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Synchronization in a network of identical discrete-time agents with uniform constant communication delay

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SUMMARY

This paper studies the synchronization problem for a network of identical discrete-time agents with unknown uniform constant communication delay. When the agents are non-introspective, the problem is solvable via a decentralized low-gain-based synchronization controller if the delay satisfies the proposed upper bound. When the agents are introspective, the synchronization problem can be solved with arbitrary bounded communication delay. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Synchronization in the network has received substantial attention in recent years. Networks of linear agents are studied in [1–4]. More complex nonlinear agents are studied in [5–11]. A more comprehensive coverage on the earlier literature can be found in [12–14].

Although this research initiates from an idealized network model, the applied nature of the synchronization problem has prompted researchers to take the network imperfectness into account, in particular, time-delay effects, which are ubiquitous in any communication scheme and/or actuator dynamic. Tremendous effort has been put into this problem, see [15–20] and [21] to name a few. This body of work, including results on linear and nonlinear agents, is largely restricted to simple agent models such as first (scalar) or second-order dynamics. In a recent paper [22] of the authors, the synchronization problem under uniform constant delay is solved for continuous-time high-order linear agents that are critically unstable.

It is demonstrated in [2, 23] that synchronization problems can be split into two classes of problems, which yield a vastly different type of analysis and design. In a network, if the agents possess absolute measurement of their own dynamics besides the relative information received from the network, they are said to be *introspective* and *non-introspective* otherwise. The synchronization problem for discrete-time agents has been studied in both introspective [24] and non-introspective cases [3, 4, 12, 25] (also, see the references in these papers). In particular for non-introspective agents, [3] introduce the concept of 'disc margin' for discrete-time Linear Quadratic Regulator (LQR), based on which a static synchronization controller can be designed for critically unstable agents using relative information of neighboring states. An observer-based distributed synchronization controller

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is constructed in [4] for general linear agents, which, however, requires communication between controllers using the same network topology.

The communication delay in the network significantly complicates the problem, although many results have emerged in the literature such as the aforementioned references, mainly for continuous-time agents. Lyapunov methods, frequency-domain approaches, and passivity theory have been utilized. To the best of our knowledge, the only relevant results in the context of synchronization of discrete-time agents are reported in [16] where first-order agents are considered, and the results are obtained using a frequency-domain approach. The goal is to extend the results in [22] to discrete-time agents.

1.1. Contribution of this paper

This paper studies the synchronization problem for a network of identical discrete-time agents subject to uniform constant communicate delays. The general philosophy underlying our handling of communication delay and the synchronization problem has roots in the seminal work of Chua [26]. The contribution of this paper lies in three aspects. The first contribution resides in the fact that a very broad set of networks is considered. To begin with, we assume that the agents are at most critically unstable but may have high-order dynamics. This significantly expands the class of agents that are normally considered in the existing literature of synchronization of discrete-time agents under time-delay, for example, [16]. Also, the agents may measure either relative state or output information of neighboring agents and the communication topology can be directed or undirected. Second, an explicit upper bound for the delay is derived on the basis of a simple frequency-domain criterion. For any tolerable delay, a *decentralized* low-gain static feedback or compensator can be constructed to achieve synchronization in the network. The upper bound and controller design only depends on the agent model and some common characteristics rather than precise information of the communication topology provided that it has a directed spanning tree. Consequently, it is possible to solve the synchronization problem for an *a priori* given set of networks. In the special case where the communication topology is undirected, the upper bound for the delay is topology-independent. Last but not least, although this paper concentrates on non-introspective agents who only measure their state or output relative to that of neighboring agents, we also consider introspective agents who also directly acquire knowledge of their own dynamics. In this case, we do not need any structural property of the agent besides observability. An assumption like right-invertibility, as used in [27], is not needed. It turns out that with this additional local measurement, we are able to solve the synchronization problem with arbitrary but bounded communication delay.

1.2. Notations and preliminaries

In this note, the following notations are used. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ denote, respectively, the sets of all complex numbers, real numbers, positive real numbers, integers, and natural numbers. For any open set, $\mathcal{G} \subset \mathbb{C}$, $\partial\mathcal{G}$, and $\overline{\mathcal{G}}$ denote its boundary and closure. For $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}^+$, $\mathcal{D}(z_0, r)$ denotes an open disc centered at z_0 with radius r. In particular, we denote

$$\mathbb{C}^{\odot} := \overline{\mathcal{D}(0,1)}, \quad \mathbb{C}^{o} := \partial\mathcal{D}(0,1).$$

For any $k_1, k_2 \in \mathbb{Z}$ and $k_1 \leqslant k_2$,

$$\overline{[k_1,k_2]} := \{k \in \mathbb{Z} \mid k_1 \leqslant k \leqslant k_2\}.$$

For column vectors $x_1, \dots, x_n$, the stacking column vector of $x_1, \dots, x_n$ is denoted by $[x_1; \dots; x_n]$. For a matrix X, $\underline{\sigma}(X)$ and $\bar{\sigma}(X)$ denote the smallest and the largest singular values of X, respectively.

A matrix $D = \{d_{ij}\}_{n\times n}$ is called a row stochastic matrix if

1. $d_{ij} \geqslant 0$ for any i, j;
2. $\sum_{j=1}^{n} d_{ij} = 1$ for $i = 1, ..., n$.

A row stochastic matrix D has at least one eigenvalue at 1 with right eigenvector $\mathbf{1}$. D can be associated with a graph $G = (\mathcal{N}, \mathcal{E})$. The number of nodes in $\mathcal{N}$ is the dimension of D and an arc $(j, i) \in \mathcal{E}$ if $d_{ij} > 0$. Let G be the graph associated with D. It is shown in [28] that 1 is a simple eigenvalue of D if and only if G contains a directed spanning tree. Moreover, the other eigenvalues are in the open unit disk if $d_{ii} > 0$ for all i.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider a network of N identical agents

$$x^i(k+1) = Ax^i(k) + Bu^i(k), \quad i = 1, ..., N, \tag{1}$$

where $x^i \in \mathbb{R}^n$ and $u^i \in \mathbb{R}^m$.

Each agent collects a delayed information of the state of neighboring agents through the network

$$z^i(k) = \sum_{j=1}^{N} d_{ij} \left[Cx^i(k-\kappa) - Cx^j(k-\kappa) \right], \tag{2}$$

where $\kappa > 0$ is an unknown constant satisfying $\kappa \in \overline{[0, \bar{\kappa}]}$. Here, $D = \{d_{ij}\}_{n \times n}$ is a row stochastic matrix whose diagonal elements are unequal to 0.

Definition 1
If C has full column rank, we refer to the network as having full-state coupling. Otherwise, the network is said to have partial-state coupling.

Remark 1
The network measurement z^i is the only information that is available to each agent for controller design. The agent does not have separate observation of its own dynamics. This is referred to as the non-introspective case.

The matrix $D = \{d_{ij}\} \in \mathbb{R}^{N \times N}$ defines a communication topology that can be captured by a directed graph $G = (\mathcal{N}, \mathcal{E})$. Note that in other papers, people have characterized the communication topology using a Laplacian matrix; the analysis in this paper carries over to that case with only very minor modifications. The following assumption is made for the associated graph G.

Assumption 1
The communication topology G contains a directed spanning tree and $d_{ii} > 0$ for all i.

Under Assumption 1, D has a simple eigenvalue at 1 associated with right eigenvector $\mathbf{1}$, and the other eigenvalues are strictly inside the unit disk. Let $\lambda_1, ..., \lambda_N$ denote the eigenvalues of D such that $\lambda_1 = 1$ and $|\lambda_i| < 1$, $i = 2, ..., N$. We can define a set of communication topologies as follows:

Definition 2
For $\delta \in (0, 1]$, let $\mathcal{G}_\delta$ denote a set of communication topologies such that $|\lambda_i| < \delta$, $i = 2, ..., N$.

Remark 2
It can be verified according to [28] that any communication topology in the set $\mathcal{G}_\delta$ for $\delta \in (0, 1]$ satisfies Assumption 1.

The following assumption on the agent dynamics is also made throughout the paper.

Assumption 2
(A, B) is stabilizable, (A, C) is detectable, and A has all its eigenvalues in the closed unit disc $\mathbb{C}^{\odot}$.

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Remark 3
The detectability and stabilizability are standard properties. In many applications, it is reasonable to assume that the synchronization dynamics are not exponentially increasing.

Definition 3
The network synchronizes if

$$\lim_{k\to\infty}\left(x^i(k)-x^j(k)\right)=0,\quad \forall i,j=1,\ldots,N.$$

The problem studied in this paper can be formulated as follows.

Problem 1
Consider a homogeneous network of the form (1) and (2). For a given set $\mathcal{G}_\delta$ and a positive integer $\bar{\kappa}$, the synchronization problem with a set of communication topologies $\mathcal{G}_\delta$ and communication delay $\bar{\kappa}$ is to design N local controllers of the form

$$\begin{cases} \chi^i(k+1)=A_c\chi^i(k)+B_c z^i(k), \\ u^i(k) \quad = C_c\chi^i(k). \end{cases} \tag{3}$$

such that synchronization can be achieved in the network with any communication topology belonging to $\mathcal{G}_\delta$ and $\kappa\in\overline{[0,\bar{\kappa}]}$.

2.1. *Stability of discrete linear time-delay systems*

Consider system

$$x(k+1)=Ax(k)+A_1x(k-\kappa), \tag{4}$$

where $x(k)\in\mathbb{R}^n$ and $\kappa\in\mathbb{N}$. Suppose $A+A_1$ is Schur stable. The following result has been proved in [29].

Lemma 1
The system (4) is asymptotically stable if

$$\det\left[zI-A-(1-\alpha)A_1-\alpha z^{-\kappa}A_1\right]\neq 0,\ \forall z\in\mathbb{C}^o,\ \forall\alpha\in[0,1]. \tag{5}$$

2.2. *H_2 low-gain state feedback and compensator*

Consider a linear uncertain system

$$\begin{cases} x(k+1)=Ax(k)+\lambda Bu(k), \quad x(0)=x_0 \\ y(k) \quad = Cx(k), \end{cases} \tag{6}$$

where $\lambda\in\mathbb{C}$ is unknown. Let Assumption 2 hold. A low-gain state feedback can be constructed as

$$F_\varepsilon=-(B'P_\varepsilon B+I)^{-1}B'P_\varepsilon A \tag{7}$$

where for $\varepsilon\in(0,1]$, P_ε is the unique positive definite solution of the H_2 algebraic Riccati equation

$$P_\varepsilon=A'P_\varepsilon A+\varepsilon I-A'P_\varepsilon B(B'P_\varepsilon B+I)^{-1}B'P_\varepsilon A. \tag{8}$$

It is known that under Assumption 2, $P_\varepsilon\to 0$, and thus $F_\varepsilon\to 0$, as $\varepsilon\to 0$. Moreover, the low-gain feedback (7) has the following robustness property, which is proved in [3].

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Lemma 2
We have that $A + \lambda BF_\varepsilon$ is Schur stable if

$$\lambda \in \Omega_\varepsilon := \left\{ z \in \mathbb{C} : \left| z - \left(1 + \frac{1}{\gamma_\varepsilon}\right) \right| < \frac{\sqrt{1+\gamma_\varepsilon}}{\gamma_\varepsilon} \right\}, \tag{9}$$

where $\gamma_\varepsilon = \bar{\sigma}(B'P_\varepsilon B)$. As $\varepsilon \to 0$, Ω_ε approaches the set $H := \left\{ z : \operatorname{Re}(z) > \frac{1}{2} \right\}$ in the sense that any compact subset of H will be contained in Ω_ε for ε is small enough.

The low-gain state feedback $u = F_\varepsilon x$ can be realized as a dynamic measurement feedback controller

$$\begin{cases} \chi(k+1) = A\chi(k) - K\left(y(k) - C\chi(k)\right), & \chi(0) = \chi_0, \\ \quad u(k) \quad = F_\varepsilon \chi(k), \end{cases} \tag{10}$$

where K is such that $A + KC$ is Schur stable, which we refer to as a low-gain compensator. A robustness property similar to Lemma 2 can be proved for (10).

Lemma 3
For any compact set $\mathcal{S} \subset H := \{z \in \mathbb{C} : \operatorname{Re}(z) > 1\}$, there exists ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the closed-loop of (6) and (10) is asymptotically stable for $\lambda \in \mathcal{S}$.

Proof
See Appendix. □

3. NETWORK WITH FULL-STATE COUPLING CASE

In this section, we consider the case where the network has full-state coupling. We assume, without loss of generality, $C = I$. For a given set of networks $\mathcal{G}_\delta$, we design a decentralized local consensus controller for each agent using a low-gain feedback as follows.

$$u^i = \beta F_\varepsilon z^i, \tag{11}$$

with the design parameter β to be chosen later and

$$F_\varepsilon = -(B'P_\varepsilon B + I)^{-1} B'P_\varepsilon A, \tag{12}$$

where for $\varepsilon \in (0, 1]$, P_ε is the unique positive definite solution of the H_2 algebraic Riccati equation

$$P_\varepsilon = A'P_\varepsilon A + \varepsilon I - A'P_\varepsilon B(B'P_\varepsilon B + I)^{-1} B'P_\varepsilon A. \tag{13}$$

The low-gain parameter ε will be chosen depending only on δ and $\bar{\kappa}$. Define

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Schur stable.} \\ \max\{\omega \in [0, \pi] \mid \det\left(e^{j\omega} I - A\right) = 0\}, & \text{otherwise} \end{cases}$$

The first main result of this paper is stated in the next theorem, which solves Problem 1 in the full-state coupling case.

Theorem 1
For a given set $\mathcal{G}_\delta$ with $\delta < 1$ and $\bar{\kappa} > 0$, consider the agents (1) and (2) with any communication topology belonging to the set $\mathcal{G}_\delta$. In that case, Problem 1 is solvable via synchronization controller (11) if

$$\omega_{\max}\bar{\kappa} < \arccos(\delta). \tag{14}$$

Specifically, for given $\mathcal{G}_\delta$ and $\bar{\kappa}$ satisfying (14), there exist $\beta > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (11) achieve synchronization for any communication topologies in $\mathcal{G}_\delta$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

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In order to prove Theorem 1, we need the following lemma.

Lemma 4
Consider networks (1) and (2) with $C = I$. The synchronization is achievable if there exists an F such that the $N-1$ systems

$$\xi^i(k+1) = A\xi^i(k) + (1-\lambda_i)BF\xi^i(k-\kappa) \tag{15}$$

are globally asymptotically stable for any eigenvalue of the matrix D, which is not equal to 1, that is, λ_i, $i = 2, ..., N$.

Proof
See Appendix. □

Proof of Theorem 1
It follows from Lemma 4 that Theorem 1 holds if there exist $\beta > 0$ and $\varepsilon^* \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon^*]$,

$$x(k+1) = Ax(k) + \lambda\beta BF_\varepsilon x(k-\kappa) \tag{16}$$

is asymptotically stable for all $\lambda \in \mathcal{D}(1, \delta)$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

Because $\bar{\kappa}$ satisfies condition (14), there exists $\beta > \frac{1}{1-\delta}$ such that

$$\omega_{\max}\bar{\kappa} < \arccos\left(\delta + \frac{1}{\beta}\right). \tag{17}$$

Note that β is independent of ε. Let this β be fixed.

For $\lambda \in \mathcal{D}(1, \delta)$ and the previously selected β, we have

$$\lambda\beta \in \mathcal{D}(\beta, \beta\delta) \subset H = \left\{z \in \mathbb{C} : \mathrm{Re}(z) > \frac{1}{2}\right\}$$

because $\beta(1-\delta) > 1$. Because $\mathcal{D}(\beta, \beta\delta)$ is contained in a compact subset of H, by Lemma 2, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, $\mathcal{D}(\beta, \beta\delta) \subset \Omega_\varepsilon$, and hence, $A + \lambda\beta BF_\varepsilon$ is Schur stable, where Ω_ε is the disc margin defined in (9). According to Lemma 1, system (16) is asymptotically stable if

$$\det\left\{e^{j\omega}I - A - \left[(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}\right]BF_\varepsilon\right\} \neq 0,$$
$$\forall\omega \in [-\pi, \pi],\ \forall\alpha \in [0, 1],\ \forall\lambda \in \mathcal{D}(1, \delta),\ \kappa \in \overline{[0, \bar{\kappa}]}. \tag{18}$$

By (17), there exists $\eta > 0$ independent of ε such that

$$\omega\bar{\kappa} < \arccos\left(\delta + \frac{1}{\beta}\right), \quad \text{for } |\omega| < \omega_{\max} + \eta.$$

We first show that there exists a $\varepsilon_2 \in (0, \varepsilon_1]$ such that for $\varepsilon \in (0, \varepsilon_2]$, (18) holds for $\pi \geqslant |\omega| \geqslant \omega_{\max} + \eta$. To see this, notice that $\det(e^{j\omega}I - A) \neq 0$ for all $\pi \geqslant |\omega| \geqslant \omega_{\max} + \eta$, which implies $\underline{\sigma}(e^{j\omega}I - A) > 0$ provided $\pi \geqslant |\omega| \geqslant \omega_{\max} + \eta$. Because $\underline{\sigma}(e^{j\omega}I - A)$ depends continuously on ω and the set $\{\pi \geqslant |\omega| \geqslant \omega_{\max} + \eta\}$ is compact, there exists a μ such that

$$\underline{\sigma}(e^{j\omega}I - A) > \mu,\ \forall\omega \text{ s.t. } \pi \geqslant |\omega| \geqslant \omega_{\max} + \eta.$$

Let $\bar{\lambda} = (1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}$. We have

$$|\bar{\lambda}| \leqslant 2\beta, \quad \forall\alpha \in [0, 1],\ \forall\lambda \in \mathcal{D}(1, \delta),\ \forall\omega,\ \forall\kappa.$$

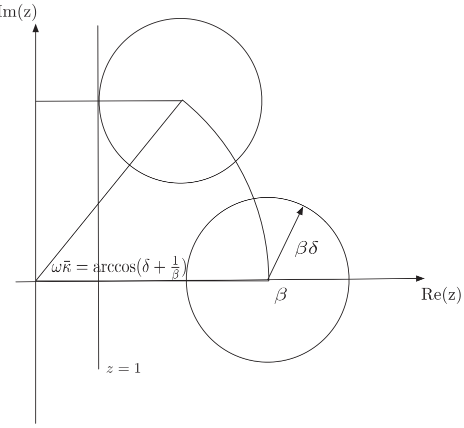

Figure 1. $\mathcal{D}(\beta e^{-j\omega\kappa}, \beta\delta)$.

Choose ε_2 such that $\|F_\varepsilon\| \leqslant \frac{\mu}{2\beta}\|B\|^{-1}$ for $\varepsilon \leqslant \varepsilon_2$. In that case,

$$\underline{\sigma}\left(e^{j\omega} I - A - \bar{\lambda} B F_\varepsilon\right) > \mu - |\bar{\lambda}|\|B\|\|F_\varepsilon\| > 0, \quad \forall |\omega| \geqslant \omega_{\max} + \eta,$$

and hence, (18) holds for $|\omega| \geqslant \omega + \eta$.

It remains to verify condition (18) for $|\omega| < \omega_{\max} + \eta$. Let us consider the gain $\lambda\beta e^{-j\omega\kappa}$. We have $\lambda\beta \in \mathcal{D}(\beta, \beta\delta)$. It is evident from Figure 1 that because

$$|\omega|\kappa \leqslant |\omega|\bar{\kappa} < \arccos\left(\delta + \frac{1}{\beta}\right),$$

we have

$$\lambda\beta e^{-j\omega\kappa} \in \mathcal{D}(\beta e^{-j\omega\kappa}, \beta\delta) \subset H, \quad \forall \kappa \in \overline{[0, \bar{\kappa}]}.$$

Because δ, β, η, $\bar{\kappa}$, and $\omega_{\max}$ are independent of ε, by Lemma 2, there exists $\varepsilon_3 \in (0, \varepsilon_2]$ such that for $\varepsilon \in (0, \varepsilon_3]$,

$$\mathcal{D}(\beta e^{-j\omega\kappa}, \beta\delta) \subset \Omega_\varepsilon, \quad \forall |\omega| < \omega_{\max} + \eta,\ \forall \kappa \in \overline{[0, \bar{\kappa}]}.$$

This is visualized in Figure 1. Obviously, we also have $\lambda\beta \in \mathcal{D}(\beta, \beta\delta) \subset \Omega_\varepsilon$ for $\varepsilon \leqslant \varepsilon_3 \leqslant \varepsilon_1$. This implies, because Ω_ε is convex, that

$$(1-\alpha)\lambda + \alpha\lambda e^{j\omega\kappa} \in \Omega_\varepsilon,\ \forall |\omega| < \omega_{\max} + \eta,\ \kappa \in \overline{[0, \bar{\kappa}]},\ \alpha \in [0, 1] \text{ and } \lambda \in \mathcal{D}(1, \delta).$$

In this case, by Lemma 2, $A + \left[(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}\right] B F_\varepsilon$ is Schur stable. Hence, condition (18) holds. □

Remark 4

Four parameters are chosen sequentially in the consensus design and analysis, namely, β, η, μ, and ε. First, we select the scaling parameter β in (17) using the given data δ and $\omega_{\max}$. Then, η is chosen on the basis of network data and the choice of β and such a δ will yield corresponding value of μ. Eventually, ε is determined by δ, β, and μ.

From Theorem 1 and its proof, when $\omega_{\max} = 0$, that is, A is either Schur stable or has all its unstable eigenvalues at 1, we immediately have the following result.

Corollary 1

For a given set $\mathcal{G}_\delta$ with $\delta < 1$ and $\bar{\kappa} > 0$, consider the agents (1) and (2) with any communication topology belonging to the set $\mathcal{G}_\delta$. Suppose $\omega_{\max} = 0$. In that case, Problem 1 is always solvable

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via synchronization controller (11). Specifically, for given $\mathcal{G}_\delta$ and any $\bar{\kappa} > 0$, there exists β and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (11) achieve synchronization for any communication topology in $\mathcal{G}_\delta$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

If the communication topology is undirected. D is symmetric and has only real eigenvalues. Of course, the result in Theorem 1 still holds, but in fact, a stronger result can be proved. It turns out that the delay tolerance is independent from network topology.

Corollary 2
For a given set $\mathcal{G}_\delta$ and $\bar{\kappa} > 0$, consider the agents (1) and (2) with any undirected communication topology belonging to the set $\mathcal{G}_\delta$. In that case, Problem 1 is solvable via synchronization controller (11) if

$$\omega_{\max}\bar{\kappa} < \frac{\pi}{2}. \tag{19}$$

Specifically, for given $\mathcal{G}_\delta$ and $\bar{\kappa}$ satisfying (14), there exist $\beta > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (11) achieve synchronization for any communication topologies in $\mathcal{G}_\delta$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

Proof
Thanks to Lemma 4 , we only need to show that for any $\delta \in (0, 1)$ and $\bar{\kappa} \in \mathbb{Z}^+$ satisfying (14), there exist $\beta > 0$ and $\varepsilon^* \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$x(k+1) = Ax(k) + \lambda\beta BF_\varepsilon x(k-\kappa) \tag{20}$$

is asymptotically stable for all $\lambda \in (1-\delta, 1+\delta)$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

For any $\delta \in (0, 1)$ and $\bar{\kappa}$ satisfying (19), there exists β such that

$$\beta(1-\delta)\cos(\bar{\kappa}\omega_{\max}) > 1. \tag{21}$$

Let this β be fixed.

First of all, because $\lambda\beta > 1$, by Lemma 2, there exists $\varepsilon_1 \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon_1]$,

$$\lambda\beta \in \Omega_\varepsilon.$$

Then, Lemma 1 says that the system (20) is asymptotically stable if

$$\det\left\{e^{j\omega}I - A - \left[(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}\right]BF_\varepsilon\right\} \neq 0,$$
$$\forall\omega \in [-\pi, \pi],\ \forall\alpha \in [0, 1],\ \forall\lambda \in (1-\delta, 1+\delta),\ \kappa \in \overline{[0, \bar{\kappa}]}. \tag{22}$$

By (21), there exists $\eta > 0$ independent of ε such that

$$\lambda\beta\cos(\omega\bar{\kappa}) > \frac{1}{2}, \quad \forall\,|\omega| < \omega_{\max} + \eta,\ \forall\,\lambda \in (1-\delta, 1+\delta).$$

With the same argument as in the proof of Theorem 1, we can show that there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that for $\varepsilon \in (0, \varepsilon_2]$, condition (22) holds for $|\omega| \in [\omega_{\max} + \eta, \pi]$.

For $|\omega| < \omega_{\max} + \eta$, by definition of η and Lemma 2, there exists $\varepsilon_3 \in (0, \varepsilon_2]$ such that for $\varepsilon \in (0, \varepsilon_3]$,

$$\lambda\beta e^{-j\omega\kappa} \in \Omega_\varepsilon, \quad \forall\lambda \in (1-\delta, 1+\delta),\ |\omega| < \omega_{\max} + \eta,\ \kappa \in \overline{[0, \bar{\kappa}]}.$$

Note that $\lambda\beta \in \Omega_\varepsilon$ and Ω_ε is convex. This implies

$$(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa} \in \Omega_\varepsilon.$$

Therefore, $A + \left[(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}\right]BF_\varepsilon$ is Schur stable for any $|\omega| < \omega_{\max} + \eta$, $\lambda \in (1-\delta, 1+\delta)$, $\kappa \in \overline{[0, \bar{\kappa}]}$ and $\alpha \in [0, 1]$. In conclusion, condition (22) holds. □

4. NETWORK WITH PARTIAL-STATE COUPLING

We next consider the case where the network has partial-state coupling. In this case, we design a decentralized synchronization controller for each agent using the low-gain compensator.

$$\begin{cases} \chi^i(k+1) = (A+KC)\chi^i(k) - Kz^i(k), \\ u^i(k) \quad = \beta F_\varepsilon \chi^i(k), \end{cases} \tag{23}$$

where $F_\varepsilon = -(B'P_\varepsilon B + I)^{-1}B'P_\varepsilon A$ and P_ε is the unique positive definite solution of the algebraic Riccati equation (13), whereas K is such that $A+KC$ is Schur stable.

Theorem 2
For a given set $\mathcal{G}_\delta$ with $\delta < 1$ and $\bar{\kappa} > 0$, consider the agents (1) and (2) with any communication topology belonging to the set $\mathcal{G}_\delta$. In that case, Problem 1 is solvable via synchronization controller (23) if

$$\omega_{\max}\bar{\kappa} < \arccos(\delta). \tag{24}$$

Specifically, for given $\mathcal{G}_\delta$ and κ satisfying (24), there exist $\beta > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (23) achieve synchronization for any communication topology in $\mathcal{G}_\delta$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

We first show the following lemma.

Lemma 5
Consider N agents of the form (1) with associated communication topology (2). Synchronization is achievable via a controller of the form

$$\begin{cases} \chi^i(k+1) = A_c\chi^i(k) + B_c z^i(k), \\ u^i(k) \quad = C_c\chi^i(k). \end{cases} \tag{25}$$

If the $N-1$ systems

$$\bar{\xi}^i(k+1) = \bar{\mathcal{A}}\bar{\xi}^i(k) + (1-\lambda_i)\bar{\mathcal{B}}\bar{\mathcal{C}}\bar{\xi}^i(k-\kappa) \tag{26}$$

are globally asymptotically stable where λ_i, $i = 2, ..., N$, are the eigenvalues of D matrix not equal to 1 and

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ B_cC & A_c \end{bmatrix}, \quad \bar{\mathcal{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{\mathcal{C}} = \begin{bmatrix} 0 & C_c \end{bmatrix}.$$

Proof
See Appendix. □

Proof of Theorem 2
It follows from Lemma 5 that Theorem 2 holds if there exists β and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the system

$$\begin{cases} x(k+1) = Ax(k) + \lambda\beta BF_\varepsilon\chi(k-\kappa), \\ \chi(k+1) = (A+KC)\chi(k) - KCx(k). \end{cases} \tag{27}$$

is asymptotically stable for any $\lambda \in \mathcal{D}(1,\delta)$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

First of all, because $\bar{\kappa}$ satisfies condition (24), there exists $\beta > \frac{2}{1-\delta}$ such that

$$\omega_{\max}\bar{\kappa} < \arccos\left(\delta + \frac{2}{\beta}\right). \tag{28}$$

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Note that β is independent of ε. Let this β be fixed. Because $(1-\delta)\beta > 2$, we have for $\lambda \in \mathcal{D}(1,\delta)$ that $\mathrm{Re}(\lambda\beta) > 2$. It follows from Lemma 3 that there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, the system (27) is asymptotically stable for $\kappa = 0$, that is, the matrix $\bar{\mathcal{A}} + \lambda\beta\bar{\mathcal{B}}\bar{\mathcal{C}}$ is Schur stable, where

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ -KC & A+KC \end{bmatrix}, \quad \bar{\mathcal{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix} \text{ and } \bar{\mathcal{C}} = \begin{bmatrix} 0 & \beta F_\varepsilon \end{bmatrix}.$$

Then, Lemma 1 tells that (27) with $\varepsilon \in (0, \varepsilon_1]$ is asymptotically stable for $\lambda \in \mathcal{D}(1,\delta)$ and $\kappa \in \overline{[0,\bar{\kappa}]}$ if

$$\det\left\{e^{j\omega}I - \bar{\mathcal{A}} - \left[(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}\right]\bar{\mathcal{B}}\bar{\mathcal{C}}\right\} \neq 0,$$
$$\forall\omega \in [-\pi,\pi],\ \lambda \in \mathcal{D}(1,\delta),\ \kappa \in \overline{[0,\bar{\kappa}]},\ \alpha \in [0,1]. \quad (29)$$

The rest of the proof basically follows along the same lines as the proof of Theorem 1. There exists $\eta > 0$ independent of ε such that

$$\omega\bar{\kappa} < \arccos\left(\delta + \frac{2}{\beta}\right), \quad \text{for } |\omega| < \omega_{\max} + \eta.$$

For $|\omega| \geqslant \omega_{\max} + \eta$, we can show that there exists $\varepsilon_2 \leqslant \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, we see that condition (29) holds using the same argument as in the proof of Theorem 1.

For $|\omega| < \omega_{\max}+\eta$, by definition of β and η, $\mathrm{Re}(\lambda\beta e^{-j\omega\kappa}) > 2$ for any $\lambda \in \mathcal{D}(1,\delta)$, $|\omega| < \omega_{\max}+\eta$ and $\kappa \in \overline{[0,\bar{\kappa}]}$. This, together with the fact that $\mathrm{Re}(\lambda\beta) > (1-\delta)\beta > 1$, implies that

$$\mathrm{Re}\left[(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}\right] > \min\{2, (1-\delta)\beta\} > 1$$
$$\lambda \in \mathcal{D}(1,\delta),\ |\omega| < \omega_{\max} + \eta,\ \kappa \in \overline{[0,\bar{\kappa}]},\ \alpha \in [0,1].$$

Obviously, for $\lambda \in \mathcal{D}(1,\delta)$ and β given by (28), $(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}$ can then be bounded in some compact set, say $\mathcal{S}$, which only depends on δ and $\bar{\kappa}$ and is located inside $\{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$. Then, by Lemma 3, we can find $\varepsilon^* \leqslant \varepsilon_2$ such that for $\varepsilon \in (0, \varepsilon^*]$, the matrix

$$\bar{\mathcal{A}} + \left[(1-\alpha)\lambda\beta + \alpha\lambda\beta e^{-j\omega\kappa}\right]\bar{\mathcal{B}}\bar{\mathcal{C}}$$

is Schur stable, and hence, condition (29) holds for $|\omega| < \omega_{\max} + \eta$, $\lambda \in \mathcal{D}(1,\delta)$, $\kappa \in \overline{[0,\bar{\kappa}]}$ and $\alpha \in [0,1]$.

When the communication topology is undirected, the following result can be proven with an argument similar to the proof of Corollary 2. □

Corollary 3

For a given set $\mathcal{G}_\delta$ and $\bar{\kappa} > 0$, consider the agents (1) and (2) with any undirected communication topology belonging to the set $\mathcal{G}_\delta$. In that case, Problem 1 is solvable via synchronization controller (23) if

$$\omega_{\max}\bar{\kappa} < \frac{\pi}{2}. \quad (30)$$

Specifically, for given $\mathcal{G}_\delta$ and $\bar{\kappa}$ satisfying (30), there exist $\beta > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (23) achieve synchronization for any communication topologies in $\mathcal{G}_\delta$ and $\kappa \in \overline{[0,\bar{\kappa}]}$.

5. NETWORK OF INTROSPECTIVE AGENTS

In this section, we consider introspective agents. In this case, the agents have the following dynamics.

$$\begin{cases} x^i(k+1) = Ax^i(k) + Bu^i(k), \\ y^i(k) \quad = C_m x^i(k). \end{cases} \quad (31)$$

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The network measurement is the same as given by (2). In other words, besides the network measurement (2), the agents also collect a local output y^i that is an absolute measurement of their own dynamics, to which the agents have instantaneous access.

For our communication topology, we assume that Assumption 1 is satisfied. For our agents, we substitute Assumption 2 with the following

Assumption 3
(A, B) is controllable, (A, C) is observable, and (A, C_m) is detectable.

Note that despite of different settings, the problem is the same as in the non-introspective case, that is essentially to synchronize the *initial conditions* among the agents although the time-delay communication scheme of the network and of course under certain conditions. The local measurement, however, provides additional freedom to remove some of those conditions that are imposed in Theorems 1 and 2 at the cost of a more limited set of synchronization trajectories. We shall prove the following result.

Theorem 3
For a given set $\mathcal{G}_\delta$ and $\bar{\kappa} > 0$, consider the agents (31) and (2) with any communication topology belonging to the set $\mathcal{G}_\delta$. In that case, there exists N controllers of the form

$$\begin{cases} \chi^i(k+1) = A_c\chi^i(k) + B_{1,c}z^i(k) + B_{2,c}y^i(k), \\ u^i(k) \quad = C_c\chi^i(k), \end{cases} \tag{32}$$

such that the agents (31) with controller (32) achieve synchronization for any communication topologies in $\mathcal{G}_\delta$ and $\kappa \in \overline{[0, \bar{\kappa}]}$.

Proof
It takes two steps to prove this theorem. First, we design a local dynamic measurement feedback for each agent to assign the close-loop eigenvalues to proper locations based on the local information y^i. The second step is to design synchronization protocol for the closed-loop agents. Consider an observer-based measurement feedback controller

$$\begin{cases} \hat{x}_1^i(k+1) = A\hat{x}_1^i(k) + Bu^i(k) - K\left(y^i(k) - C_m\hat{x}_1^i(k)\right), \\ u^i(k) \quad = F\hat{x}_1^i(k) + v^i(k), \end{cases} \tag{33}$$

where $A + KC_m$ is Schur stable and F is such that $A + BF$ has desired eigenvalues in the closed unit disc that condition (14) is satisfied and that $(A + BF, C)$ is observable. Such K and F always exist under Assumption 3. Note that because (A, C) is observable, we can always guarantee that $(A + BF, C)$ is observable by an arbitrary small perturbation of F. Define $e^i = x^i - \chi_1^i$ and $\bar{\xi}^i = [x^i; e^i]$. The closed-loop of (31) and (33) can be written in terms of x^i and e^i as follows

$$\bar{\xi}^i(k+1) = \bar{A}\xi^i(k) + \bar{B}v^i(k), \tag{34}$$

where

$$\bar{A} = \begin{bmatrix} A + BF & -BF \\ 0 & A + KC_m \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

The network measurement becomes

$$z^i(k) = \sum_{j=1}^{N} d_{ij}\bar{C}\left(\bar{\xi}^i(k-\kappa) - \bar{\xi}^j(k-\kappa)\right), \tag{35}$$

where

$$\bar{C} = \begin{bmatrix} C_m & 0 \end{bmatrix}.$$

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Note that the aforementioned construction of F and K guarantees that $(\bar{A}, \bar{C})$ is detectable.

Then, we find that (34) with (35) can be viewed as a network of non-introspective agents with partial state coupling. Moreover, condition (24) in Theorem 2 is always satisfied. Therefore, the second-step design follows straightforwardly from Theorem 2 and results in the following controller

$$\begin{cases} \hat{x}_2^i(k+1) = (\bar{A} + \bar{K}\bar{C})\hat{x}_2^i(k) - \bar{K}z \\ v^i(k) \quad = \beta \bar{F}\hat{x}_2^i(k), \end{cases} \tag{36}$$

where $\bar{K}$ is such that $\bar{A} + \bar{K}\bar{C}$ is Schur stable and $\bar{F}$ is designed as in the proof of Theorem 2.

Eventually, the composite controller of (33) and (36), which is (32), will solve the synchronization problem for the introspective agents (31) with arbitrarily given delay and we have

$$A_c = \begin{bmatrix} A + KC_m + BF & \beta \bar{F} \\ 0 & \bar{A} + \bar{K}\bar{C} \end{bmatrix}, \quad B_{c,1} = \begin{bmatrix} 0 \\ -\bar{K} \end{bmatrix}, \quad B_{c,2} = \begin{bmatrix} K \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} F & \beta \bar{F} \end{bmatrix}.$$

□

6. CONCLUSION

In this paper, we study the synchronization problem in a homogeneous network of discrete-time agents with unknown uniform constant communication delay. When agents do not possess absolute measurement of their own dynamics, we find an upper bound of tolerable delay and for delay satisfying the proposed conditions, a decentralized synchronization controller can be designed using low-gain technique. On the other hand, if the agents are introspective—that is, they acquire separate observation of their own states—the synchronization problem can be solved with arbitrary but bounded communication delay.

APPENDIX A

Proof of Lemma 3

Define $e = x - \chi$. The closed-loop of (6) and (10) in terms of x and e can be written as

$$\begin{cases} x(k+1) = (A + \lambda BF_\varepsilon)x(k) - \lambda BF_\varepsilon e(k) \\ e(k+1) = (A + KC - \lambda BF_\varepsilon)e(k) + \lambda BF_\varepsilon x(k). \end{cases} \tag{37}$$

First of all, by (8), we have

$$\begin{aligned} &(A + \lambda BF_\varepsilon)^* P_\varepsilon (A + \lambda BF_\varepsilon) - P_\varepsilon \\ &\qquad = -\varepsilon I + F_\varepsilon' \left[|1-\lambda|^2 (B' P_\varepsilon B + I) - |\lambda|^2 I \right] F_\varepsilon \\ &\qquad \leqslant -\varepsilon I + \left[(1+\gamma_\varepsilon)|1-\lambda|^2 - |\lambda|^2 \right] F_\varepsilon' F_\varepsilon, \end{aligned}$$

where γ_ε is defined in Lemma 2. Define a set

$$\Omega_\varepsilon := \left\{ z \in \mathbb{C} : \left| z - \left(1 + \frac{1}{\gamma_\varepsilon}\right) \right| \leqslant \frac{1}{\gamma_\varepsilon} \right\}. \tag{38}$$

It is easy to see that for any compact set $\mathcal{S} \subset H$, there exists ε_1 such that $\mathcal{S} \subset \Omega_\varepsilon$ for $\varepsilon \in (0, \varepsilon_1]$. Moreover, $\lambda \in \Omega_\varepsilon$ is equivalent to

$$(1+\gamma_\varepsilon)|1-\lambda|^2 - |\lambda|^2 \leqslant -1,$$

and hence,

$$(A + \lambda BF_\varepsilon)^* P_\varepsilon (A + \lambda BF_\varepsilon) - P_\varepsilon \leqslant -\varepsilon I - F_\varepsilon' F_\varepsilon. \tag{39}$$

Next, let Q be the positive definite solution of Lyapunov equation

$$(A+KC)'Q(A+KC)-Q+4I=0.$$

Because $F_\varepsilon \to 0$ as $\varepsilon \to 0$, for any compact set $\mathcal{W}$, there exists a $\varepsilon_2 \leqslant \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$

$$(A+KC-\lambda BF_\varepsilon)'Q(A+KC-\lambda BF_\varepsilon)-Q+3I \leqslant 0.$$

Then, consider the function $V_1(k)=e(k)^*Qe(k)$. Let $\mu(k)=F_\varepsilon x(k)$. To ease our presentation, we shall omit the time label (k) whenever this will not cause any confusion.

$$\begin{aligned} V_1(k+1)-V_1(k) & \\ \leqslant & -3\|e\|^2+2\,\mathrm{Re}\left(\lambda^*\mu^*B'Q[A+KC-\lambda BF_\varepsilon]e\right)+|\lambda|^2\mu^*B'QB\mu \\ \leqslant & -3\|e\|^2+M_1\|\mu\|\|e\|+M_2\|\mu\|^2, \end{aligned} \tag{40}$$

where

$$M_1=2\|B'Q\|\max_{\substack{\lambda\in\mathcal{S}\\ \varepsilon\in[0,1]}}\{|\lambda|\|A+KC-\lambda BF_\varepsilon\|\}, \quad M_2=\|B'QB\|\max_{\lambda\in\mathcal{S}}|\lambda|^2.$$

It should be noted that M_1 and M_2 are independents of specific ε and λ.

Also, consider $V_2(k)=x(k)'P_\varepsilon x(k)$. We have that

$$\begin{aligned} V_2(k+1)-V_2(k) \leqslant & -\varepsilon\|x\|^2-\|\mu\|^2+2\,\mathrm{Re}\left(\lambda^*e^*F_\varepsilon'B'P_\varepsilon[A+\lambda BF_\varepsilon]x\right) \\ & +|\lambda|^2e^*F_\varepsilon'BP_\varepsilon BF_\varepsilon e. \end{aligned}$$

Note that

$$\begin{aligned} 2\left|\lambda^*e^*F_\varepsilon'B'P_\varepsilon[A+\lambda BF_\varepsilon]x\right| &= 2\left|\lambda^*e^*F_\varepsilon'B'P_\varepsilon Ax+|\lambda|^2e^*F_\varepsilon'B'P_\varepsilon B\mu\right| \\ &= 2\left|-\lambda^*e^*F_\varepsilon'(B'P_\varepsilon B+I)\mu+|\lambda|^2e^*F_\varepsilon'B'P_\varepsilon B\mu\right| \\ &\leqslant \theta_1(\varepsilon)\|e\|\|\mu\|, \end{aligned}$$

where

$$\theta_1(\varepsilon)=2(1+\gamma_\varepsilon)\|F_\varepsilon\|\max_{\lambda\in\mathcal{S}}\{|\lambda|\}+2\gamma_\varepsilon\|F_\varepsilon\|\max_{\lambda\in\mathcal{S}}\{|\lambda|^2\}.$$

Then,

$$V_2(k+1)-V_2(k) \leqslant -\varepsilon\|x\|^2-\|\mu\|^2+\theta_1(\varepsilon)\|e\|\|\mu\|+\theta_2(\varepsilon)\|e\|^2, \tag{41}$$

where

$$\theta_2(\varepsilon)=\gamma_\varepsilon\|F_\varepsilon\|^2\max_{\lambda\in\mathcal{S}}\{|\lambda|^2\}.$$

Finally, consider a Lyapunov candidate $V(k)=V_1(k)+cV_2(k)$ with

$$c=M_2+M_1^2.$$

In view of (40) and (41), we obtain

$$V(k+1)-V(k) \leqslant -\varepsilon c\|x\|^2-M_1^2\|\mu\|^2-[3-c\theta_2(\varepsilon)]\|e\|^2+[M_1+c\theta_1(\varepsilon)]\|\mu\|\|e\|.$$

There exists a $\varepsilon^* \leqslant \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$,

$$3-c\theta_2(\varepsilon) \geqslant 2, \quad M_1+c\theta_1(\varepsilon) \leqslant 2M_1.$$

This yields that

$$V(k+1)-V(k) \leqslant -\varepsilon c\|x\|^2-\|e\|^2-(\|e\|-M_1\|\mu\|)^2.$$

Therefore, for $\varepsilon \in (0, \varepsilon^*]$, the system (37) is globally asymptotically stable. □

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Proof of Lemma 4
We choose a consensus controller for agent i as

$$u^i = Fz^i.$$

for some matrix $F \in \mathbb{R}^{m\times n}$. Define $\tilde{x} = [x^1; \cdots; x^N]$. The overall dynamics of the N agents can be written as

$$\tilde{x}(k+1) = (I_N \otimes A)\tilde{x}(k) + [(I-D)\otimes BF]\,\tilde{x}(k-\kappa).$$

Define $\xi = [\xi^1; \cdots; \xi^N] = (T \otimes I_n)\tilde{x}$, where $\xi^i \in \mathbb{C}^n$ and T is such that $J = T(I-D)T^{-1}$ is in the Jordan canonical form and $J(1,1) = 0$. In the new coordinates, the dynamics of ξ can be written as

$$\xi(k+1) = (I_N \otimes A)\xi(k) + (J \otimes BF)\xi(k-\kappa).$$

The network synchronization is achieved if $\xi^i \to 0$ as $k \to \infty$ for $i = 2, ..., N$. To see this, let $\eta(k) = [\xi^1(k); 0; \cdots; 0]$. If $\xi(k) \to \eta(k)$, then $\tilde{x}(k) \to (T^{-1} \otimes I_n)\eta(k)$. Note that the columns of T^{-1} comprise all the right eigenvectors and generalized eigenvectors of $I - D$. The first column of T^{-1} is vector $\mathbf{1}$. This implies that for $i = 1, ..., N$

$$x^i(k) \to \xi^1(k).$$

The sub-dynamics of $\bar{\xi}(k) = \left[\xi^2(k); \cdots; \xi^N(k)\right]$ are

$$\bar{\xi}(k+1) = (I_{N-1} \otimes A)\bar{\xi}(k) + (\bar{J} \otimes BF)\bar{\xi}(k-\kappa) \tag{42}$$

where $\bar{J}$ is such that

$$J = \begin{bmatrix} 0 & \\ & \bar{J} \end{bmatrix}.$$

The eigenvalues of system (42) are given by the roots of its characteristic equation

$$\det\left\{zI - (I_{N-1} \otimes A) - z^{-\kappa}(\bar{J} \otimes BF)\right\} = 0,$$

which, due to the upper-triangular structure of $I_{N-1} \otimes A$ and $\bar{J} \otimes BF$, are the union of the eigenvalues of the $N-1$ systems

$$\xi^i(k+1) = A\xi^i(k) + (1-\lambda_i)BF\xi^i(k-\kappa), \quad i = 2, ..., N.$$

The result in Lemma 4 follows. □

Proof of Lemma 5
Let $\bar{x}^i = [x^i; \chi^i]$. Then, for each agent, the closed-loop dynamics are

$$\begin{cases} \bar{x}^i(k+1) = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \bar{x}^i(k) + \begin{bmatrix} 0 \\ B_c \end{bmatrix} z^i(k) \\ \quad z^i(k) \quad = \sum_{j=1}^N d_{ij} \begin{bmatrix} C & 0 \end{bmatrix} \left(\bar{x}^i(k-\kappa) - \bar{x}^j(k-\kappa)\right). \end{cases}$$

Define $\tilde{x} = [\bar{x}^1; \cdots; \bar{x}^N]$,

$$\mathcal{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}.$$

The overall dynamics of the N agents can be written as

$$\tilde{x}(k+1) = (I_N \otimes \mathcal{A})\tilde{x}(k) + [(I-D)\otimes \mathcal{B}\mathcal{C}]\,\tilde{x}(k-\kappa).$$

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Similarly as in the proof of Lemma 4, by introducing the transformation

$$\xi = \left[\xi^i; \ldots; \xi^N\right] = (T \otimes I_n)\tilde{x},$$

we can prove that the synchronization is achieved if the $N-1$ system

$$\xi^i(k+1) = \mathcal{A}\xi^i(k) + (1-\lambda_i)\mathcal{B}\mathcal{C}\xi^i(k-\kappa), \quad i = 2, ..., N \tag{43}$$

is globally asymptotically stable, where λ_i are the eigenvalues of D not equal to 1. It remains to show that the stability of (26) is equivalent to that of (43). Note that

$$\begin{aligned}
&\begin{bmatrix} \frac{1}{\lambda}z^{\kappa}I & \\ & I \end{bmatrix} \left(zI - \bar{A} - \lambda z^{-\kappa}\bar{\mathcal{B}}\bar{\mathcal{C}}\right) \begin{bmatrix} \lambda z^{-\kappa}I & \\ & I \end{bmatrix} \\
&\quad = \begin{bmatrix} \frac{1}{\lambda}z^{\kappa}I & \\ & I \end{bmatrix} \begin{bmatrix} zI - A & -\lambda z^{-\kappa}BC_c \\ -B_cC & zI - A_c \end{bmatrix} \begin{bmatrix} \lambda z^{-\kappa}I & \\ & I \end{bmatrix} \\
&\quad = \begin{bmatrix} zI - A & -BC_c \\ -\lambda z^{-\kappa}B_cC & zI - A_c \end{bmatrix} \\
&\quad = z - \mathcal{A} - \lambda z^{-\kappa}\mathcal{B}\mathcal{C}.
\end{aligned}$$

This implies that the characteristic function of each system has the same number of zeros outside the unit circle. Hence, the stability of the two systems are equivalent. □

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