Simplified Calculus for the Design of a Cryogenic Current Comparator

Javier Sesé, Elena Bartolomé, Agustín Camón, Jaap Flokstra, Gert Rietveld, and Conrado Rillo

***Abstract*—The calculation of inductances of superconducting structures like the cryogenic current comparator (CCC) is not straightforward due to image effects and unknown current distributions. By treating the problem as a magnetic circuit, an approximate analytical expression for the self-inductance of the shielded CCC is obtained. This formula can be used to derive a "rule of thumb" for maximizing the value of an inductance inside a superconducting shield. Using this rule the design of an optimum CCC is simplified.**

***Index Terms*—Cryogenic current comparator (CCC), inductance calculations, SQUID, superconductors.**

I. INTRODUCTION

THE cryogenic current comparator (CCC) [1] allows the control of the ratio of two currents $I_1/I_2 = N_2/N_1$ (where N_1 and N_2 are integer numbers) with high precision. Resistance bridges based on the CCC allow resistance comparisons to be made at the highest metrological level. They are, at present, widely used to transfer the value of the quantum Hall resistance standard to very stable resistors [2]. More recently, the use of a CCC toward the realization of a quantum current standard is being investigated [3]–[5]. There, it will be used for the accurate amplification of quantized currents, in the pA range, produced by single electron tunneling devices.

II. DESIGN OF AN OPTIMUM CCC

The ultimate current resolution of a CCC per turn in the primary windings that is ideally coupled to a SQUID is given by [6]

$$\langle i_p^2 \rangle = \frac{8\varepsilon_{SQ}}{k_{SQ}^2 L_{\mathrm{CCC,eff}}} \tag{1}$$

where $\varepsilon_{SQ}/k_{SQ}^2$ is the coupled energy resolution of the SQUID and $L_{\mathrm{CCC,eff}}$ is the effective inductance of the CCC overlapped tube inside the superconducting shield (Fig. 1). The situation of maximum coupling (optimum sensitivity) occurs if the self-inductances of the sensing coil and the SQUID input coil L_i match. This matching condition can be closely achieved by constructing a sensing coil with N_S turns wound very close to the overlapped tube with $N_S = \sqrt{L_i/L_{\mathrm{CCC,eff}}}$.

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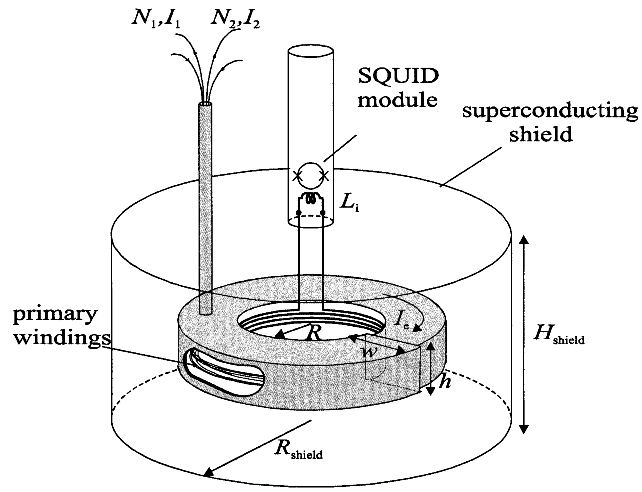
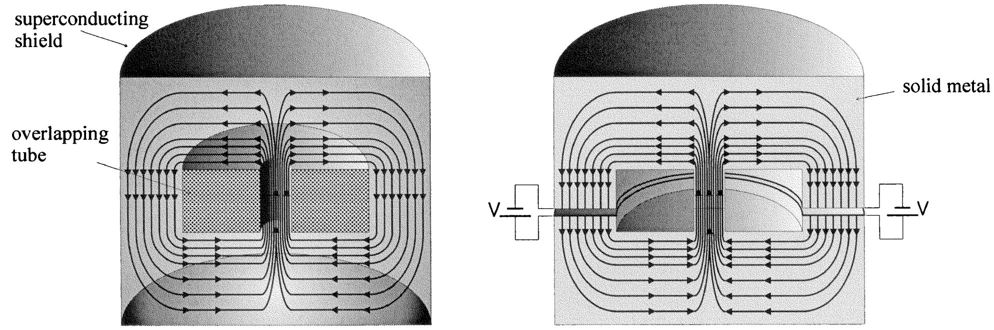


Fig. 1. Schematics of a CCC. The overlapped tube has cross section $h \cdot w$ and internal radius R. The flux created by the Meissner current $I_e = N_1 I_1 - N_2 I_2$ is picked up by a sensing coil connected to a SQUID. The whole is surrounded by a superconducting shield with dimensions R_{shield} and H_{shield}.

For a given number of turns, the cross section $h \cdot w$ is fixed and the optimum CCC will be the one with maximum $L_{\mathrm{CCC,eff}}$ allowed in the available diameter. The optimization of $L_{\mathrm{CCC,eff}}$ with respect to the shields, based on different numerical methods, has been the subject of several papers [7]–[10]. Numerical methods can be applied to any geometry and have proven to give accurate results but are time-consuming, not straightforward, and the underlying physics is hardly seen. It would be desirable to have an approximate analytical formula that is valid for the typical dimensions of a CCC. In this work, such a formula is derived using the theory of magnetic circuits.

III. SHIELDED CCC AS A MAGNETIC CIRCUIT

Magnetic circuit theory (see, for example, [11]) establishes a perfect analogy between the conduction of magnetic flux through a medium with permeability μ and the conduction of current through a medium of conductivity σ, where the two media have the same geometry. Fig. 2 shows this analogy. On the left side of Fig. 2, a current circulating in the overlapping tube creates a magnetic field that is contained inside the superconducting shield. The right part of Fig. 2 represents a solid piece of metal with the same dimensions as the space between the superconducting shield and the overlapping tube. A small slit has been introduced that allows a voltage difference V to be applied. The distribution of the current density inside the piece of metal is the same as the distribution of magnetic field in the previous case. The analogous quantity to the electrical



(a) (b)

Fig. 2. Equivalence between (a) magnetic and (b) electrical circuits.

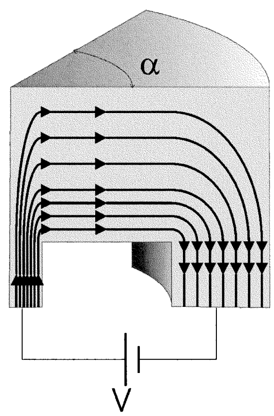


Fig. 3. Section of Fig. 2(b) for the calculation of electrical resistance. This section has electrical resistance that is π/α [radians] times the resistance of Fig. 2(b).

resistance R_{el} is the magnetic reluctance $\Re$. Even more, a magnetic circuit can be considered as a parallel or series connection of reluctances. For our case, the self-inductance L of the magnetic circuit is the inverse of its reluctance. L, $\Re$, and R_{el} are related by

$$L = \frac{1}{\Re} = \frac{\mu_0}{\sigma R_{el}} \tag{2}$$

where μ_0 is the permeability of vacuum and σ is the conductivity of the metal used. Hence, with the magnetic circuit theory, it is possible to deduce the self-inductance of a superconducting structure by making an electrical resistance measurement. Note that taking advantage of the symmetry, a simple piece like the one in Fig. 3 would be sufficient for performing the calculations. Nevertheless, for the design of a CCC it is still desirable to have an approximate analytical expression for the calculation of $L_{\text{CCC,eff}}$.

IV. Analytical Expression for the Effective Self-Inductance

In order to find an analytical expression, the magnetic circuit has been split in several pieces that approximate the real geometry. These are shown in Fig. 4.

Hence, our formula for $L_{\text{CCC,eff}}$ is

$$L_{\text{CCC,eff}} = \frac{1}{\Re_1 + \Re_2 + 2\Re_3}. \tag{3}$$

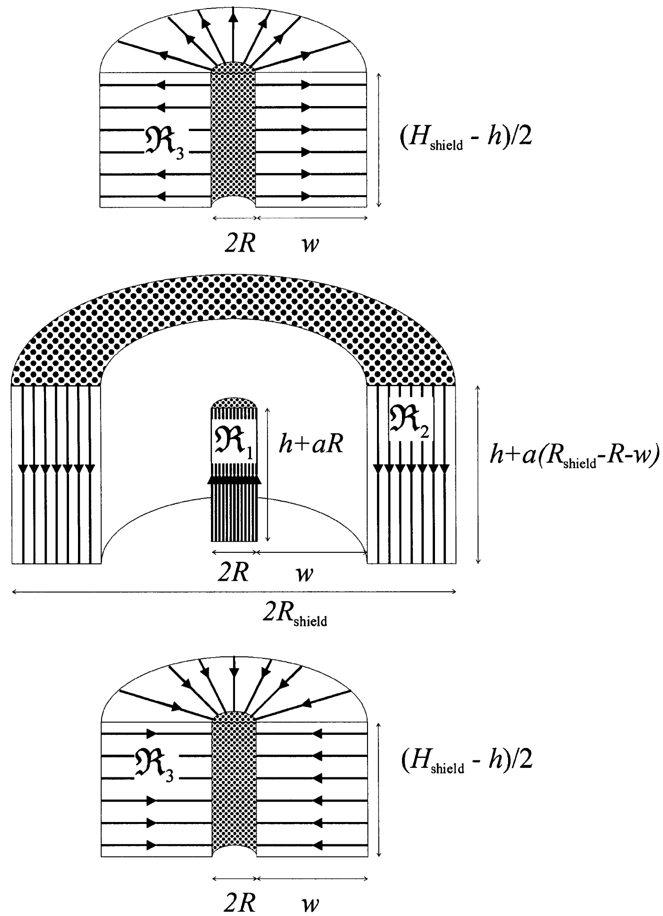


Fig. 4. Section of the schematics of Fig. 2 for the calculation of the reluctance. $\Re_1$ and $\Re_2$ are approximations for the reluctance of the central hole and the space between the overlapping tube and the superconducting shield, respectively. $\Re_3$ is the approximation of the reluctance for the top and bottom parts of Fig. 2. The arrows indicate the direction of flow of the magnetic flux.

When the magnetic field is homogeneous, the reluctance of a piece can be calculated as $\Re = l/\mu_0 S$ where l and S are the length and the cross section of the piece. A homogeneous magnetic field is expected to be a good approximation when the dimensions h and w are of the same order of magnitude as R_{shield} and H_{shield}. Hence, with this assumption, $\Re_1$ and $\Re_2$ are expressed as

$$\Re_1 = \frac{h + aR}{\mu_0 \pi R^2}; \quad \Re_2 = \frac{h + a\,(R_{\text{shield}} - R - W)}{\mu_0 \pi\,(R_{\text{shield}}^2 - (R+W)^2)}. \tag{4}$$

TABLE I
COMPARISON BETWEEN THE VALUE OF $L_{\mathrm{CCC,eff}}$ MAXIMUM OBTAINED WITH THE ANALYTICAL FORMULA USING $a = 0.8$ AND WITH NUMERICAL METHODS

H_{shield} (mm)	R_{shield} (mm)	h (mm)	w (mm)	R_{opt} (mm)	$L_{CCC,eff}$ maximum (nH)		
					analytical	numerical	difference (%)
80	30	10	10	15	24.18	22.65	6.3
80	40	10	10	23	41.76	40.60	2.8
80	50	10	10	31	60.55	60.90	-0.6
80	60	10	10	39	80.10	82.90	-3.5
80	30	15	15	12	13.27	12.30	7.3
80	40	15	15	19	27.01	25.80	4.5
80	50	15	15	27	42.51	41.80	1.7
80	60	15	15	35	58.94	59.30	-0.6
80	30	20	20	8	6.16	5.80	5.9
80	40	20	20	16	17.06	16.20	5.0
80	50	20	20	23	29.73	29.10	2.1
80	60	20	20	31	43.65	43.60	0.1
100	30	20	20	8	6.27	5.85	6.7
100	40	20	20	16	17.57	16.25	7.5
100	50	20	20	23	30.95	29.31	5.3
100	60	20	20	31	45.75	44.31	3.2
30	40	20	20	16	10.77	11.30	-4.9
40	40	20	20	16	13.83	14.60	-5.6
200	40	20	20	16	18.50	17.65	4.6

The parameter a is of the order of 1 and it takes into account the addition of some reluctance in the corners where the flux lines have to bend. An approximate value of a will be given in Section V. $\Re_3$ is a piece with a variable cross section, hence the reluctance is the result of an integral $\Re = \int (1/\mu_0 S(l))dl$ and it is given by

$$\Re_3 = \frac{\ln\left(1+\frac{w}{R}\right)}{\mu_0 \pi \left(H_{\mathrm{shield}} - h\right)}. \tag{5}$$

Note that the reluctance $\Re_3$ is counted twice in (3) because it appears in the top and in the bottom part of the field pattern.

V. COMPARISON OF ANALYTICAL FORMULA AND NUMERICAL RESULTS

We expect our formula to give good results as far as the assumption of homogeneous magnetic field still holds. Typical h and w dimensions are between 10 and 20 mm and typical R_{shield} and H_{shield} dimensions are not more than five times larger. To obtain the best value for a, we have compared expression (3) with numerical methods for the different geometries shown in Table I. Since the value obtained with the analytical approximation for $0.7 < a < 0.9$ is always within 11% of the value obtained with numerical methods, $a = 0.8$ is used in the rest of the paper. The numerical methods have earlier proved to be exact within 3% [7]. The numerical calculations involved in this table took some hours in a Pentium III PC at 866 MHz with 256 Mb RAM.

For a fixed R_{shield}, there is an optimum value for R for which $L_{\mathrm{CCC,eff}}$ is maximum. Fig. 5 shows a good agreement between the results obtained with expression (3) and with numerical methods for $h = w = 20$ mm, $H_{\mathrm{shield}} = 80$ mm, and $R_{\mathrm{shield}} = 60$ mm. In fact, the maximum occurs approximately when the internal area of the CCC, $A_{\mathrm{int}} = \pi R^2$, is equal to the area A_{ext} between the CCC outer side and the shield: $A_{\mathrm{ext}} = \pi R_{\mathrm{shield}}^2 - \pi(R+w)^2$. This criterion can be used as a simple "rule of thumb" to obtain the maximum value of $L_{\mathrm{CCC,eff}}$. It easily follows from the analytical formula (3) when we make the approximations $\Re_1 \approx (h/\mu_0 A_{\mathrm{int}})$, $\Re_2 \approx (h/\mu_0 A_{\mathrm{ext}})$ and $A_{\mathrm{int}} + A_{\mathrm{ext}} = \mathrm{constant}$. The rule of thumb is saying that the minimum reluctance (and hence

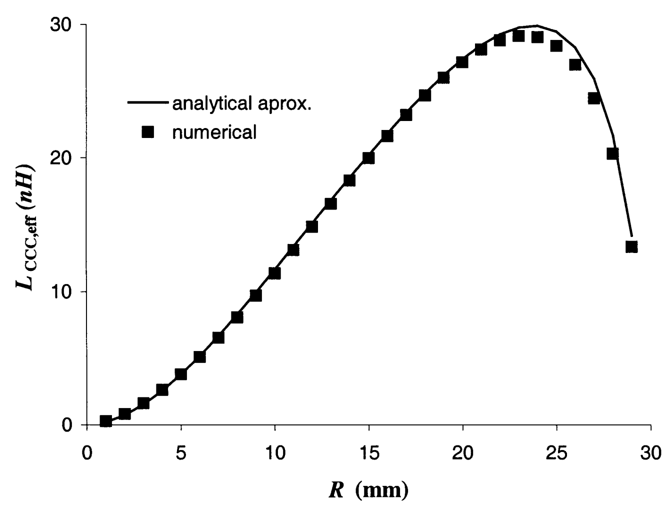


Fig. 5. Value of $L_{\mathrm{CCC,eff}}$ as a function of R for $h = w = 20$ mm, $H_{\mathrm{shield}} = 80$ mm and $R_{\mathrm{shield}} = 50$ mm. Squares correspond to numerical calculation based on finite element method. The line corresponds to the analytical approximation deduced in this work with $a = 0.8$. The analytical line deviates at most 10% from the squares.

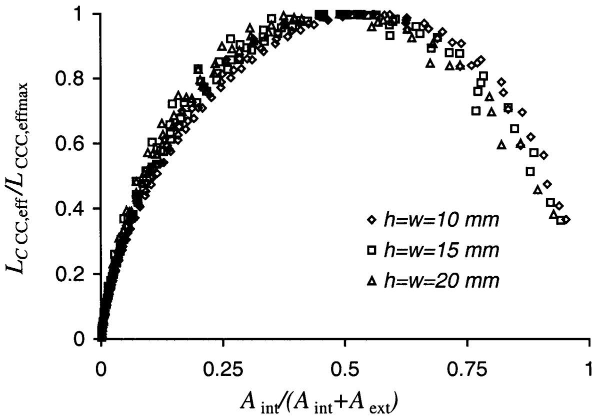


Fig. 6. Numerical calculation of $L_{\mathrm{CCC,eff}}$ for typical CCC dimensions, normalized to its maximum $L_{\mathrm{CCC,effmax}}$. In all studied cases, the maximum is reached when $A_{\mathrm{int}} = A_{\mathrm{ext}}$.

the maximum inductance) occurs when $\Re_1$ and $\Re_2$ have the same contribution to the total reluctance. Fig. 6 shows the value of $L_{\mathrm{CCC,eff}}/L_{\mathrm{CCC,effmax}}$ versus $A_{\mathrm{int}}/(A_{\mathrm{int}} + A_{\mathrm{ext}})$ as obtained with numerical calculations for different parameters h, w, and R_{shield}. The dependence is not very strong in the region of the maximum, in all the cases studied, the value of $L_{\mathrm{CCC,eff}}/L_{\mathrm{CCC,effmax}}$ was already higher than 0.95 when $A_{\mathrm{int}}/(A_{\mathrm{int}} + A_{\mathrm{ext}})$ is between 0.45 and 0.55.

VI. Conclusion

We have used the theory of magnetic circuits to deduce an approximate formula for the calculation of the effective self-inductance of the superconducting overlapped tube of a CCC inside a superconducting shield. We have validated the accuracy of the formula using numerical methods for typical dimensions of the CCC. By using this analytical formula, there is no need to use numerical methods anymore; hence the design of an optimum CCC is simplified. Even more, we have derived a general "rule of thumb," namely that the maximum value of the overlapping tube inductance of a CCC inside a superconducting shield occurs when the internal and external areas of the CCC are equal.

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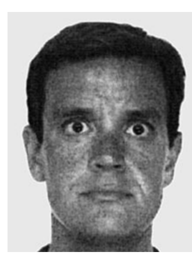

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