[7] A. R. Teel, R. M. Murray, and G. Walsh, "Nonholonomic control systems: From steering to stabilization with sinusoids," in *Proc. IEEE Contr. Decision Conf.*, 1992.
[8] J.-B. Pomet, "Explicit design of time-varying stabilizing control laws of controllable systems without drift," *Syst. Contr. Lett.*, vol. 18, no. 2, pp. 147–158, 1992.
[9] J.-M. Godhavn and O. Egeland, "Lyapunov based time varying control for exponential stabilization of a unicycle," in *IFAC Triennial World Congr.*, San Francisco, CA, 1996.
[10] R. T. M'Closkey and R. M. Murray, "Extending exponential stabilizers for nonholonomic systems from kinematic controllers to dynamic controllers," in *Proc. Preprints Fourth IFAC Symp. Robot Contr.*, 1994.
[11] M. Kawski, "Geometric homogeneity and stabilization," in *Proc. Preprints Nonlinear Contr. Syst. Design Symp.*, 1995.
[12] R. M. Murray and S. S. Sastry, "Nonholonomic motion planning: Steering using sinusoids," *IEEE Trans. Automat. Contr.*, 1993.
[13] M. Vidyasagar, *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
[14] P. Morin and C. Samson, "Time-varying exponential stabilization of a rigid spacecraft with two controls," in *Proc. 34th IEEE Contr. Decision Conf.*, New Orleans, LA, 1995, pp. 3988–3993.

A Dual Formulation of Mixed μ and on the Losslessness of (D, G) Scaling

Gjerrit Meinsma, Yash Shrivastava, and Minyue Fu

***Abstract*—This paper studies the mixed structured singular value, μ, and the well-known (D, G)-scaling upper bound, ν. A dual characterization of μ and ν is derived, which intimately links the two values. Using the duals it is shown that ν is guaranteed to be lossless (i.e., equal to μ) if and only if $2(m_r + m_c) + m_C \leq 3$, where $m_r, m_c,$ and m_C are the numbers of repeated real scalar blocks, repeated complex scalar blocks, and full complex blocks, respectively. The losslessness result further leads to a variation of the well-known Kalman–Yakubovich–Popov lemma and Lyapunov inequalities.**

***Index Terms*—Duality, Kalman–Yakubovich–Popov lemma, linear matrix inequalities, mixed structured singular values.**

I. Introduction

In two adjoining papers, Doyle [1] and Safonov [2] coined the structured singular value as a tool to test for robust stability of closed-loop systems. The D-scaling upper bound introduced in the very first paper on structured singular values [1] is to date still the most widely used upper bound of the structured singular value. As claimed in [3], D-scaling for complex structures with full blocks is in practice close to the actual structured singular value (or μ for short), and for several nontrivial complex structures the D-scaling upper bound is proved to be lossless [1], [3].

Progress in the theory of mixed μ has been slower. Mixed real/complex μ is an extension of μ that allows the structure to consist of real and complex parts. Such mixed structures arise, for example, if robust stability is to be tested with respect to parametric uncertainties. In Fan *et al.* [4], an upper bound for mixed μ is presented, but, unlike its pure complex counterpart, this upper bound—which we call ν—can be far from the actual mixed μ [5].

So far not much is known about losslessness of ν for mixed structures. Fan *et al.* [4] have shown that ν is lossless if there is one *nonrepeated* real scalar and one full complex block. Young [6] showed that ν is lossless for rank-one matrices.

In this paper we show that the upper bound ν of mixed μ is lossless iff $2(m_r+m_c)+m_C \leq 3$, where m_r, m_c, and m_C are the numbers of repeated real scalar blocks, repeated complex scalar blocks, and full complex blocks, respectively. These losslessness results come as a by-product of a dual formulation of μ and ν that we derive. The duality theory forms the bulk of this paper and is of independent interest. It is partly based on Rantzer's proof of the Kalman–Yakubovich–Popov (KYP) lemma [7], which in turn is influenced by [3].

Section II introduces notation and a few well established results. In Section III the dual characterizations of μ and ν are derived. As an example of the use of these dual results, we show that $\mu(M) = \nu(M)$ if M has rank one (the proof is a substantial simplification compared to that in Young [6]). The dual characterizations are applied in Section IV to prove the losslessness of ν for the mentioned structures. In Section V we give examples that show that for all other structures ν is not lossless. Section VI is about a variation of the KYP lemma and Lyapunov inequalities.

II. Notation and (D, G) Scaling

The norm $\|T\|$ of a matrix $T \in C^{m\times n}$ is in this paper the spectral norm. The Euclidean norm of T is denoted as $\|T\|_2$. T^{H} is the complex conjugate transpose of T, and $\mathrm{He}\, T$ is the Hermitian part of T defined as

$$\mathrm{He}\, T = \frac{1}{2}(T + T^{\mathrm{H}}).$$

Given a subset X of $C^{n\times n}$, the *(mixed) structured singular value* of $M \in C^{n\times n}$ is denoted by $\mu_X(M)$ and is defined as

$$\mu_X(M) = \frac{1}{\inf\{\|\Delta\| : I - \Delta M \text{ is singular and } \Delta \in X\}}.$$

$\mu_X(M)$ is set to zero if $I - \Delta M$ is nonsingular for every $\Delta \in X$. Obviously $\mu_X(M)$ depends on the "structure" X. Whenever $\mu_X(M)$ is used it is implicitly assumed that some structure X is given. Invariably, X is assumed block-diagonal of the form

$$X = \mathrm{diag}(RI_{k_1}, \cdots, RI_{k_{m_r}},\ CI_{l_1}, \cdots, CI_{l_{m_c}}, C^{f_1\times f_1}, \cdots, C^{f_{m_C}\times f_{m_C}}) \tag{1}$$

where $m_r, m_c,$ and m_C are the number of repeated real scalar blocks, repeated complex scalar blocks, and full complex blocks, respectively.

A. (D, G) Scaling

Let H^q denote the set of $q\times q$ Hermitian matrices, and denote its subset of positive definite elements by P^q. Given the structure X of (1), the sets $\mathcal{D}_X$ and $\mathcal{G}_X$ are defined as

$$\mathcal{D}_X = \mathrm{diag}(P^{k_1}, \cdots, P^{k_{m_r}},\ P^{l_1}, \cdots, P^{l_{m_c}}, P\, I_{f_1}, \cdots, P\, I_{f_{m_C}})$$
$$\mathcal{G}_X = \mathrm{diag}(H^{k_1}, \cdots, H^{k_{m_r}}\ 0_{l_1\times l_1}, \cdots, 0_{l_{m_c}\times l_{m_c}}, 0_{f_1\times f_1}, \cdots, 0_{f_{m_C}\times f_{m_C}}).$$

Manuscript received December 18, 1995; revised September 5, 1996 and October 14, 1996.

G. Meinsma is with the Department of Applied Mathematics, University of Twente, 7500 AE Enschede, The Netherlands.
Y. Shrivastava is with the Department of Electrical Engineering, The University of Sydney, NSW 2006, Australia.
M. Fu is with the Department of Electrical and Computer Engineering, The University of Newcastle, Callaghan, NSW 2308, Australia.

Publisher Item Identifier S 0018-9286(97)04315-8.

Given $M \in C^{n\times n}$ and $\alpha \in R$, the matrix function $\Phi_\alpha(D,G)$ is defined as

$$\Phi_\alpha(D,G) = M^{\rm H}DM + j(GM - M^{\rm H}G) - \alpha^2 D.$$

This notation is a bit different from that of [4], where Fan *et al.* showed that $\mu_X(M) < \alpha$ if $\alpha > 0$ and $\Phi_\alpha(D,G) < 0$ for some $D \in \mathcal{D}_X$ and $G \in \mathcal{G}_X$. The infimal α for which such D and G can be found is thus an upper bound of $\mu_X(M)$, and this upper bound is denoted throughout as $\nu_X(M)$, i.e.,

$$\nu_X(M) = \inf_{\alpha>0}\{\alpha : \exists D \in \mathcal{D}_X, G \in \mathcal{G}_X \text{ s.t. } \Phi_\alpha(D,G) < 0\}.$$

Computation of $\nu_X(M)$ can be done efficiently (in polynomial time).

It may be verified that

$$\Phi_\alpha(D,G) = \text{ He}(M^{\rm H} + \alpha I)\left(D + \frac{j}{\alpha}G\right)(M - \alpha I).$$

This allows us to characterize ν_X somewhat more compactly. Given α, any element E of $\mathcal{D}_X + j\mathcal{G}_X$ can be uniquely decomposed as $E = D + (j/\alpha)G$ with $D \in \mathcal{D}_X$ and $G \in \mathcal{G}_X$ (namely, take $D = \text{He}\, E$ and let $(j/\alpha)G$ be the skew-Hermitian matrix $E - \text{He}\, E$). Therefore

$$\nu_X(M) < \alpha \Leftrightarrow \exists E \in \mathcal{D}_X + j\mathcal{G}_X \text{ such that } \text{He}(M^{\rm H} + \alpha I)E(M - \alpha I) < 0.$$

This we use frequently.

III. Dual Characterization of μ and ν

In this section we give a dual characterization of μ_X and ν_X. In the next section we use these results to prove that $\mu_X = \nu_X$ for structures of the form $X = \text{diag}(RI_m, C^{p\times p})$. The dual characterizations of μ_X and ν_X that we present are easy, and they are remarkably similar. They may have wider use than just the next sections, and they may, for example, be used to simplify Young's proof [6] of the fact that $\mu_X(M) = \nu_X(M)$ for rank-one matrices M, irrespective of the structure. This is done at the end of this section.

The characterizations presented are dual in that they are an application of a duality argument for convex sets. The following preparatory result is in essence standard (see Boyd *et al.* [8, p. 29]). For completeness we give a proof.

Lemma III.1 (Separating Hyper-Planes): Suppose $F(E) \in C^{m\times m}$ depends affinely on $E \in C^{n\times n}$. Let $\mathcal{E}$ be some convex subset of $C^{n\times n}$. Then, no $E \in \mathcal{E}$ exists for which

$$\text{He}\, F(E) < 0$$

iff there is a nonzero $W = W^{\rm H} \geq 0$ such that

$$\operatorname{Re}\operatorname{tr} WF(E) \geq 0, \qquad \forall E \in \mathcal{E}. \tag{2}$$

Proof: Suppose $W = W^{\rm H} \geq 0, W \neq 0$ satisfies (2). Then, for any $E \in \mathcal{E}$

$$\operatorname{tr} W^{1/2}(\text{He}\, F(E))W^{1/2} = \operatorname{Re}\operatorname{tr} WF(E) \geq 0.$$

This excludes the possibility that $\text{He}\, F(E)$ is negative definite as that would have implied that $\operatorname{tr} W^{1/2}(\text{He}\, F(E))W^{1/2} < 0$.

Conversely, suppose that $\text{He}\, F(\mathcal{E})$ contains no negative definite element. Stated differently, $-\text{He}\, F(\mathcal{E})$ does not intersect the cone of positive definite matrices

$$\mathcal{P} := \{P : P = P^{\rm H} > 0\}.$$

Since both $-\text{He}\, F(\mathcal{E})$ and $\mathcal{P}$ are convex, there is a separating hyper-plane ([9, Th. 3, p. 133]), that is, a nonzero $H \in C^{m\times m}$ exists and a number $a \in R$ such that

$$\operatorname{Re}\langle H, -\text{He}\, F(\mathcal{E})\rangle \leq a \leq \operatorname{Re}\langle H, \mathcal{P}\rangle.$$

The inner product here is $\langle A, B\rangle = \operatorname{tr} A^{\rm H}B$. For any $P \in \mathcal{P}$ we have that $\langle H, P\rangle = \operatorname{Re}\operatorname{tr} HP = \operatorname{tr} P^{1/2}(\text{He}\, H)P^{1/2}$, and this shows that $a \leq \langle H, \mathcal{P}\rangle$ implies $\text{He}\, H \geq 0$ and $a \leq 0$. Define $W = \text{He}\, H$. Then, $-\operatorname{Re}\operatorname{tr} WF(\mathcal{E}) = \operatorname{tr} - W\text{He}\, F(\mathcal{E}) = \operatorname{Re}\langle H, -\text{He}\, F(\mathcal{E})\rangle \leq a \leq 0.$ □

Lemma III.2 (Dual Characterization of ν_X): Let X be any structure (1). Then, $\nu_X(M) \geq \alpha$ iff there is a nonzero $W = W^{\rm H} \geq 0$ such that

$$\operatorname{Re}\operatorname{tr}(M - \alpha I)W(M^{\rm H} + \alpha I)E \geq 0, \qquad \forall E \in \mathcal{D}_X + j\mathcal{G}_X. \tag{3}$$

Proof: $\nu_X(M) \geq \alpha$ iff no $E \in \mathcal{D}_X + j\mathcal{G}_X$ exists for which

$$\text{He}\,(M^{\rm H} + \alpha I)E(M - \alpha I) < 0.$$

By Lemma III.1, that is the case iff there is a nonzero $W = W^{\rm H} \geq 0$ such that $\operatorname{Re}\operatorname{tr} W(M^{\rm H} + \alpha I)E(M - \alpha I) \geq 0$ for all such E. The traces of $W(M^{\rm H} + \alpha I)E(M - \alpha I)$ and $(M - \alpha I)W(M^{\rm H} + \alpha I)E$ are the same. □

We next reformulate this characterization of ν_X without using E. To this end we partition E and $(M - \alpha I)W(M^{\rm H} + \alpha I)$ compatible with X as

$$E = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & E_{m_r+m_c+m_C} \end{bmatrix}$$
$$(M - \alpha I)W(M^{\rm H} + \alpha I) = \begin{bmatrix} Z_1 & ? & ? \\ ? & \ddots & ? \\ ? & ? & Z_{m_r+m_c+m_C} \end{bmatrix}. \tag{4}$$

A "?" denotes an irrelevant entry. Varying E over all elements of $\mathcal{D}_X + j\mathcal{G}_X$ can be done by varying each block E_i independently of the other blocks, and as each block may be arbitrarily close to zero, we have that (3) holds iff for every $i \in \{1, \cdots, m_r + m_c + m_C\}$ we have that

$$\operatorname{Re}\operatorname{tr}\, Z_i E_i \geq 0 \tag{5}$$

for all E_i in the appropriate sets.

Lemma III.3 (Another Dual Characterization of ν_X): Let X be any structure (1). Then, $\nu_X(M) \geq \alpha$ iff there is a nonzero $W = W^{\rm H} \geq 0$ such that

$$\begin{cases} Z_i \text{ is Hermitian and } \geq 0, & \forall i \in \{1, \cdots, m_r\}, \\ \text{He}\, Z_i \geq 0, & \forall i \in \{m_r + 1, \cdots, m_r + m_c\}, \\ \operatorname{Re}\operatorname{tr} Z_i \geq 0, & \forall i \in \{m_r + m_c + 1, \cdots\}. \end{cases} \tag{6}$$

Here Z_i is the ith block on the diagonal of $(M - \alpha I)W(M^{\rm H} + \alpha I)$ as shown in (4).

Proof: $\nu_X(M) \geq \alpha$ iff (5) holds for all i and E_i in the appropriate sets. We distinguish three cases.

Case 1) If $i \in \{1, \cdots, m_r\}$, then E_i is any matrix whose Hermitian part $\text{He}\, E_i$ is positive definite. For all such E_i we have that $\operatorname{Re}\operatorname{tr} Z_i E_i \geq 0$ iff Z_i is Hermitian and ≥ 0. This may be seen as follows. If $Z_i = Z_i^{\rm H} \geq 0$, then

$$\operatorname{Re}\operatorname{tr} Z_i E_i = \operatorname{tr} Z_i^{1/2}(\text{He}\, E_i)Z_i^{1/2} \geq 0$$

since $\text{He}\, E_i > 0$. If Z_i is Hermitian but not ≥ 0, then many $E_i = E_i^{\rm H} > 0$ exist for which $\operatorname{Re}\operatorname{tr} Z_i E_i = \operatorname{tr} E_i^{1/2} Z_i E_i^{1/2} < 0$. Finally, suppose Z_i is not Hermitian. Decompose Z_i in a Hermitian and skew-Hermitian part,

$Z_i = \mathrm{He}\, Z_i + S_i$, and note that the skew-Hermitian part S_i is nonzero. Let $E_i := \varepsilon I - S_i^{\mathrm{H}}$, depending on $\varepsilon > 0$. The Hermitian part of E_i is $\varepsilon I > 0$, and for small enough $\varepsilon > 0$ we have

$$\begin{aligned}\mathrm{Retr}\, Z_i E_i &= \mathrm{Retr}(\mathrm{He}\, Z_i + S_i)\big(\varepsilon I - S_i^{\mathrm{H}}\big)\\ &= \mathrm{tr}\big(\varepsilon(\mathrm{He}\, Z_i) - S_i S_i^{\mathrm{H}}\big)\\ &= \varepsilon\, \mathrm{tr}\, \mathrm{He}\, Z_i - \|S_i\|_{\mathrm{F}}^2 < 0.\end{aligned}$$

The $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm.

Case 2) For $i \in \{m_r+1, \cdots, m_r+m_c\}$ the E_i is any Hermitian positive definite matrix. $\mathrm{Retr}\, Z_i E_i \geq 0$ for all such E_i iff $\mathrm{He}\, Z_i \geq 0$. This follows from the fact that $\mathrm{Retr}\, Z_i E_i$ equals $\mathrm{tr}\, E_i^{1/2}(\mathrm{He}\, Z_i)E_i^{1/2}$.

Case 3) For $i \in \{m_r + m_c + 1, \cdots, m_r + m_c + m_C\}$ the E_i is any matrix of the form $E_i = d_i I$ with $0 < d_i \in R$. $\mathrm{Retr}\, Z_i E_i \geq 0$ for all such E_i iff $\mathrm{Retr}\, Z_i \geq 0$. □

Next we derive a characterization of μ_X in similar terms. For that we need the following lemma, which is readily proved.

Lemma III.4 (Three Little Lemmas): Let f, g be two column vectors of the same dimension.

1) $(f-g)(f+g)^{\mathrm{H}}$ is Hermitian and ≥ 0 iff $g = \delta f$ for some $\delta \in [-1, 1]$.
2) The Hermitian part of $(f-g)(f+g)^{\mathrm{H}}$ is ≥ 0 iff $g = \delta f$ for some $\delta \in C$ with $|\delta| \leq 1$.
3) $\mathrm{Retr}\,(f-g)(f+g)^{\mathrm{H}} = \|f\|_2^2 - \|g\|_2^2$. Hence $\mathrm{Retr}\,(f-g)(f+g)^{\mathrm{H}} \geq 0$ iff $g = \Delta f$ for some matrix Δ with $\|\Delta\| \leq 1$.

Lemma III.5 (Dual Characterization of μ_X): Let X be any structure (1). Then, $\mu_X(M) \geq \alpha$ iff there is a nonzero vector $t \in C^n$ such that

$$\mathrm{Retr}\,(M - \alpha I)tt^{\mathrm{H}}(M^{\mathrm{H}} + \alpha I)E \geq 0, \qquad \forall E \in \mathcal{D}_X + j\mathcal{G}_X. \tag{7}$$

Equivalently, $\mu_X(M) \geq \alpha$ iff there is a nonzero vector $t \in C^n$ such that

$$\begin{cases} Z_i \text{ is Hermitian and } \geq 0, & \forall i \in \{1, \cdots, m_r\},\\ \mathrm{He}\, Z_i \geq 0, & \forall i \in \{m_r+1, \cdots, m_r+m_c\},\\ \mathrm{Retr}\, Z_i \geq 0, & \forall i \in \{m_r + m_c + 1, \cdots\}.\end{cases} \tag{8}$$

Here Z_i is the ith block on the diagonal of $(M - \alpha I)tt^{\mathrm{H}}(M^{\mathrm{H}} + \alpha I)$ as partitioned compatibly with X.

Proof: The equivalence of (7) and (8) was shown earlier. We prove that $\mu_X(M) \geq \alpha$ iff (8) holds. Note that each Z_i can be written as the product of a column vector and a row vector as

$$\begin{aligned}Z_i =& [M_{i1} \cdots M_{i(i-1)} M_{ii} - \alpha I \quad M_{i(i+1)} \cdots]t\\ &\times ([M_{i1} \cdots M_{i(i-1)} \quad M_{ii} + \alpha I \quad M_{i(i+1)} \cdots]t)^{\mathrm{H}}.\end{aligned} \tag{9}$$

We will formulate necessary and sufficient conditions for (8) to hold. We distinguish the three cases.

Case 1) Let $i \in \{1, \cdots, m_r\}$. By Lemma III.4-1) we have that (9) is Hermitian and ≥ 0 iff $[0 \cdots 0 \;\; \alpha I \;\; 0 \cdots 0]t = \delta_i [M_{i1} \;\cdots\; M_{i(m_r+m_c+m_C)}]\, t$ for some $\delta_i \in [-1, 1]$.

Case 2) Let $i \in \{m_r+1, \cdots, m_r+m_c\}$. By Lemma III.4-2) we have that the Hermitian part of (9) is ≥ 0 iff $[0 \cdots 0\; \alpha I\; 0 \cdots 0]t = \delta_i[M_{i1} \;\cdots\; M_{i(m_r+m_c+m_C)}]t$ for some $\delta_i \in C$ with $|\delta_i| \leq 1$.

Case 3) Let $i \in \{m_r + m_c + 1, \cdots, m_r + m_c + m_C\}$. By Lemma III.4-3) the real part of the trace of (9) is ≥ 0 iff $[0 \cdots 0 \;\; \alpha I \;\; 0 \cdots 0]t = \Delta_i[M_{i1} \;\cdots\; M_{i(m_r+m_c+m_C)}]t$ for some Δ_i with $\|\Delta_i\| \leq 1$.

The three cases combined show that there is a nonzero t that satisfies (8) if and only if $(\alpha I - \Delta M)t = 0$ for some t and some $\Delta = \mathrm{diag}(\delta_1, \cdots, \delta_{m_r}, \delta_{m_r+1}, \cdots, \delta_{m_r+m_c}, \Delta_1, \cdots, \Delta_{m_C}) \in X$ with $\|\Delta\| \leq 1$, i.e., iff $\mu_X(M) \geq \alpha$. □

In summary, the results in this section say that $\nu_X(M) \geq \alpha$ iff a nonzero $W = W^{\mathrm{H}} \geq 0$ exists with certain properties (6) and that $\mu_X(M) \geq \alpha$ iff W can be chosen to have rank one. Another interpretation, which is more in line with that of Packard and Doyle [3] and Rantzer [7], is as follows. The set

$$\{(M - \alpha I)W(M^{\mathrm{H}} + \alpha I) : W = W^{\mathrm{H}} \geq 0\}$$

is the convex hull of the set

$$\Theta := \{(M - \alpha I)tt^{\mathrm{H}}(M^{\mathrm{H}} + \alpha I) : t \in C^n\}.$$

Therefore, $\nu_X(M) \geq \alpha$ iff the convex hull of Θ has certain properties, whereas $\mu_X(M) \geq \alpha$ iff Θ itself has those properties.

We end this section with an application which shows the potential of the dual characterizations. Young [6] was the first to prove the following lemma, but whereas his proof is rather cumbersome, the proof based on dual characterizations is only a few lines.

Lemma III.6 (Rank-One Matrices, cf. [6]): $\mu_X(M) = \nu_X(M)$ if M has rank one.

Proof: It suffices to show that $\nu_X(M) \geq \alpha$ implies $\mu_X(M) \geq \alpha$.

Suppose $\nu_X(M) \geq \alpha$. Therefore, there is a nonzero nonnegative definite $W = W^{\mathrm{H}}$ for which $(M - \alpha I)W(M^{\mathrm{H}} + \alpha I)$ satisfies (6). Let $x, y \in C^n$ be such that $M = xy^{\mathrm{H}}$, and decompose W compatibly with that as

$$W = tt^{\mathrm{H}} + \widehat{W}, \text{ in which } \widehat{W}y = 0, \widehat{W} = \widehat{W}^{\mathrm{H}} \geq 0 \text{ and, } t \in C^n.$$

If $W^{1/2}y = 0$ the above is satisfied for $t = 0, \widehat{W} = W$, and if $W^{1/2}y \neq 0$, we can take $t = (1/\sqrt{y^{\mathrm{H}}Wy})Wy$ and $\widehat{W} = W - (1/y^{\mathrm{H}}Wy)Wyy^{\mathrm{H}}W$. Then we have

$$\begin{aligned}(M - \alpha I)tt^{\mathrm{H}}(M^{\mathrm{H}} + \alpha I) &= (xy^{\mathrm{H}} - \alpha I)(W - \widehat{W})(yx^{\mathrm{H}} + \alpha I)\\ &= (M - \alpha I)W(M^{\mathrm{H}} + \alpha I) + \alpha^2\widehat{W}.\end{aligned}$$

By assumption, $(M - \alpha I)W(M^{\mathrm{H}} + \alpha I)$ satisfies (6), but then so does $(M - \alpha I)tt^{\mathrm{H}}(M^{\mathrm{H}} + \alpha I)$ because $\alpha^2\widehat{W}$ is Hermitian and ≥ 0. Moreover, the vector t is nonzero, because otherwise $(M - \alpha I)W(M^{\mathrm{H}} + \alpha I) = -\alpha^2\widehat{W} = -\alpha^2 W \leq 0$ which would have contradicted (6). Hence $\mu_X(M) \geq \alpha$. □

IV. The Case $X = \mathrm{diag}(RI_m, C^{p\times p})$

In this section we prove that $\mu_X = \nu_X$ if the structure has the form $X = \mathrm{diag}(RI_m, C^{p\times p})$. A straightforward application of Lemma III.3 and Lemma III.5 is as follows.

Corollary IV.1: Let $X = \mathrm{diag}(RI_m, C^{p\times p})$. Then $\nu_X(M) \geq \alpha$ iff a nonzero Hermitian $W \geq 0$ exists such that

$$\begin{cases}[M_{11} - \alpha I_m \quad M_{12}]W\begin{bmatrix} M_{11}^{\mathrm{H}} + \alpha I_m \\ M_{12}^{\mathrm{H}}\end{bmatrix} \text{ is Hermitian and } \geq 0,\\ \mathrm{Retr}\,[M_{21} \quad M_{22} - \alpha I_p]W\begin{bmatrix} M_{21}^{\mathrm{H}} \\ M_{22}^{\mathrm{H}} + \alpha I_p\end{bmatrix} \geq 0.\end{cases} \tag{10}$$

Moreover, $\mu_X(M) \geq \alpha$ iff a $W = W^{\mathrm{H}} \geq 0$ of rank one exists with these properties (10).

Lemma IV.2 (Technical Result): Let F and G be complex matrices of the same dimensions.

1) If F and G have full column rank, then $FG^{\mathrm{H}} = GF^{\mathrm{H}} \geq 0$ if and only if $F = GQ$ for some $Q = Q^{\mathrm{H}} > 0$.
2) If a $W = W^{\mathrm{H}} \geq 0$ of rank n is such that

$$FWG^{\mathrm{H}} \text{ is Hermitian and } \geq 0$$

then there exist n nonzero column vectors t_k such that $W = \sum_{k=1}^{k=n} t_k t_k^{\rm H}$ and

$$Ft_k t_k^{\rm H} G^{\rm H} \text{ is Hermitian and } \geq 0, \qquad \text{for all } k \in \{1, \cdots, n\}.$$

Proof:

1) If $F = GQ$ for some $Q = Q^{\rm H} > 0$, then obviously $FG^{\rm H} \geq 0$. Conversely, suppose F and G have full column rank and $FG^{\rm H} = GF^{\rm H} \geq 0$. Then $FG^{\rm H}G = GF^{\rm H}G$ so that $F = GQ$ with $Q = (F^{\rm H}G)(G^{\rm H}G)^{-1}$. $Q = Q^{\rm H} \geq 0$ because $G^{\rm H}G$ is invertible and $(G^{\rm H}G)Q(G^{\rm H}G) = (G^{\rm H}G)F^{\rm H}G = G^{\rm H}(GF^{\rm H})G \geq 0$. Q is in fact nonsingular because $F = GQ$ has full column rank.
2) By induction in n (which is the rank of W), if $n = 1$, or $n = 0$, then obviously the result is correct. Now consider $n > 1$. Write W as $W = VV^{\rm H}$ with V having n columns.

First consider the case that $FWG^{\rm H} = (FV)(GV)^{\rm H}$ has rank strictly less than n. Then either FV or GV does not have full column rank. So there is a unitary vector t such that either FVt or GVt is zero. Define $\widehat{W}$ as

$$\widehat{W} = W - Vtt^{\rm H}V^{\rm H} = V(I - tt^{\rm H})V^{\rm H}$$

and note that $\widehat{W} \geq 0$, that it has rank $n-1$, and that $F\widehat{W}G^{\rm H} = FWG^{\rm H} - FVtt^{\rm H}V^{\rm H}G^{\rm H} = FWG^{\rm H}$ is nonnegative definite. By induction, therefore, there exist nonzero $\{t_k\}$ such that

$$\widehat{W} = \sum_{k=1}^{n-1} t_k t_k^{\rm H}$$

and such that $Ft_k t_k^{\rm H} G^{\rm H}$ is Hermitian and ≥ 0 for all $k \in \{1, \cdots, n-1\}$. Finally, with t_n defined as $t_n := Vt$ we have that

$$W = \widehat{W} + t_n t_n^{\rm H} = \sum_{k=1}^{n} t_k t_k^{\rm H}$$

and that $Ft_k t_k^{\rm H} G^{\rm H}$ is Hermitian and ≥ 0 for every $k \in \{1, \cdots, n\}$.

Now suppose that $FWG^{\rm H} = (FV)(GV)^{\rm H}$ has rank n. Then by 1) we have that $FV = GVQ$ for some $Q = Q^{\rm H} > 0$. Let $Q = UDU^{\rm H}$ be an eigenvalue decomposition of Q. Therefore, $FVU = GVUD$. Define t_k as the kth column of VU. Then $W = VV^{\rm H} = (VU)(VU)^{\rm H} = \sum_k t_k t_k^{\rm H}$, and since $Ft_k = Gt_k d_k$ (where $0 < d_k$ is the kth diagonal entry of D), we also have that $Ft_k t_k^{\rm H} G^{\rm H}$ is Hermitian and ≥ 0. □

Theorem IV.3:

$$\mu_X = \nu_X \text{ if } X = \text{diag}(RI_m, C^{p \times p}).$$

Proof: Since $\mu_X(M) \leq \nu_X(M)$ it suffices to prove that $\nu_X(M) \geq \alpha$ implies $\mu_X(M) \geq \alpha$.

Suppose $\nu_X(M) \geq \alpha$. Then by Corollary IV.1 there is a nonzero $W = W^{\rm H} \geq 0$ that satisfies (10). By Lemma IV.2-2) we can write this W as $W = \sum_k t_k t_k^{\rm H}$ such that for all k

$$[M_{11} - \alpha I \quad M_{12}] t_k t_k^{\rm H} \begin{bmatrix} M_{11}^{\rm H} + \alpha I \\ M_{12}^{\rm H} \end{bmatrix} \text{ is Hermitian and } \geq 0.$$

Since $W = \sum_k t_k t_k^{\rm H}$ satisfies (10), there is at least one index k, say $k = l$, that also satisfies

$$\text{Re tr}\,[\text{M}_{21} \quad \text{M}_{22} - \alpha \text{I}] \text{t}_l \text{t}_l^{\rm H} \begin{bmatrix} \text{M}_{21}^{\rm H} \\ \text{M}_{22}^{\rm H} + \alpha \text{I} \end{bmatrix} \geq 0.$$

Hence $W := t_l t_l^{\rm H}$ is a rank-one matrix that satisfies (10) so that $\mu_X(M) \geq \alpha$. □

TABLE I
WHEN $\mu_X = \nu_X$ IS GUARANTEED

	$m_C = 0$	$m_C = 1$	$m_C = 2$	$m_C = 3$
$m_r = 0$, $m_c = 0$		Yes. See [3]	Yes. See [3]	Yes. See [3]
$m_r = 0$, $m_c = 1$	Yes. See [3]	Yes. See [3]	No. See [3]	
$m_r = 1$, $m_c = 0$	Yes. Thm. IV.3	Yes. Thm. IV.3	No. Ex. V.1	
$m_r = 1$, $m_c = 1$	No. Ex. V.2			
$m_r = 0$, $m_c = 2$	No. See [3]			
$m_r = 2$, $m_c = 0$	No. See [10]			

V. LOSSLESSNESS OF (D, G) SCALING

Building on work by [1], Packard and Doyle [3] showed that $\mu_X = \nu_X$ whenever

$$m_r = 0, \qquad 2m_c + m_C \leq 3. \tag{11}$$

Together with the results of Section IV, we thus have that $\mu_X = \nu_X$ for any of the structures X for which

$$2(m_r + m_c) + m_C \leq 3. \tag{12}$$

Packard and Doyle [3] further show by examples that $\mu_X(M) < \nu_X(M)$ can occur within any complex structure with $2m_c + m_C > 3$ (and $m_r = 0$). For $m_r = 2$ it is possible to find 2×2 matrices $M \in C^{2\times 2}$ such that $\mu_X(M)$ is less than $\nu_X(M)$ (see [10]). Table I details (12) and gives references for the different cases.

In this section we give two more examples that complete the picture in that they—combined with the other examples—show that for any structure X that violates (12) there exist matrices M such that $\mu_X(M) < \nu_X(M)$.

Example V.1: Let $X = \text{diag}(R, C, C)$ and take

$$M = \begin{bmatrix} 0 & 1 & j \\ j & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The claim is that $\mu_X(M) = 1$ and that $\nu_X(M) = \sqrt{3}$.

The spectral norm of M is $\|M\| = \sqrt{3}$, so we have $\nu_X(M) \leq \sqrt{3}$. Furthermore, for the Hermitian nonnegative definite W defined as

$$W = \begin{bmatrix} 2 & 1 & -j \\ 1 & 2 & j \\ j & -j & 2 \end{bmatrix}$$

we have that

$$(M - \sqrt{3}I)W(M^{\rm H} + \sqrt{3}I) = \begin{bmatrix} 0 & ? & ? \\ ? & 0 & 0 \\ ? & 0 & 0 \end{bmatrix}. \tag{13}$$

(The "?" denotes an irrelevant entry.) Since all diagonal entries of (13) are zero, it follows from Lemma III.3 that $\nu_X(M) \geq \sqrt{3}$. Hence $\nu_X(M) = \sqrt{3}$.

Calculation shows that $I_3 - \text{diag}(\delta_1, \delta_2, \delta_3)M$ is singular iff

$$\delta_2\delta_3 + j\delta_1(\delta_2 + \delta_3) - 1 = 0. \tag{14}$$

Suppose $\delta_1 \in [-1, 1]$ and that $|\delta_3| < 1$. Then the δ_2 for which (14) holds equals $\delta_2 = (1 - j\delta_1\delta_3)/(j\delta_1 + \delta_3)$, and it satisfies

$$|\delta_2|^2 = \left|\frac{1 - j\delta_1\delta_3}{j\delta_1 + \delta_3}\right|^2 = \frac{1 + \delta_1^2|\delta_3|^2 - 2\delta_1 \text{Im}(\delta_3)}{\delta_1^2 + |\delta_3|^2 - 2\delta_1 \text{Im}(\delta_3)} \geq 1.$$

Therefore, $\max_i |\delta_i| \geq 1$ for every solution of (14). Since $(\delta_1, \delta_2, \delta_3) = (1, j, -j)$ is a solution of (14) for which $\max_i |\delta_i| = 1$, we have that $\mu_X(M) = 1$.

Example V.2: Let $X = \operatorname{diag}(R, CI_2)$ and take the same M as in the previous example. From (13) we infer also that for this structure $\nu_X(M) \geq \sqrt{3}$. Since $\nu_X(M) \leq \|M\| = \sqrt{3}$, we have, again, that $\nu_X(M) = \sqrt{3}$. It further follows from the previous example that $I_3 - \operatorname{diag}(\delta_1, \delta_2 I_2)M$ is singular iff

$$\delta_2^2 + j2\delta_1\delta_2 - 1 = 0.$$

The solutions $\delta_2 = -j\delta_1 \pm \sqrt{1-\delta_1^2}$ have absolute value one for every $\delta_1 \in [-1, 1]$. Hence $\mu_X(M) = 1 < \nu_X(M)$.

VI. A Variation of the KYP Lemma

An application of the KYP lemma [11]–[13], called the strictly positive real lemma, is known to be equivalent to the fact that $\mu_X = \nu_X$ for the complex structures $X = \operatorname{diag}(CI_m, C^{p\times p})$ [3]. In this section we rephrase Theorem IV.3 as a KYP-type result.

Lemma VI.1: Suppose G is a square rational matrix with realization $G(s) = C(sI - A)^{-1}B + D$, and consider the following linear matrix inequality in P:

$$\begin{bmatrix} PA + A^{\mathrm{H}}P^{\mathrm{H}} & PB - C^{\mathrm{H}} \\ -C + B^{\mathrm{H}}P^{\mathrm{H}} & -D - D^{\mathrm{H}} \end{bmatrix} < 0. \tag{15}$$

Then:

1) there is a Hermitian $P = P^{\mathrm{H}} > 0$ that satisfies (15) iff A is stable and $G(s) + [G(s)]^{\mathrm{H}} > 0$ for all s in the closed right half-plane, including ∞;
2) there is a P (Hermitian or not) with $\operatorname{He} P > 0$ that satisfies (15) iff A has no eigenvalues on the positive real line $[0, \infty)$ and $G(s) + [G(s)]^{\mathrm{H}} > 0$ for all $s \in [0, \infty)$, including ∞;
3) there is a P (Hermitian or not) that satisfies (15) iff A is nonsingular and $G(s) + [G(s)]^{\mathrm{H}} > 0$ at $s = 0$ and $s = \infty$.

Proof: Item 1) is the strictly positive real lemma in an inequality version (see, e.g., [14]). Item 3) can be shown using [15, Lemma 3.1]. Items 1) and 3) are included for comparison only. We prove Item 2). Define E and N as

$$E = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}, \qquad N = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$$

and note that (15) is nothing but $EN + N^{\mathrm{H}}E^{\mathrm{H}} = 2\operatorname{He} EN < 0$. If $\operatorname{He} P > 0$ and P satisfies (15), then $I - N$ is invertible because $\operatorname{He} E(I-N) = \operatorname{He} E - \operatorname{He} EN > 0$. Define $M = (I-N)^{-1}(I+N)$. Then $M + I = 2(I-N)^{-1}$ is invertible and $N = (M-I)(M+I)^{-1}$. Let $X = \operatorname{diag}(RI_m, C^{p\times p})$. We have that $\exists P$ such that (15) holds and $\operatorname{He} P > 0$:

- $\Leftrightarrow$ $\exists E \in \mathcal{D}_X + j\mathcal{G}_X$ s.t. $\operatorname{He} EN < 0$;
- $\Leftrightarrow$ $\exists E \in \mathcal{D}_X + j\mathcal{G}_X$ s.t. $\operatorname{He} E(M - I)(I + M)^{-1} < 0$;
- $\Leftrightarrow$ $\exists E \in \mathcal{D}_X + j\mathcal{G}_X$ s.t. $\operatorname{He} (I + M^{\mathrm{H}})E(M - I) < 0$;
- $\Leftrightarrow$ $\nu_X(M) < 1 \Leftrightarrow \mu_X(M) < 1$;
- $\Leftrightarrow$ $I - M \operatorname{diag}(\delta_1 I_m, \Delta_2)$ is nonsingular for all $\delta_1 \in [-1, 1]$ and $\|\Delta_2\| \leq 1$;
- $\Leftrightarrow$ $(I - N) - (I + N) \operatorname{diag}(\delta_1 I_m, \Delta_2)$ is nonsingular for all $\delta_1 \in [-1, 1]$ and $\|\Delta_2\| \leq 1$;
- $\Leftrightarrow$ $\begin{bmatrix} (1-\delta_1)I - (1+\delta_1)A & -B(I+\Delta_2) \\ (I+\delta_1)C & I - \Delta_2 + D(I+\Delta_2) \end{bmatrix}$ is nonsingular for all $\delta_1 \in [-1, 1]$ and $\|\Delta_2\| \leq 1$;
- $\Leftrightarrow$ A has no eigenvalues on $[0, \infty)$, and $(I - \Delta_2) + G(\frac{1-\delta_1}{1+\delta_1})(I + \Delta_2)$ is nonsingular for all $\delta_1 \in [-1, 1]$ and $\|\Delta_2\| \leq 1$;
- $\Leftrightarrow$ A has no eigenvalues on $[0, \infty)$, and $I - (I + G(s))^{-1}(I - G(s))\Delta_2$ is nonsingular for all $s \in [0, \infty) \cup \infty$ and $\|\Delta_2\| \leq 1$;
- $\Leftrightarrow$ A has no eigenvalues on $[0, \infty)$ and $G(s) + [G(s)]^{\mathrm{H}} > 0$ for all $s \in [0, \infty) \cup \infty$. □

Lemma VI.1 remains valid if the matrices B, C, and D are void. In that case, Lemma VI.1-2) reads as a variation of a Lyapunov stability condition.

Corollary VI.2: A matrix $A \in C^{n\times n}$ has no eigenvalues on the positive real line $[0, \infty)$ iff there is $P \in C^{n\times n}$ such that

$$PA + A^{\mathrm{H}}P^{\mathrm{H}} < 0, \qquad \operatorname{He} P > 0.$$

Acknowledgment

The authors would like to thank P. Young for his comments on the losslessness of (D, G)-scaling.

References

[1] J. C. Doyle, "Analysis of feedback systems with structured uncertainties," *Proc. Inst. Elec. Eng.*, vol. 129, Pt. D, pp. 242–250, 1982.
[2] M. G. Safonov, "Stability margins of diagonally perturbed multivariable feedback systems," *Proc. Inst. Elec. Eng.*, vol. 129, Pt. D, pp. 251–256, 1982.
[3] A. Packard and J. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 1, pp. 71–109, 1993.
[4] M. Fan, A. Tits, and J. Doyle, "Robustness in the presence of joint parametric uncertainty and unmodeled dynamics," *IEEE Trans. Automat. Contr.*, vol. 36, no. 1, pp. 25–38, 1991.
[5] P. M. Young, M. P. Newlin, and J. C. Doyle, "Let's get real," in *Robust Control Theory*. New York: Springer-Verlag, 1995, pp. 143–174.
[6] P. M. Young, "The rank one mixed μ problem and 'Kharitonov-type' analysis," *Automatica*, vol. 30, no. 12, pp. 1899–1911, 1994.
[7] A. Rantzer, "A note on the Kalman–Yacubovich–Popov lemma," *Syst. Contr. Lett.*, vol. 28, no. 1, pp. 7–10, 1996.
[8] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.
[9] D. G. Luenberger, *Optimization by Vector Space Methods*. New York: Wiley, 1969.
[10] G. Meinsma, Y. Shrivastava, and M. Fu, "Some properties of an upper bound of μ," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1326–1330, 1996.
[11] R. E. Kalman, "Lyapunov functions for the problem of lur'e in automatic control," in *Proc. Nat. Acad. Sci.*, 1963, vol. 49, pp. 201–205.
[12] V. A. Yakubovich, "Solution of certain matrix inequalities in the stability theory of nonlinear systems," *Dokl. Akad. Nauk.*, vol. 143, no. 6, pp. 1304–1307, 1962; English translation in *Soviet Math. Dokl.*, pp. 620–623, 1962.
[13] V. M. Popov, "Absolute stability of nonlinear systems of automatic control," *Automat. Remote Contr.*, vol. 22, pp. 857–875, 1962.
[14] J. C. Willems, "Least squares stationary optimal control and the algebraic Riccati equation," *IEEE Trans. Automat. Contr.*, vol. 16, pp. 621–634, 1971.
[15] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to $\mathcal{H}_\infty$ control," *Int. J. Robust Nonlinear Contr.*, vol. 4, pp. 421–448, 1994.