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ORIGINAL PAPER

An Exact Formula for all Star-Kipas Ramsey Numbers

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Abstract Let G_1 and G_2 be two given graphs. The Ramsey number $R(G_1, G_2)$ is the least integer r such that for every graph G on r vertices, either G contains a G_1 or $\overline{G}$ contains a G_2. A complete bipartite graph $K_{1,n}$ is called a star. The kipas $\widehat{K}_n$ is the graph obtained from a path of order n by adding a new vertex and joining it to all the vertices of the path. Alternatively, a kipas is a wheel with one edge on the rim deleted. Whereas for star-wheel Ramsey numbers not all exact values are known to date, in contrast we determine all exact values of star-kipas Ramsey numbers.

Keywords Ramsey number · Star · Kipas · Wheel

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1 Introduction

Throughout this paper, all graphs are finite and simple. For a pair of graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$, is defined as the smallest integer r such that for every graph G on r vertices, either G contains a G_1 or $\overline{G}$ contains a G_2, where $\overline{G}$ is the complement of G. We denote by P_n a path, and by C_n a cycle on n vertices, respectively. A complete bipartite graph $K_{1,n}$ ($n \geq 2$) is called a star. The kipas $\widehat{K}_n$ ($n \geq 2$) is the graph obtained from a path P_n by adding one new vertex and joining it to all the vertices of the P_n. The term kipas as well as its notation are adopted from [8]. Kipas is the Malay word for fan; the motivation for the term kipas is that the graph looks like a hand fan (especially, if the path P_n is drawn as part of a circle) but that the term fan is already in use for another type of graph. The wheel W_n ($n \geq 3$) is the graph obtained from a cycle C_n by adding one new vertex and joining it to all the vertices of the C_n.

Ramsey numbers for stars versus wheels have been studied intensively, but a complete solution for all star-wheel Ramsey numbers is still lacking. Hasmawati [4] determined all exact values of $R(K_{1,n}, W_m)$ for $n \geq 2$ and $m \geq 2n$, and Chen et al. [1] determined $R(K_{1,n}, W_m)$ for all odd m with $m \leq n + 2$, later extended to all odd m with $m \leq 2n - 1$ by Hasmawati et al. [5]. For even m, the small cases were solved in papers by Surahmat and Baskoro ($m = 4$, [9]), Chen et al. ($m = 6$, [1]), and Zhang et al. ($m = 8$, [10,11]). A new breakthrough for even m appeared in a recent paper [7], in which Li and Schiermeyer solve the case that m is even and $n + 2 \leq m \leq 2n - 2$. The remaining case that m is even and $m \leq n + 1$ seems to be very difficult.

In contrast, although the kipas and wheel of the same order differ by only one edge on the rim, the Ramsey numbers of stars versus kipases are much easier to determine, as will be shown in this paper. In the sequel we prove the following result, establishing an exact formula for all star-kipas Ramsey numbers.

Theorem 1 *Suppose that* $n, m \geq 2$.

(1) *If* $m \geq 2n$, *then*

$$R(K_{1,n}, \widehat{K}_m) = \begin{cases} n + m - 1, & \textit{if both } n \textit{ and } m \textit{ are even}; \\ n + m, & \textit{otherwise}. \end{cases}$$

(2) *If* $m \leq 2n - 1$, *then*

$$R(K_{1,n}, \widehat{K}_m) = \begin{cases} 2n + \lfloor m/2 \rfloor - 1, & \textit{if both } n \textit{ and } \lfloor m/2 \rfloor \textit{ are even}; \\ 2n + \lfloor m/2 \rfloor, & \textit{otherwise}. \end{cases}$$

2 Some Useful Results

We start by presenting some known results that we find useful for our purposes. We first list the following two results on star-star Ramsey numbers and star-wheel Ramsey numbers.

Theorem 2 (Harary [3]) *For $n, m \geq 2$,*

$$R(K_{1,n}, K_{1,m}) = \begin{cases} n+m-1, & \textit{if both } n \textit{ and } m \textit{ are even;} \\ n+m, & \textit{otherwise.} \end{cases}$$

Theorem 3 (Hasmawati [4]) *For $n \geq 2$ and $m \geq 2n$,*

$$R(K_{1,n}, W_m) = \begin{cases} n+m-1, & \textit{if both } n \textit{ and } m \textit{ are even;} \\ n+m, & \textit{otherwise.} \end{cases}$$

Noting that $K_{1,m} \subset \widehat{K}_m \subset W_m$, it is obvious that $R(K_{1,n}, K_{1,m}) \leq R(K_{1,n}, \widehat{K}_m) \leq R(K_{1,n}, W_m)$. Hence, using Theorems 2 and 3, we immediately obtain that for $n \geq 2$ and $m \geq 2n$,

$$R(K_{1,n}, \widehat{K}_m) = \begin{cases} n+m-1, & \text{if both } n \text{ and } m \text{ are even;} \\ n+m, & \text{otherwise,} \end{cases}$$

establishing (1) of Theorem 1.

We will use the following two results on the existence of long cycles in graphs and bipartite graphs in the proof of (2) of Theorem 1. For a graph G, we denote by $\nu(G)$ the order of G, and by $\delta(G)$ the minimum degree of G.

Theorem 4 (Dirac [2]) *Every 2-connected graph G has a cycle of order at least* $\min\{2\delta(G), \nu(G)\}$.

Theorem 5 (Jackson [6]) *Let G be a bipartite graph with partition sets X and Y, and with $|X| \geq 2$. If for every vertex $x \in X$, $d(x) \geq \max\{|X|, |Y|/2+1\}$, then G has a cycle of order* $2|X|$.

From Theorems 4 and 5, we obtain the following results, respectively.

Lemma 1 *Every connected graph G has a path of order at least* $\min\{2\delta(G)+1, \nu(G)\}$.

Proof If G has only one vertex, then the assertion is trivially true. Next assume $\nu(G) \geq 2$, and let G' be the graph obtained from G by adding a new vertex x and joining it to all the vertices of G. Since G is connected and x is adjacent to every vertex of G, G' is 2-connected. Note that $\delta(G') = \delta(G)+1$. By Theorem 4, G' has a cycle C of order at least $\min\{2\delta(G)+2, \nu(G)+1\}$. Thus $G = G' - x$ has a path $C - x$ of order at least $\min\{2\delta(G)+1, \nu(G)\}$. □

Lemma 2 *Let G be a bipartite graph with partition sets X and Y. If for every vertex $x \in X$, $d(x) \geq \max\{|X|+1, (|Y|+1)/2\}$, then G has a path of order* $2|X|+1$.

Proof If $|X| = 1$, then the assertion is trivially true. Now we assume that $|X| \geq 2$. Let G' be the bipartite graph obtained from G by adding a new vertex y and joining it to every vertex in X. Set $Y' = Y \cup \{y\}$. Note that for every vertex $x \in X$, $d_{G'}(x) \geq d(x)+1 \geq \max\{|X|+2, (|Y|+1)/2+1\} \geq \max\{|X|, |Y'|/2+1\}$. By Theorem 5,

G' has a cycle of order $2|X|$. Let $C = x_1y_1x_2y_2\cdots x_{|X|}y_{|X|}x_1$ be such a cycle. We may assume that $y \in V(C)$; otherwise, we can replace one of y_i by y. Now assume without loss of generality that $y = y_{|X|}$. Since $d(x) \geq |X|+1$ for every vertex $x \in X$, in G we can find a neighbor y_0 of x_1 in $Y\backslash\{y_i : 1 \leq i \leq |X|\}$ and a neighbor $y'_{|X|}$ of $x_{|X|}$ in $Y\backslash\{y_i : 0 \leq i \leq |X|-1\}$. Then $P = y_0x_1y_1x_2\cdots x_{|X|}y'_{|X|}$ is a path of order $2|X|+1$ in G. □

We will also make use of the following lemma that was proved in [7].

Lemma 3 *Let k and n be two integers with $n \geq k+1$ and either k or n is even. Then there exists a k-regular graph of order n each component of which is of order at most $2k+1$.*

3 Proof of Theorem 1

Recall that statement (1) of Theorem 1 follows immediately from Theorems 2 and 3, as we noted in the beginning of the previous section.

So from now on, we assume that $m \leq 2n-1$. For convenience, we define the parameter θ such that $\theta = 1$ if both n and $\lfloor m/2 \rfloor$ are even, and $\theta = 0$ otherwise. To prove (2) of Theorem 1, it suffices to show that $R(K_{1,n}, \widehat{K}_m) = 2n + \lfloor m/2 \rfloor - \theta$.

We first show that $R(K_{1,n}, \widehat{K}_m) \geq 2n + \lfloor m/2 \rfloor - \theta$ by providing example graphs, using Lemma 3.

Suppose first that m is even. Note that either $m/2-1$ or $n+m/2-\theta-1$ is even. By Lemma 3, there exists an $(m/2-1)$-regular graph H of order $n+m/2-\theta-1$ such that each component of H has order at most $m-1$. Let $G = K_n \cup \overline{H}$. Then $\nu(G) = 2n+m/2-\theta-1$. One can check that G contains no $K_{1,n}$, and that $\overline{G}$ contains no $\widehat{K}_m$. This implies that $R(K_{1,n}, \widehat{K}_m) \geq 2n + m/2 - \theta$. If m is odd, then we have $R(K_{1,n}, \widehat{K}_m) \geq R(K_{1,n}, \widehat{K}_{m-1}) \geq 2n + \lfloor m/2 \rfloor - \theta$.

Now we will prove that $R(K_{1,n}, \widehat{K}_m) \leq 2n + \lfloor m/2 \rfloor - \theta$. Note that it is sufficient to consider the case that m is odd. Let G be a graph of order

$$\nu(G) = 2n + \frac{m-1}{2} - \theta. \tag{1}$$

Suppose that $\overline{G}$ contains no $K_{1,n}$, i.e.,

$$\delta(G) \geq n + \frac{m-1}{2} - \theta. \tag{2}$$

We will prove that G contains a $\widehat{K}_m$. We assume to the contrary that G contains no $\widehat{K}_m$, and derive at contradictions in all cases. We choose G such that it has the smallest number of edges among all candidates.

Let u be a vertex of G with maximum degree. We prove two claims. Here is our first claim.

Claim $d(u) \geq n + (m-1)/2$*; and for every* $v \in N(u)$, $d(v) = n + (m-1)/2 - \theta$.

Proof If $\theta = 0$, then by (2), $d(u) \geq n + (m-1)/2$. If $\theta = 1$, then n and $(m-1)/2$ are both even. Thus $\nu(G)$ is odd by (1). If every vertex of G has degree $n + (m-1)/2 - 1$, then G will have an odd number of vertices with odd degree, a contradiction. This implies $d(u) \geq n + (m-1)/2$.

Let v be a vertex in $N(u)$. Then $d(v) \geq \delta(G) \geq n + (m-1)/2 - \theta$. If $d(v) \geq n + (m-1)/2 - \theta + 1$, then $d(u) \geq d(v) \geq n + (m-1)/2 - \theta + 1$. Thus $G' = G - uv$ has fewer edges than G while $\delta(G') \geq n + (m-1)/2 - \theta$. Since G' is a subgraph of G, it contains no $\widehat{K}_m$, a contradiction to the choice of G. □

Set $H = G[N(u)]$ and $L = G - H$. Note that $\nu(H) = d(u)$. Using the above Claim, we assume that

$$\nu(H) = n + \frac{m-1}{2} + \tau, \tag{3}$$

where $\tau \geq 0$; and thus

$$\nu(L) = n - \theta - \tau. \tag{4}$$

Let v be an arbitrary vertex of H. By the above Claim and (4),

$$d_H(v) \geq d(v) - \nu(L) = \left(n + \frac{m-1}{2} - \theta\right) - (n - \theta - \tau) = \frac{m-1}{2} + \tau.$$

This implies that

$$\delta(H) \geq \frac{m-1}{2} + \tau. \tag{5}$$

If H has a component with order at least m, then by Lemma 1, H contains a path P_m. Since u is adjacent to every vertex of the P_m, G contains a $\widehat{K}_m$, a contradiction. So we conclude that every component of H has order at most $m - 1$. By (3) and the fact that $m \leq 2n - 1$, $\nu(H) \geq m$, which implies that H is disconnected. Let C be a component of H with minimum order. Then $\nu(C) \leq \min\{m - 1, \nu(H)/2\}$, i.e.,

$$\nu(C) \leq \min\left\{m - 1, \frac{2n + m - 1 + 2\tau}{4}\right\}. \tag{6}$$

Let v be a vertex in $V(C)$. Then $d_C(v) \geq (m-1)/2 + \tau$. Let X be the set of $(m-1)/2$ neighbors of v in C and $Y = N_L(v)$. We construct a bipartite graph B with partition sets X and Y such that for any $x \in X$ and $y \in Y$, $xy \in E(B)$ if and only if $xy \in E(G)$. Note that

$$|X| = \frac{m-1}{2} \text{ and } |Y| = n + \frac{m-1}{2} - \theta - d_H(v).$$

Here is our second claim.

Claim *For every* $x \in X$, $d_Y(x) \geq \max\{|X|+1, (|Y|+1)/2\}$.

Proof Let w be an arbitrary vertex in $X \subset N_H(v)$. Then

$$d_Y(w) = |N_L(v) \cap N_L(w)| \geq d(v) + d(w) - d_H(v) - d_H(w) - \nu(L)$$

We distinguish two cases by comparing $m-1$ with $(2n+m-1+2\tau)/4$.

Case 1 $m-1 \leq (2n+m-1+2\tau)/4$, i.e., $n \geq (3m-3)/2 - \tau$.

Note that $d_H(v) \leq m-2$ and $d_H(w) \leq m-2$. By our first Claim and (4),

$$\begin{aligned} d_Y(w) &\geq 2\left(n + \frac{m-1}{2} - \theta - m + 2\right) - (n - \theta - \tau) \\ &= n - m + 3 - \theta + \tau \\ &\geq \left(\frac{3m-3}{2} - \tau\right) - m + 3 - \theta + \tau \\ &= \frac{m-1}{2} + 2 - \theta \\ &\geq |X| + 1; \end{aligned}$$

and

$$\begin{aligned} 2d_Y(w) &\geq 4\left(n + \frac{m-1}{2} - \theta\right) - 3(m-2) - d_H(v) - 2(n - \theta - \tau) \\ &= 2n - m + 4 - 2\theta + 2\tau - d_H(v) \\ &\geq n + \left(\frac{3m-3}{2} - \tau\right) - m + 4 - 2\theta + 2\tau - d_H(v) \\ &= n + \frac{m-1}{2} - \theta - d_H(v) + 3 - \theta + \tau \\ &\geq |Y| + 1. \end{aligned}$$

Case 2 $m-1 > (2n+m-1+2\tau)/4$, i.e., $n < (3m-3)/2 - \tau$.

Note that $d_H(v) \leq (2n+m-1+2\tau)/4 - 1 = (2n+m-5+2\tau)/4$ and $d_H(w) \leq (2n+m-5+2\tau)/4$. By our first Claim and (4),

$$\begin{aligned} d_Y(w) &\geq 2\left(n + \frac{m-1}{2} - \theta - \frac{2n+m-5+2\tau}{4}\right) - (n - \theta - \tau) \\ &= \frac{m-1}{2} + 2 - \theta \\ &\geq |X| + 1; \end{aligned}$$

and

$$
\begin{aligned}
2d_Y(w) &\geq 4\left(n + \frac{m-1}{2} - \theta\right) - 3 \cdot \frac{2n+m-5+2\tau}{4} - d_H(v) - 2(n-\theta-\tau) \\
&= \frac{n}{2} + \frac{5m+7}{4} - 2\theta + \frac{\tau}{2} - d_H(v) \\
&= \frac{n}{2} + \left(\frac{3m-3}{4} - \frac{\tau}{2}\right) + \frac{m-1}{2} + 3 - 2\theta + \tau - d_H(v) \\
&> n + \frac{m-1}{2} - \theta - d_H(v) + 3 - \theta + \tau \\
&\geq |Y| + 1.
\end{aligned}
$$

This completes the proof of our second claim. □

By Lemma 2, B contains a path P_m. Since v is adjacent to all the vertices of the P_m, G contains a $\widehat{K}_m$, our final contradiction.

4 Conclusions

In this paper, we established an exact formula for all star-kipas Ramsey numbers. Although the difference between a wheel and a kipas of the same order is just one edge, and although star-wheel Ramsey numbers have been studied intensively by different groups of researchers, a complete solution for all star-wheel Ramsey numbers is still lacking. The remaining case of determining the Ramsey numbers of $R(K_{1,n}, W_m)$ for even m with $m \leq n+1$ seems to be very difficult. This might require sharpening or extending the results on the existence of long cycles in graphs and bipartite graphs that we have used, as presented in Sect. 2.

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