy}) &= m_{21}s_{xx} - m_{23}(s_{xx} + s_{yy}) \\ &= m_{31}s_{xx} + m_{32}s_{yy} \quad \forall s_{xx}, s_{yy} \end{aligned} \tag{20}$$

Combining factors with s_{xx} and s_{yy} gives:

$$\begin{aligned} -m_{13}s_{xx} + (m_{12} - m_{13})s_{yy} &= (m_{21} - m_{23})s_{xx} - m_{23}s_{yy} \\ &= m_{31}s_{xx} + m_{32}s_{yy} \quad \forall s_{xx}, s_{yy} \end{aligned} \tag{21}$$

such that the requirements for m_{ij} become

$$-m_{13} = m_{21} - m_{23} = m_{31} \quad \text{and} \quad m_{12} - m_{13} = -m_{23} = m_{32} \tag{22}$$

With 6 parameters m_{ij} and 4 constraints, we are left with 2 degrees of freedom. We can choose arbitrarily $m_{12} = \alpha$ and $m_{23} = \beta$, then it follows from Eq. 22 that $m_{32} = -\beta$, $m_{13} = \alpha + \beta$, $m_{31} = -\alpha - \beta$, $m_{21} = -\alpha - \beta + \beta = -\alpha$. We can then write $\mathbf{M}$ as function of α and β:

$$\mathbf{M} = \alpha \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

This is exactly similar to the parameter space defined by the eigenvectors corresponding to zero eigenvalues as presented in Eq. 14. By inspection, it is easily verified that multiplication of $\mathbf{M}$ with any deviatoric stress $\mathbf{s}$ indeed gives a hydrostatic contribution. Now, a parameter set $\mathbf{C}^{*\prime} = \mathbf{C}' + \mathbf{M}$ and $\mathbf{C}^{*\prime\prime} = \mathbf{C}'' + \mathbf{M}$ will give exactly the same value for ϕ for all possible deviatoric stresses as the parameter set $\mathbf{C}'$ and $\mathbf{C}''$.

This observation can be used to reduce the dimension of the parameter set by 2. Arbitrarily choosing $\alpha = c'_{12} - 1$ and $\beta = c'_{13} - c'_{12}$ will make ${c^*_{12}}' = {c^*_{13}}' = 1$. Any other choice would give equivalent results, but in this way, isotropic behaviour will result in all parameters ${c^*_{ij}}' = {c^*_{ij}}'' = 1$ being equal to 1. Notice that for fitting to experiments or crystal plasticity data, the parameters c'_{12} and c'_{13} can be set to unity and the remaining 16 parameters can then be fitted to the required behaviour.

By negating $\mathbf{C}''$ and adapting the pre-factor, Yld2004-18p equals the recently introduced Yld2011-18p model [10]. Therefore, the same non-uniqueness is found in this yield function.

Conclusion

A sensitivity analysis demonstrated that for a large number of parameter sets, 2 directions in parameter space have no influence on the value of the yield function. Analysis of the yield function showed that these directions can be related to a hydrostatic component in the transformed stress tensor that is cancelled out in the function evaluation. This means that from 18 parameters, only 16 independently affect the yield function. Consequently, at least 16 data points are required to determine the model parameters and not 18, as is commonly assumed. For the plane stress version of Yld2004-18p, 12 out of 14 parameters have an independent effect. In 3D and plane stress, 2 parameters can be fixed independent of the material behaviour. It is suggested to set the parameters c'_{12} and c'_{13} to unity.

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