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Discrete Applied Mathematics 130 (2003) 495–501
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Note

Which matrices are immune against the transportation paradox?

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Received 6 March 2001; received in revised form 6 August 2002; accepted 2 January 2003

Dedicated to Prof. Rainer E. Burkard on the occasion of his 60th birthday

Abstract

We characterize the $m \times n$ cost matrices of the transportation problem for which there exist supplies and demands such that the transportation paradox arises. Our characterization is fairly simple and can be verified within $O(mn)$ computational steps. Moreover, we discuss the corresponding question for the algebraic transportation problem.

Keywords: Transportation problem; Transportation paradox; More-for-less paradox; Combinatorial optimization; Algebraic transportation problem

1. Introduction

An instance of the classical transportation problem is specified by an $m \times n$ matrix $C = (c_{ij})$, an m-dimensional vector $a = (a_i)$, and an n-dimensional vector $b = (b_j)$; all numbers c_{ij}, a_i, and b_j are nonnegative real numbers. This data has the following meaning: There are m sources and n sinks; at the ith source there is a supply of a_i units, and at the jth sink there is a demand of b_j units. It is assumed that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$,

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[1] Supported by the Spezialforschungsbereich F 003 "Optimierung und Kontrolle", Projektbereich "Diskrete Optimierung".
[2] Supported by the START program Y43-MAT of the Austrian Ministry of Science.

doi:10.1016/S0166-218X(03)00327-5

i.e., that the total supply equals the total demand. The cost for transporting one unit from the ith source to the jth sink is c_{ij}. The goal is to find a transportation plan that satisfies all the demand and that minimizes the overall transportation cost:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = a_i \quad \text{for } i=1,\ldots,m, \\ & \sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } j=1,\ldots,n, \\ & x_{ij} \geqslant 0 \quad \text{for } i=1,\ldots,m,\ \ j=1,\ldots,n. \end{aligned}$$

Here x_{ij} denotes the quantity shipped from source i to sink j. Let X denote the set of all transportation plans $x=(x_{ij})$ that fulfill the transportation constraints above. Moreover, let $\mathrm{TP}(C,a,b)$ denote the optimal objective value of the transportation instance specified by C, a, and b. We refer the reader to the book by Ahuja et al. [1] for more information on the transportation problem and its applications.

The source of the so-called *transportation paradox* is unclear. Apparently, several researchers have discovered independently from each other the following behaviour of the transportation problem: In certain cases of the transportation problem, an *increase* in the supplies and demands may lead to a *decrease* in the optimal transportation cost. In other words, by moving bigger amounts of goods around, one may save a lot of money. This surely sounds paradoxical!

Most papers on the transportation paradox cite the papers of Charnes and Klingman [6] (these authors use the term *more-for-less paradox*) and Szwarc [9] as sources of the transportation paradox. Charnes and Klingman [6] write "The paradox was first observed in the early days of linear programming history (by whom no one knows) and has been a part of the folklore known to some (e.g. A. Charnes and W.W. Cooper) but unknown to the great majority of workers in the field of linear programming". An anonymous referee brought a paper by Appa [2] to our attention. This paper deals with variants of the transportation problem and led to a series of comments which appeared subsequently in the same journal as Appa's article. In one of these comments, Appa [3] mentions that the transportation paradox is known as "Doig paradox" at the London School of Economics (LSE), named after Alison Doig who first talked about it around 1959. (Doig did not publish her discovery, but used it in exams at the LSE.)

To describe the transportation paradox more precisely, we introduce the following notation: We say that for two vectors $v=(v_i)$ and $v'=(v'_i)$ of equal dimensions, the vector v' *dominates* the vector v if and only if $v'_i \geqslant v_i$ holds for all i; we denote this by $v' \geqslant v$. Now let $C=(c_{ij})$, $a=(a_i)$, and $b=(b_j)$ be an instance of the transportation problem, and let $a' \geqslant a \geqslant 0$ and $b' \geqslant b \geqslant 0$ be two other supply and demand vectors. Then this situation constitutes a transportation paradox if and only if $\mathrm{TP}(C,a,b) > \mathrm{TP}(C,a',b')$.

Example 1. Consider the following instance of the transportation problem with $m=3$ sources and $n=3$ sinks: $a_1=0,\ a_2=1,\ a_3=1$ and $b_1=1,\ b_2=1,\ b_3=0$. The transportation cost c_{ij} equals $2^{|i-j|}$. It is easily verified that the value of the optimal solution is $\mathrm{TP}(C,a,b)=4$. By increasing the supplies to $a_1'=1,\ a_2'=1,\ a_3'=1$ and by increasing the demands to $b_1'=1,\ b_2'=1,\ b_3'=1$ (note that $\sum_{i=1}^{3} a_i' = \sum_{j=1}^{3} b_j'$ holds), the value of the optimal solution drops to $\mathrm{TP}(C,a',b')=3$.

We call a cost matrix C *immune against the transportation paradox* if regardless of the choice of the supplies and demands the transportation paradox does not arise. In other words, for all supply vectors a and a' with $a' \geqslant a$ and for all demand vectors b and b' with $b' \geqslant b$ an immune matrix C satisfies $\mathrm{TP}(C,a,b) \leqslant \mathrm{TP}(C,a',b')$. In this note, we give an exact and simple characterization of all cost matrices that are immune against the transportation paradox. This characterization leads to an $\mathrm{O}(mn)$ recognition algorithm for $m \times n$ immune matrices.

Several researchers (see e.g. [2,3,6,7,9]) investigated the following related question: Given a cost matrix C *and* given a supply vector a and a demand vector b, do there exist supply and demand vectors $a' \geqslant a$ and $b' \geqslant b$ such that the transportation paradox arises?

This question has been answered both from the theoretical point of view and from the algorithmic point of view. For the theoretical characterization suppose that the given transportation problem is written in nonredundant form (i.e. one of the equality constraints is deleted to arrive at a constraint matrix of full rank). The transportation paradox arises if and only if in *every* optimal solution of the dual of the nonredundant formulation of the transportation problem there exists a variable with negative value. (Related characterizations and reformulations of this criterion in terms of the primal problem for the nondegenerate case can be found in [6,7,9].) It is easy to see that the question raised by Charnes et al. can be answered algorithmically by solving a slightly modified version of the given transportation problem (see e.g. [2,3,6,9]).

Note that the computational effort to solve a transportation problem is higher than the effort needed for the recognition of immune matrices. Thus, the question raised in this paper can be answered more efficiently.

2. The characterization of immune matrices

Consider some fixed $m \times n$ cost matrix $C=(c_{ij})$. We say that four integers q,r,s,t with $1 \leqslant q,\ s \leqslant m$ and $1 \leqslant r,t \leqslant n$ (where $q \neq s$ and $r \neq t$) form a *bad quadruple* if

$$c_{qt} + c_{sr} < c_{qr}. \tag{1}$$

Lemma 2. *If there exists a bad quadruple for the cost matrix* $C=(c_{ij})$, *then* C *is not immune against the transportation paradox.*

Proof. Consider the supply vector a that has supply 1 at source q and supply 0 at every other source, and the demand vector b that has demand 1 at sink r and demand

0 at every other sink. Then $\mathrm{TP}(C,a,b)=c_{qr}$. Let the supply vector a' result from a by increasing the supply at source s to 1, and let the demand vector b' result from b by increasing the demand at sink t to 1. Clearly, $a' \geqslant a$ and $b' \geqslant b$. In the resulting new instance one can send one unit directly from source q to sink t, and another unit directly from source s to sink r. This yields $\mathrm{TP}(C,a',b')=c_{qt}+c_{sr}$. By inequality (1) we have $\mathrm{TP}(C,a,b) > \mathrm{TP}(C,a',b')$. □

Lemma 3. *If the cost matrix $C=(c_{ij})$ is not immune against the transportation paradox, then there exists some bad quadruple for C.*

Proof. By the assumptions of the lemma, there exist two supply vectors a and a' with $a' \geqslant a$ and two demand vectors b and b' with $b' \geqslant b$ such that $\mathrm{TP}(C,a,b) > \mathrm{TP}(C,a',b')$. We denote the corresponding optimal transportation plans by $x=(x_{ij})$ and by $x'=(x'_{ij})$. It is convenient to translate this situation into a bipartite multigraph G where one vertex class is formed by the sources and the other vertex class is formed by the sinks. A nonzero value x_{ij} yields a black edge with weight x_{ij} between source i and sink j, and a nonzero value x'_{ij} yields a corresponding red edge with weight x'_{ij}. The cost of a (red or black) edge between source i and sink j is c_{ij}. Some readers might prefer to use a flow interpretation and regard $x'-x$ as flow in the residual graph with respect to x. (The red edges correspond to the forward arcs and the black edges to the backward arcs in the residual graph.)

It is well known from flow theory (see e.g. the book by Ahuja et al. [1]) that we can decompose the multigraph G into a finite number $\mathscr{P}_1,\dots,\mathscr{P}_k$ of simple paths and a finite number $\mathscr{Q}_1,\dots,\mathscr{Q}_l$ of simple cycles that satisfy the following properties: Every cycle has an even number of edges and alternatingly consists of red and black edges. Every path starts in a source and ends in a sink, starts with a red edge and ends with a red edge, and alternatingly consists of red and black edges. There exist nonnegative real ξ-values $\xi(\mathscr{P})$ and $\xi(\mathscr{Q})$ for every path $\mathscr{P}$ and for every cycle $\mathscr{Q}$ such that

(i) for every black edge $[i,j]$ the value x_{ij} equals the sum of the ξ-values of all paths and cycles containing $[i,j]$, and
(ii) for every red edge $[i,j]$ the value x'_{ij} equals the sum of the ξ-values of all paths and cycles containing $[i,j]$.

For every path $\mathscr{P}$ and for every cycle $\mathscr{Q}$ we define the costs $c(\mathscr{P})$ and $c(\mathscr{Q})$ as the sum of the costs of all black edges in $\mathscr{P}$ and $\mathscr{Q}$, respectively, and we define the costs $c'(\mathscr{P})$ and $c'(\mathscr{Q})$ as the sum of the costs of all red edges in $\mathscr{P}$ and $\mathscr{Q}$, respectively. Then clearly

$$\mathrm{TP}(C,a,b)=\sum_{i=1}^{k} c(\mathscr{P}_i)\xi(\mathscr{P}_i)+\sum_{j=1}^{\ell} c(\mathscr{Q}_j)\xi(\mathscr{Q}_j) \tag{2}$$

and

$$\mathrm{TP}(C,a',b')=\sum_{i=1}^{k} c'(\mathscr{P}_i)\xi(\mathscr{P}_i)+\sum_{j=1}^{\ell} c'(\mathscr{Q}_j)\xi(\mathscr{Q}_j). \tag{3}$$

Since $\mathrm{TP}(C,a,b) > \mathrm{TP}(C,a',b')$ holds, we get from (2) and (3) that there exists a cycle $\mathscr{Q}$ among $\{\mathscr{Q}_j, j=1,\ldots,\ell\}$ with $c(\mathscr{Q}) > c'(\mathscr{Q})$ or there exists a path $\mathscr{P}$ among $\{\mathscr{P}_i, i=1,\ldots,k\}$ with $c(\mathscr{P}) > c'(\mathscr{P})$.

Suppose for the sake of contradiction that $c(\mathscr{Q}) > c'(\mathscr{Q})$ holds for some cycle $\mathscr{Q}$. We show that in this case $x=(x_{ij})$ is not an optimal transportation plan. Indeed, consider a new transportation plan $y=(y_{ij})$ for the instance C,a,b where y results from x by decreasing all values x_{ij} along black edges of $\mathscr{Q}$ by some $\varepsilon > 0$, and from increasing all values x_{ij} along red edges of $\mathscr{Q}$ by the same amount ε. This new transportation plan y is also feasible for the instance C,a,b, but its objective value is by $c(\mathscr{Q})\varepsilon - c'(\mathscr{Q})\varepsilon > 0$ smaller than the objective value of plan x; that clearly is a contradiction. (An alternative way of showing that the case $c(\mathscr{Q}) > c'(\mathscr{Q})$ cannot arise, would have been to make use of the negative cycle theorem for minimum cost flow problems, see [1].)

Consequently, $c(\mathscr{P}) > c'(\mathscr{P})$ must hold for some path $\mathscr{P}$. Since all costs c_{ij} are nonnegative, the path $\mathscr{P}$ consists of at least three edges. Without loss of generality we assume that $\mathscr{P}$ starts in a source and ends in a sink. We assume that the first vertex on $\mathscr{P}$ is the source s, the second vertex is the sink r, the last but one vertex is the source q, and the last vertex is the sink t. Consider the transportation plan $z=(z_{jj})$ for the instance C,a,b that results from $x=(x_{ij})$ in the following way: We decrease all values x_{ij} along black edges of $\mathscr{P}$ by an $\varepsilon > 0$. We increase all values x_{ij} along red edges of $\mathscr{P}$ except the first red edge $[s,r]$ and the last red edge $[q,t]$ by the same ε. We increase the value x_{qr} by ε. Then the resulting transportation plan z is feasible for the instance C,a,b. (Note that if $\mathscr{P}$ has length 3, we will have $z=x$.) Since x is an optimal transportation plan, the change in the objective value from x to z must be nonnegative. Therefore,

$$0 \leqslant -c(\mathscr{P})\varepsilon + (c'(\mathscr{P}) - c_{qt} - c_{sr})\varepsilon + c_{qr}\varepsilon$$

$$= (c'(\mathscr{P}) - c(\mathscr{P}))\varepsilon + (c_{qr} - c_{qt} - c_{sr})\varepsilon < (c_{qr} - c_{qt} - c_{sr})\varepsilon.$$

Here the last inequality follows from $c(\mathscr{P}) > c'(\mathscr{P})$. Since $\varepsilon > 0$, this yields $c_{qr} - c_{qt} - c_{sr} > 0$. Hence, the four numbers q,r,s,t form a bad quadruple. □

Theorem 4. *An $m \times n$ cost matrix $C=(c_{ij})$ is immune against the transportation paradox, if and only if for all q,r,s,t with $1 \leqslant q,\ s \leqslant m, 1 \leqslant r,t \leqslant n, q \neq s$ and $r \neq t$ the inequality*

$$c_{qr} \leqslant c_{qt} + c_{sr} \tag{4}$$

is satisfied. Moreover it can be checked in $O(mn)$ *time whether C fulfills all these conditions.*

Proof. The (only if) part follows from Lemma 2, and the (if) part follows from Lemma 3. Hence, it remains to prove the claimed algorithmic result. In a preprocessing step, we determine and store for every row and for every column of C the two smallest entries. This can be done in $O(n)$ time for each of the m rows and in $O(m)$ time for each of the n columns. So the preprocessing takes $O(mn)$ time.

For every q and r with $q \in \{1,\ldots,m\}$ and $r \in \{1,\ldots,n\}$, we then check whether they may participate in a violation of (4). In the case of violation, there must be an entry

c_{qt} in the same row and an entry c_{sr} in the same column that together are smaller than c_{qr}. The most dangerous candidate for c_{qt} is either the smallest entry in row q, or the second smallest entry in this row (in case the smallest entry is c_{qr}). Analogously, the most dangerous candidate for c_{sr} is the smallest or second smallest entry in column r. Because of the preprocessing step, we can check these cases in constant time per entry c_{qr}. This yields the claimed running time of $O(mn)$. □

Finke [7] observed empirically that by allowing additional shipments (i.e., by making use of the transportation paradox) the transportation cost can be reduced considerably. In his experiments with randomly generated instances of the transportation problem of size 100×100 he obtained average cost reductions 18.6%. (The total additional shipments amounted to 20.5%.) The restrictiveness of the condition for the immuneness of a cost matrix in Theorem 4 can serve as explanation for the observation of Finke [7].

3. The paradox for the algebraic transportation problem

Our characterization and our arguments from the preceding section can be translated to the more general case of the algebraic transportation problem, ATP for short. The ATP is stated as follows (see [4,5]): Let $(H, \oplus, \preccurlyeq)$ be a totally ordered commutative semigroup such that the operation $\oplus$ on the set H is compatible with the total order $\preccurlyeq$, i.e.

$$a \preccurlyeq b \Rightarrow a \oplus c \preccurlyeq b \oplus c \quad \text{for all } a, b, c, \in H.$$

Let $\otimes : H \times \mathbb{R}_0^+ \to H$ be an outer composition that satisfies the distributive laws

$$(a \oplus b) \otimes z = (a \otimes z) \oplus (b \otimes z) \quad \text{for all } a, b \in H \quad \text{and} \quad z \in \mathbb{R}_0^+,$$

$$a \otimes (z + z') = (a \otimes z) \oplus (a \otimes z') \quad \text{for all } a \in H \quad \text{and} \quad z, z' \in \mathbb{R}_0^+$$

and that is compatible with the total order $\preccurlyeq$

$$a \preccurlyeq b \Rightarrow a \otimes z \preccurlyeq b \otimes z \quad \text{for all } a, b \in H \quad \text{and } z \in \mathbb{R}_0^+.$$

Now, given a cost matrix $C = (c_{ij})$ with entries c_{ij} taken from H, the ATP consists in minimizing the objective function

$$(c_{11} \otimes x_{11}) \oplus (c_{12} \otimes x_{12}) \oplus \cdots \oplus (c_{mn} \otimes x_{mn})$$

over all feasible solutions $x = (x_{ij}) \in X$; note that the minimization is done with respect to the total order $\preccurlyeq$. The ATP belongs to the class of algebraic optimization problems. The reader can find more information on this area in the book by Zimmermann [10]. The ATP generalizes several types of transportation problems from the literature. The classical transportation problem arises from the ATP framework by setting $(H, \oplus, \preccurlyeq) = (\mathbb{R}, +, \leqslant)$ and by choosing the multiplication in $\mathbb{R}$ as $\otimes$. Another frequently considered special case of the ATP is the *bottleneck transportation problem* where one searches for a transportation plan $x \in X$ that minimizes the objective function

$\max_{x_{ij}>0} c_{ij}$. The bottleneck transportation problem arises from the ATP framework by setting $(H, \oplus, \preccurlyeq) = (\mathbb{R} \cup \{-\infty\}, \max, \leqslant)$ and by defining $a \otimes z = a$ for $z > 0$ and $a \otimes 0 = -\infty$. Along the lines of the proof of Theorem 4, the following analogue of Theorem 4 can be obtained for the algebraic transportation problem.

Theorem 5. *An $m \times n$ cost matrix $C = (c_{ij})$ with entries taken from the semigroup $(H, \oplus, \preccurlyeq)$ is immune against the transportation paradox for the algebraic transportation problem if and only if $c_{qr} \preccurlyeq c_{qt} \oplus c_{sr}$ holds for all q, r, s, t* with $1 \leqslant q \neq s \leqslant m$ *and* $1 \leqslant r \neq t \leqslant n$.

It is easy to see that for the special case of the bottleneck transportation problem, the conditions in Theorem 5 turn into the following set of equations:

$$\max\{c_{qr}, c_{st}\} = \max\{c_{qt}, c_{sr}\} \quad \text{for all } 1 \leqslant q < s \leqslant m \quad \text{and} \quad 1 \leqslant r < t \leqslant n. \tag{5}$$

Finke [8] observed empirically that the transportation paradox arises quite frequently for randomly generated bottleneck transportation problems. The restrictiveness of condition (5) can again serve as explanation for Finke's observation.

Acknowledgements

We wish to thank Gautam Appa, Ailsa Land, Alison Harcourt (née Doig) and an anonymous referee for providing us with historical facts about the source of the transportation paradox.

References

[1] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, Network Flows: Theory, Algorithms and Applications, Prentice-Hall, Englewood Cliffs, NJ, 1993.
[2] G.M. Appa, The transportation problem and its variants, Oper. Res. Quarterly 24 (1973) 79–99.
[3] G.M. Appa, Reply to L.G. Proll, Oper. Res. Quarterly 24 (1973) 636–639.
[4] R.E. Burkard, A general Hungarian method for the algebraic transportation problem, Discrete Math. 22 (1978) 219–232.
[5] R.E. Burkard, U. Zimmermann, Weakly admissible transformations for solving algebraic assignment and transportation problems, Math. Programming Stud. 12 (1980) 1–18.
[6] A. Charnes, D. Klingman, The more-for-less paradox in the distribution model, Cahiers du Centre d'Etudes de Recherche Operationelle 13 (1971) 11–22.
[7] G. Finke, A unified approach to reshipment, overshipment and postoptimization problems, Lecture Notes in Control and Information Science 7, Springer, Berlin, 1978, pp. 201–208.
[8] G. Finke, Minimizing overshipments in bottleneck transportation problems, INFOR 21 (1983) 121–135.
[9] W. Szwarc, The transportation paradox, Naval Res. Logist. Quarterly 18 (1971) 185–202.
[10] U. Zimmermann, Linear and Combinatorial Optimization in Ordered Algebraic Structures, in: Annals of Discrete Mathematics, Vol. 10, North-Holland, Amsterdam, 1981.