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Variational inequality problems (VIP) are an important class of mathematical problems that appear in many practical situations. So, it is important to find efficient and robust numerical solution methods. An appealing idea is to embed the VIP into a one-parametric problem which, then, can be solved numerically by a path-following method. In this article, we study two different types of embeddings and we analyse their generic properties. The non-linear complementarity problem and box-constrained VIP are discussed as special cases.

Keywords: genericity; non-linear complementarity constraints; one-parametric embedding; regularity; variational inequality problem

AMS Subject Classifications: 49J40; 65K15

1. Introduction

We deal with variational inequality problems (VIP)

$$\text{VIP}(\Phi, Y): \quad \text{Find } y \in Y \text{ such that}: \ \Phi(y)^T(z-y) \geq 0 \quad \forall z \in Y \tag{1}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ are given. VIP represents an important class of mathematical problems with many applications; e.g. Nash equilibrium models in economics and traffic engineering (see e.g. [1–7]).

Under certain convexity conditions, the existence of a solution of VIP can generally be guaranteed. The following result of the e.g. is well known (see e.g. [6]). If Y is convex and Φ is continuous then a solution y of VIP(Φ, Y) exists if one of the following two conditions is satisfied.

(1) Y is compact.
(2) Φ is strongly monotone, i.e. $\exists\kappa$, $\kappa > 0$, such that for all $y_1, y_2 \in Y$ we have $(\Phi(y_1) - \Phi(y_2))^T(y_1 - y_2) \geq \kappa\|y_1 - y_2\|^2$. (In this case the solution is also unique)

Although the paper mostly deals with general sets of feasible solutions, due to its importance, some remarks on the convex case will also be done.

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Several methods for solving VIP have been proposed, such as fixed point approaches [6], methods using merit functions [8] or approaches employing regularization techniques [9]. We also refer to [3] for a detailed discussion of these methods.

There are several papers on so-called interior point methods for solving VIP. This approach is based on the KKT optimality conditions of the formulation of VIP as an optimization problem (see Section 2) and assumes convexity of the set Y. The method defines and analyses a one-parametric embedding (perturbation) of the KKT conditions similar to the interior point method for linear programs (see e.g. [10,11]).

In the present paper, we study another embedding approach for solving (1). In this approach, the original problem VIP(Φ, Y) is directly extended to a one-parametric variational inequality VIP(t), $t \in [0, 1]$, such that a solution of VIP(t) at $t = 0$ is easily available and VIP(t) for $t = 1$ coincides with VIP(Φ, Y). The idea is then to start with VIP(0) and to reach the original problem VIP(1) = VIP(Φ, Y) by applying a path-following method (see e.g. [12,13]). In this method, we need not to assume that Y is convex. We discuss two types of embedding methods and analyse the generic properties of the corresponding parametric problems VIP(t). The analysis is based on the fact that a solution of a variational inequality can be seen as a solution of a corresponding optimization problem.

Embedding methods for solving special types of optimization problems have been discussed, e.g. in [14–17]. Here, we analyse how this embedding approach can be applied to solve (VIP) problems of the form (1). The paper is organized as follows.

In Section 2, we introduce VIP and the connection to optimization theory. Section 3 considers one-parametric VIPs and summarizes the genericity results for general one-parametric problems VIP(t) obtained by Gómez [18]. In Section 4, two types of one-parametric embeddings for solving VIP(Φ, Y) are discussed and analysed. The general genericity results of Section 3 are then extended to the corresponding specially structured parametric problems. We also consider the application of this approach to two special instances of VIP, namely the non-linear complementarity problem (NLCP) in which $Y = \mathbb{R}^n_+$ and the box-constrained VIP, where $Y = [0, 1]^n$.

2. Preliminaries on VIP

We reconsider VIP(Φ, Y) in the form:

$$\text{Find } y \in Y \text{ such that} \quad 0 \le \Phi(y)^T(z - y), \quad \forall z \in Y. \tag{2}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n, h : \mathbb{R}^n \to \mathbb{R}^m, g : \mathbb{R}^n \to \mathbb{R}^s$ and $Y = \{y \in \mathbb{R}^n \mid h(y) = 0,\ g(y) \ge 0\}$, i.e. the set Y is given by means of finitely many equalities and inequalities,

$$h_i(y) = 0,\ i \in I := \{1, \dots, m\}, \quad g_j(y) \ge 0,\ j \in J := \{1, \dots, s\}.$$

We denote such a variational inequality by VIP(Φ, h, g) and we assume throughout that $Y \neq \varnothing$ and $(\Phi, h, g) \in C^2(\mathbb{R}^n, \mathbb{R}^{n+m+s})$.

In view of $\Phi(y)^T(y - y) = 0$ and $y \in Y$, y solves (2) if and only if it is a solution of the (parametric) optimization problem (in the variable z):

$$\min_z \ \Phi(y)^T z \quad \text{s.t. } z \in Y \tag{3}$$

This formulation relates VIP to optimization theory. So, as usual in optimization, we introduce the active index set $J_0(y) = \{j \in J \mid g_j(y) = 0\}$ and the (derivative of) the Lagrangian function

$$\overline{L}(y; \lambda_0, \lambda, \mu) = \lambda_0 \Phi(y) - \sum_{i \in I} \lambda_i Dh_i(y) - \sum_{j \in J_0(y)} \mu_j Dg_j(y). \tag{4}$$

The following necessary optimality conditions are well known in optimization.

Proposition 1 (see e.g. [18]) *For any solution y of $VIP(\Phi, h, g)$, there exist multipliers $(\lambda_0, \lambda, \mu) \neq 0$ such that $\overline{L}(y; \lambda_0, \lambda, \mu) = 0$. Moreover, the Hessian (wrt. the variable z)*

$$-\sum_{i \in I} \lambda_i D^2 h_i(y) - \sum_{j \in J_0(y)} \mu_j D^2 g_j(y) \text{ is positive semidefinite on } T_y Y$$

where $T_y Y := \{\xi \mid Dh_i(y)^T \xi = 0, i \in I; Dg_j(y)^T \xi = 0; j \in J_0(y)\}$.

If $LICQ$ holds at y, i.e. if the gradients $Dh_i(y)$, $i \in I$; $Dg_j(y)$, $j \in J_0(y)$ are linear independent, then $\overline{L}(y; \lambda_0, \lambda, \mu) = 0$ holds with $\lambda_0 \neq 0$, i.e. we can assume $\lambda_0 = 1$.

If the problem (3) is convex, that is if the functions h_i are (affine) linear and the components $-g_j$ are convex (the objective $\Phi(y)^T z$ is linear in z), then the validity of the KKT-system $\overline{L}(y; 1, \lambda, \mu) = 0$, $\mu \geq 0$, is a sufficient condition for $y \in Y$ to be a solution of (3). Under LICQ, optimality and KKT-condition are even equivalent.

We now introduce some definitions and regularity conditions concerning candidate solutions y for (3) and thus for VIP(Φ, h, g).

Definition 1 A point $y \in Y$ is called a generalized critical point (gc-point, denoted by $y \in \Sigma_{gc}$) if $\exists \lambda_0, \lambda_i, i \in I, \mu_j, j \in J_0(y)$ not all zero such that $\overline{L}(y; \lambda_0, \lambda, \mu) = 0$. If LICQ holds at $y \in \Sigma_{gc}$ it is called a critical point, i.e. $y \in \Sigma_{crit}$. In this case $\overline{L}(y; 1, \lambda, \mu) = 0$ holds with uniquely determined λ, μ. *If* $\mu_j \geq 0$, $j \in J_0(y)$, $y \in \Sigma_{crit}$ is said to be a stationary point, 'notation $y \in \Sigma_{stat}$'.

Definition 2 A generalized critical point y is called non-degenerate, notation $y \in \Sigma^1_{gc}$, if the following regularity conditions hold:

- *V-a* LICQ holds at y; so $\overline{L}(y; 1, \lambda, \mu) = 0$ holds with unique λ, μ.
- *V-b* $\mu_j \neq 0$ holds for all $j \in J_0(y)$.
- *V-c* The reduced Jacobian $D_y \overline{L}(y; 1, \lambda, \mu) \mid_{T_y Y}$ is non-singular.

VIP(Φ, h, g) *is called* regular *if all its solutions y satisfy [V-a]–[V-c].*

As in the case of optimization problems, a non-degenerate critical point is an isolated critical point. However, there is an important difference with standard optimization. Due to the fact that $D\Phi(y)$ is not always symmetric, the Jacobian $D_y \overline{L}|_{T_y Y}$ need not to be symmetric. Consequently, this matrix may have negative or even complex eigenvalues at the solution. The next example illustrates this situation.

Example 1 Consider VIP(Φ, Y), with $Y = \mathbb{R}^3$ and $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\Phi(y) := Ay - (-1, 0, 0)^T \quad \text{with} \quad A := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Then $y = [1, 0, 0]$ is the solution but the eigenvalues of $D_y\overline{L}([1, 0, 0])_{|_{\mathbb{R}^3}} = A$ are $-1, i, -i$.

The fact that the matrix $D_y\overline{L}|_{T_yY}$ may be non-symmetric and thus may have non-real eigenvalues makes the analysis of (sharp) second-order optimality conditions more complicated than in the case of standard optimization problems. Such an analysis is beyond the aim of this paper and is a topic of future research.

3. Parametric VIP

In this section, we give a brief introduction into parametric VIP and into genericity results. In the next section, these results will be extended to the proposed special parametric embeddings for solving a non-parametric problem VIP(Φ, h, g). We consider the following one-parametric VIP denoted by VIP($\Psi, H, G; t$)): for $t \in [0, 1]$ find

$$y \in Y(t) \text{ such that } \begin{cases} 0 \leq \Psi(y, t)^T(z - y) \quad \forall z \in Y(t) \\ Y(t) = \left\{ y \in \mathbb{R}^n \,\middle|\, \begin{array}{rcl} H(y, t) & = & 0 \\ G(y, t) & \geq & 0 \end{array} \right\} \end{cases} \tag{5}$$

Here, we assume $(\Psi, H, G) \in C^3(\mathbb{R}^n \times \mathcal{T}, \mathbb{R}^{n+m+s})$ where $\mathcal{T} \subset \mathbb{R}$ is an interval (open or closed, depending on the context). It is clear that all definitions of the previous section such as J_0, $\overline{L}$, stationary points, generalized-, non-degenerate critical points can directly be extended to the one-parametric case. The functions now depend on (y, t) instead merely on y. The first-order condition necessary for (y, t) with $y \in Y(t)$ now reads $\overline{L}(y, t; \lambda_0, \lambda, \mu) = 0$, where

$$\overline{L}(y, t; \lambda_0, \lambda, \mu) = \lambda_0 \Psi(y, t) - \sum_{i \in I} \lambda_i D_y H_i(y, t) - \sum_{j \in J_0(y,t)} \mu_j D_y G_j(y, t)$$

and $\Sigma_{gc}(\Psi, H, G)$ denotes the set of gc-points of VIP($\Psi, H, G; t$). At a non-degenerate critical point $(\overline{y}, \overline{t})$ standard results of parametric optimization guaranties, locally near $\overline{t}$, the existence of an unique curve $(y(t), t)$ of non-degenerated critical points of VIP($\Psi, H, G; t$). Unfortunately only special parametric VIPs are such that for all t all solution (y, t) of VIP($\Psi, H, G; t$) are non-degenerate. However, we can analyse the generic behaviour of such a problem. Genericity results reveal to us the precise types of degeneracies that are to be encountered in solving a generic ('normal') problem (to be defined later on).

Genericity results for one-parametric finite optimization problems have been developed by Jongen et al. In [19,20] they have proven the famous result of the 'five types'. In Gómez [18] these techniques have been applied to parametric VIP. Here are the five types of critical points. A non-degenerate critical point (y, t) is a point of Type 1 (V-1) and the points of Types 2–5, (V-2–V-5) are points where precisely one of the conditions V-a–V-c in Definition 2 fails:

- Points of Type 2, V-2: Condition (V-b) does not hold at (y, t).
- Points of Type 3, V-3: condition (V-c) fails.
- Points of Type 4, V-4: LICQ fails and $m + |J_0(y, t)| \leq n$.
- Points of Type 5, V-5: LICQ fails and $m + |J_0(y, t)| = n + 1$.

Denoting the sets of gc-points of Type i by $\Sigma^i_{gc}(\Psi, H, G)$, we introduce the subset of VIP(t) problems with $\mathcal{T} \subset [0, 1]$,

$$\mathcal{F}_{|\mathcal{T}} = \left\{ (\Psi, H, G) \in C^3(\mathbb{R}^n \times \mathcal{T}, \mathbb{R}^{n+m+s}) \,\middle|\, \begin{array}{l} (y, t) \in \Sigma_{gc}(\Psi, H, G) \\ t \in \mathcal{T} \end{array} \right. \\ \left. \Rightarrow (y, t) \in \cup_{i=1}^5 \Sigma^i_{gc} \right\}.$$

We emphasize that the set of one-parametric variational inequalities VIP$(\Psi, H, G; t), t \in \mathcal{T}$, can be identified with the set of functions $(\Psi, H, G) \in C^3(\mathbb{R}^n \times \mathcal{T}, \mathbb{R}^{n+m+s})$. Therefore, we often write VIP$(\Psi, H, G; t) \in \mathcal{F}_{|\mathcal{T}}$ instead of $(\Psi, H, G) \in \mathcal{F}_{|\mathcal{T}}$.

A problem VIP$(\Psi, H, G; t)$ in $\mathcal{F}_{|\mathcal{T}}$ has the (nice) properties that only at a discrete subset $\mathcal{T}_0 \subset \mathcal{T}$ a degenerate gc-point may occur and these gc-points $(y, t), t \in \mathcal{T}_0$ are of precisely one of the Types V-2–V-5. At all points $t \in \mathcal{T} \setminus \mathcal{T}_0$, all gc-points (y, t) are of Type 1. So in $\mathcal{F}_{|\mathcal{T}}$ only very specific cases of (degenerate) gc-points can occur.

In the sequel, we assume that the set of C^k functions is endowed with the so-called strong topology C^k_S. In this topology, a neighbourhood $N_\varepsilon(f)$ of a function $f \in C^k(\mathbb{R}, \mathbb{R})$, for example, is given by continuous functions $\varepsilon(x) > 0$, $x \in \mathbb{R}$, instead of constants $\varepsilon > 0$: $g \in N_\varepsilon(f)$ if

$$|f^{(r)}(x) - g^{(r)}(x)| < \varepsilon(x), \quad \forall x \in \mathbb{R},\ 0 \leq r \leq k.$$

In what follows, we say that $\mathcal{C} \subset C^k$ is generic w.r.t. the topology C^k_S if $\mathcal{C}$ is dense and open in C^k_S or more generally if $\mathcal{C} = \cap_{\nu \in \mathbb{N}} \mathcal{C}_\nu$ with sets $\mathcal{C}_\nu$ which are dense and open in C^k_S.

The next theorem presents the genericity result for VIP(t) (cf. [18] for details).

Proposition 2 *If $\mathcal{T}$ is a closed and bounded interval, then the set $\mathcal{F}_{|\mathcal{T}}$ is open and dense wrt. the C^3_S-topology.*

Note that in [18] this genericity result is proven for $\mathcal{T} = \mathbb{R}$. It can be shown that the result also is true for any interval $\mathcal{T} \subset \mathbb{R}$.

Figure 1 gives a sketch of curves $(y(t), t)$ of gc-points around $(\bar{y}, \bar{t}) \in \Sigma^i_{gc}$, $i = 1, \ldots, 5$, for a problem in $\mathcal{F}_{|[0,1]}$. Note that the whole set of gc-points consists of a union of such pieces (countably many). Let us assume that the problem VIP$(\Psi, H, G; t)$ is in $\mathcal{F}_{|[0,1]}$.

The local structure of the set of gc-points of Types 1, 2, 4 and 5 is the same as in standard optimization, in which the number of positive and negative multipliers and the sign of the determinant of $D_y\overline{L}|_{T_yY}$ will change as described in [21]. This does not hold for gc-points of Type 3 in general. At a gc-point of Type 3, the matrix $D_y\overline{L}|_{T_yY}$ becomes singular. However, since the matrices $D_y\overline{L}|_{T_yY}$ need not to be symmetric, either a real or a complex eigenvalue becomes zero. In the first case, the number of positive and negative eigenvalues changes as in the non-linear optimization problem. In the second case, the behaviour of the eigenvalues is more complex, although as shown in [22], the determinant of $D_y\overline{L}_{|T_yY}$ will change too when passing this type of singular point.

For the particular class of convex problems VIP(t), certain types of solutions can be excluded.

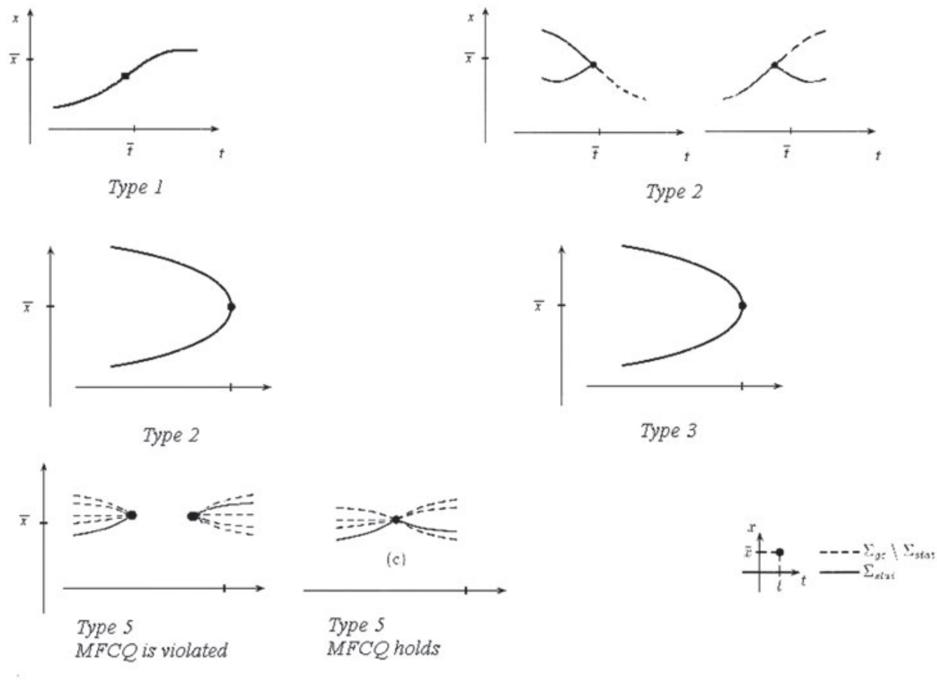

Figure 1. The behaviour of Σ_{stat} around the singularities.

Proposition 3 *Let $VIP(\Psi, H, G; t)$ be such that $Y(t)$ is convex (i.e. for any fixed t the functions $H_i(y,t)$ are linear, and $-G_j(y,t)$ are convex in y) and suppose that for all t, $\Psi(y,t)$ is a strongly monotone operator. Then there does not exist a solution (stationary point) (y,t) of Type 3.*

Proof In view of convexity, a stationary point yields a solution (y,t) (see Section 2). On the other hand, as $\Psi(y,t)$ is a strongly monotone operator, $D_y\Psi(y,t) \succ 0$, $\forall (y,t)$. So, in view of the second-order condition in Proposition 1 and the positive definiteness of $D_y\Psi$, the matrix

$$D_y\Psi - \sum_{i\in I} \lambda_i D^2_{yy} H_i - \sum_{j\in J_0} \mu_j D^2_{yy} G_j$$

is positive definite on the tangent space T_yY. So, the gc-point (y,t) cannot be of Type 3. □

Remark 1 The density property in Proposition 2 is based on the following fact [cf. [18]]: For almost every linear perturbation in y of the functions $\Psi(y,t)$, $H(y,t)$, $G(y,t)$, the (corresponding perturbed) parametric problem VIP($\Psi, H, G; t$) is in $\mathcal{F}_{|[0,1]}$.

4. Embeddings for VIP

In this section, we introduce the embedding approach for solving non-parametric VIP. Embedding methods have been successfully used to solve (non-parametric) optimization problems; see [15–17,23–26]. In our context the idea of this method is as follows. Let be given a non-parametric problem VIP(Φ, h, g). We define an appropriate one-parameter family of variational inequalities VIP($\Psi, H, G; t$), $t \in [0, 1]$ such that

- For $t = 0$, a solution y_0 of VIP($\Psi, H, G; 0$) is easily available.
- There is a solution of VIP($\Psi, H, G; t$) $\forall t \in [0, 1]$.
- VIP($\Psi, H, G; 1$) coincides with VIP(Φ, h, g).

Starting with y_0, the idea is to try to obtain a solution of problem VIP($\Psi, H, G; 1$) $=$ VIP(Φ, h, g) by following the path of solutions of VIP(t) from $t = 0$ to $t = 1$. Of course, a piecewise-continuous path connecting y_0 and a solution of VIP($\Psi, H, G, 1$) does not always exist. However, if the parametric problem lies in the generic set of 'nice' problems, then the solution paths can be tracked (at least locally) by applying continuation methods, (see e.g. [12,13]), to solve the corresponding parametric system of equations.

In the next subsections, we propose and analyse two types of embeddings VIP$^1(t)$ and VIP$^2(t)$ to solve a non-parametric VIP and apply the techniques of Section 3 to extend the genericity results to both embeddings. We notice that also from a practical viewpoint these genericity results are important. They precisely describe the (generic) structural situations a (generic) path-following method for solving VIP should be able to detect and to deal with numerically.

4.1. *Standard embedding for VIP*

This embedding is as follows: given the non-parametric problem VIP(Φ, h, g), we define a parametric variational inequality as follows:

$$(t\Phi(y) + (1-t)(y-y_0))^T (z-y) \geq 0 \quad \forall z \in Y^1(t) \tag{6}$$

where y_0 is a point of the feasible set Y of VIP(Φ, h, g) and

$$Y^1(t) := \left\{ y \in \mathbb{R}^n \;\middle|\; \begin{array}{cc} th_i(y) + (1-t) \geq 0, & i = 1, \ldots, m, \\ -t\sum_{i=1}^m h_i(y) + (1-t) \geq 0, & \\ tg_j(y) + (1-t) \geq 0, & j = 1, \ldots, s. \end{array} \right\}$$

This defines a parametric variational inequality VIP($\Psi, H, G; t$) with functions

$$F^1(\Phi, h, g; y, t) := \begin{pmatrix} \Psi(y,t) \\ G(y,t) \end{pmatrix} \tag{7}$$

where

$$\begin{aligned} \Psi(y,t) &= t\Phi(y) + (1-t)(y-y_0) \\ G(y,t) &= \begin{pmatrix} -t\sum_{i=1}^m h_i(y) + (1-t) \\ th_1(y) + (1-t) \\ \vdots \\ th_m(y) + (1-t) \\ tg_1(y) + (1-t) \\ \vdots \\ tg_s(y) + (1-t)) \end{pmatrix} \end{aligned} \tag{8}$$

This parametric variational inequality has a special structure. The embedding only contains inequality constraints. Moreover, it only depends on the functions $\Phi(y)$, $h_i(y)$, $g_j(y)$ of the original non-parametric problem VIP(Φ, h, g). We, therefore, denote the special parametric problem (6) by $\mathrm{VIP}^1(\Phi, h, g; t)$.

On one hand, the point $y_0 \in Y$ yields a solution of $\mathrm{VIP}^1(\Phi, h, g; t_0)$ for $t_0 = 0$ and obviously $\mathrm{VIP}^1(\Phi, h, g; 1)$ coincides with the original problem VIP(Φ, h, g). On the other hand, the following holds

PROPOSITION 4 *If Φ is a monotone operator $h_1, \ldots, h_m$, a linear function and $-g_1, \ldots, -g_s$ are convex, then $VIP^1(\Phi, h, g; t)$ has a solution for all $t \in [0, 1)$*

Proof In this case the set $Y^1(t)$ is a convex set. As $t\Phi(y) + (1-t)(y - y_0)$ is the sum of a monotone operator and a strongly monotone operator, it is a strongly monotone operator, and hence $\mathrm{VIP}^1(\Phi, h, g; t)$ has a solution. □

We are interested in genericity results for this special embedding. Unfortunately, the genericity results for the general parametric VIP in Section 3 cannot directly be applied because this problem $\mathrm{VIP}^1(t)$ has a special structure. So, the proof of the genericity result in Proposition 2 has to be adapted to this special structure and we have to formulate the genericity results in terms of the original problem VIP(Φ, h, g). Here is the genericity statement.

PROPOSITION 5 *The set of problem functions $(\Phi, h, g) \in C^3(\mathbb{R}^n, \mathbb{R}^{n+m+s})$, such that $VIP^1(\Phi, h, g; t) \in \mathcal{F}_{|_{t\in(0,1)}}$ holds, is generic wrt. the C^3_S-topology in $C^3(\mathbb{R}^n, \mathbb{R}^{n+m+s})$.*

Proof We briefly outline the proof and refer to [27] for details. It will be shown that for any $k > 2$, the set

$$I_k = \left\{(\Phi, h, g) \in C^3(\mathbb{R}^n, \mathbb{R}^{n+m+s}) \mid \mathrm{VIP}^1(\Phi, h, g; t) \in \mathcal{F}_{|_{t\in[\frac{1}{k}, 1-\frac{1}{k}]}}\right\} \tag{9}$$

is open and dense (generic) w.r.t. the C^3_S-topology.

I_k is open: Let $(\overline{\Phi}, \overline{h}, \overline{g}) \in I_k$ and define $\mathcal{T} = [1/k, 1-1/k]$. By Proposition 2, the set $\mathcal{F}_{|_{[a,b]}}$ is open for any $a, b,\ 0 < a < b < 1$. So, there is a strong C^3_S-neighbourhood U of functions $(\Psi, G) \in C^3(\mathbb{R}^n \times \mathcal{T}, \mathbb{R}^{n+1+m+s})$ near $F^1(\Phi, h, g; y, t)$ (cf. (7)) such that $U \subset \mathcal{F}_{|_{[1/k,1-1/k]}}$. Note that a strong neighbourhood is defined by a continuous function $\varepsilon(y, t) > 0$, $y \in \mathbb{R}^n$, $t \in \mathcal{T}$. Clearly, a function $(\Psi, G_1, \ldots, G_{s+m}) \in C^3(\mathbb{R}^n \times \mathcal{T}, \mathbb{R}^{n+1+m+s})$ is in U if and only if for all $(y, t) \in \mathbb{R}^n \times [\frac{1}{k}, 1 - \frac{1}{k}]$

$$\begin{aligned}
\|\Psi(y,t) - \left[t\overline{\Phi}(y) + (1-t)(y-y_0)\right]\| &< \varepsilon(y,t), \\
\left\|G_1(y,t) - \left[-t\sum_{i=1}^{m}\overline{h}_i(y) + (1-t)\right]\right\| &< \varepsilon(y,t), \\
\left\|G_{i+1}(y,t) - \left[t\overline{h}_i(y) + (1-t)\right]\right\| &< \varepsilon(y,t), \quad i = 1, \ldots, m, \\
\|G_{s+1+j}(y,t) - \left[t\overline{g}_j(y) + (1-t)\right]\| &< \varepsilon(y,t), \quad j = 1, \ldots, s,
\end{aligned}$$

and analogous relations hold for the partial derivatives up to order 3. Now, we consider a strong neighbourhood $\hat{U}$ of functions (Φ, h, g) near $(\overline{\Phi}, \overline{h}, \overline{g}) \in C^3(\mathbb{R}^n, \mathbb{R}^{n+m+s})$ defined by

$$\hat{\varepsilon}(y) = \begin{cases} \min\limits_{t\in[\frac{1}{k},1-\frac{1}{k}]} \frac{\varepsilon(y,t)}{m} & \text{if } m \neq 0, \\ \min\limits_{t\in[\frac{1}{k},1-\frac{1}{k}]} \varepsilon(y,t) & \text{if } m = 0. \end{cases}$$

As the minimum is taken over a compact set and $\varepsilon(y,t)$ is a continuous and positive function, also $\hat{\varepsilon}(y)$ is positive and continuous in y. Let (Φ, h, g) be an element in the neighbourhood of $(\overline{\Phi}, \overline{h}, \overline{g})$ defined by $\hat{\varepsilon}(y)$. We claim that the corresponding function $F^1(\Phi, h, g; y, t)$ in (7) is an element of the neighbourhood U. Indeed, for $m \geq 1$ we obtain, for $t \in [\frac{1}{k}, 1-\frac{1}{k}]$:

$$\begin{aligned} \|t\Phi(y) + (1-t)(y-y_0) - \left[t\overline{\Phi}(y) + (1-t)(y-y_0)\right]\| &= t\|\Phi(y) - \overline{\Phi}(y)\| \\ &< t\hat{\varepsilon}(y) \leq \frac{\varepsilon(y,t)}{m} \leq \varepsilon(y,t) \end{aligned}$$

and for $m = 0$ in the same way

$$\|t\Phi(y) + (1-t)(y-y_0) - \left[t\overline{\Phi}(y) + (1-t)(y-y_0)\right]\| < t\hat{\varepsilon}(y) \leq \varepsilon(y,t).$$

Similarly, it is easy to see that $\|th_i(y) + (1-t) - \left[t\overline{h}_i(y) + (1-t)\right]\| < \varepsilon(y,t)$ and $\|tg_j(y) + (1-t) - \left[t\overline{g}_j(y) + (1-t)\right]\| < \varepsilon(y,t)$. The partial derivatives (up to order 3) of (Φ, h, g) satisfy an analogous inequality.

For $m \neq 0$ we also have to consider the bound for H_0: for $t \in [\frac{1}{k}, 1-\frac{1}{k}]$

$$\begin{aligned} \left\| -t\sum_{i=1}^m h_i(y) + (1-t) - \left[-t\sum_{i=1}^m \overline{h}_i(y) + (1-t)\right]\right\| \\ = t\left\|\sum_{i=1}^m h_i(y) - \sum_{i=1}^m \overline{h}_i(y)\right\| \\ \leq t\sum_{i=1}^m \|h_i(y) - \overline{h}_i(y)\| < m\hat{\varepsilon}(y) \leq \varepsilon(y,t). \end{aligned}$$

That means, we have found a strong neighbourhood $\hat{U}$ of $(\overline{\Phi}, \overline{h}, \overline{g})$ given by $\hat{\varepsilon}(y)$ such that $\hat{U} \subset I_k$. Hence, I_k is open.

I_k is dense: To show the density part we first fix the functions (Φ, h, g) and consider the linearly perturbed function $(\Phi + Ay + b, h + C_h y + d_h, g + C_g y + d_g)$, where $(A, b, C_h, d_h, C_g, d_g) \in \mathbb{R}^{n^2+n+mn+m+sn+s}$. We prove the subset of $(A, b, C_h, d_h, C_g, d_g)$ such that

$$(\Phi + Ay + b, h + C_h y + d_h, g + C_g y + d_g) \notin I_k$$

has zero Lebesgue measure (cf. Remark 1). The proof follows the technique appearing in [28].

We begin by considering the g.c. points (y,t) where $LICQ$ fails, i.e. there exists $\mu \neq 0$ such that:

$$\sum_{j\in J_0(y,t)} \mu_j D_y G_j(y,t) = 0 \tag{10}$$

where G is defined in (8). We will show that for almost all (C_h, d_h, C_g, d_g) for the parametric problem $\text{VIP}^1(\Phi, h, g; t)$ corresponding to the perturbed problem functions it holds:

(a) LICQ fails only in a discrete set of feasible points (y, t), $y \in Y^1(t)$ with $t \in [\frac{1}{k}, 1 - \frac{1}{k}]$. We denote this set by Y_0.
(b) For all $(y, t) \in Y_0$ and (y, t, μ) solving (10), it holds that $\mu_j \neq 0$ for all $j \in J_0(y, t)$.
(c) For all $(y, t) \in Y_0$ and (y, t, μ) solving (10) and $j^* \in J_0(y, t)$ the matrix

$$\begin{pmatrix} \sum_{j\in J_0(y,t)} \mu_j D_{(y,t)}[D_y G_j(y,t)]^T & D_{(y,t)} G_{J_0(y,t)}(y,t) \\ [D_y G_{J_0(y,t)\setminus\{j^*\}}]^T(y,t) & 0 \end{pmatrix}$$

is non-singular.

Note that the feasible set $Y^1(t)$ has a special structure because the same functions h_i appear also in the $(m+1)$th-constraint. For $y \in Y^1(t)$, $0 < t < 1$, the first $m+1$ inequality constraints $G_j(y, t) \geq 0$, $j = 1, \ldots, m+1$ cannot be active simultaneously. Indeed if the first m constraints are active,

$$th_i(y) + (1 - t) = 0, \quad i = 1, \ldots, m,$$

then

$$-t \sum_{i=1}^{m} h_i(y) + (1 - t) = (m + 1)(1 - t) > 0\,.$$

For $i = 1, \ldots, m+1$, we consider the sets $Y_i^1(t)$ which are obtained from $Y^1(t)$ by skipping the ith-inequality $G_i \geq 0$. Then, in particular, the following holds: for all $t \in \left[\frac{1}{k}, 1 - \frac{1}{k}\right]$ if $y \in Y^1(t)$ then $y \in Y_i^1(t)$ for some $i \in \{1, \ldots, m\}$, and the ith-inequality is strictly positive at y.

Fixing $i \in \{1, \ldots, m+1\}$, and following the ideas of the proof of Lemma 6.17, p.119, in [29], we obtain that for almost all perturbations (C_h, d_h, C_g, d_g), for the correspondingly perturbed problem the set $Y_0 \cap \{(y, t) \mid y \in Y_i^1(t)\}$ is a discrete set, see (a), and for its elements, conditions (b) and (c) hold. Now if we consider all possible indices $i = 1, \ldots, m+1$ and intersect the resulting sets of perturbations, we find that, for almost all parameters (C_h, d_h, C_g, d_g), conditions (a)–(c) are fulfilled for $Y^1(t)$ leading to the desired result.

Now we fix the parameters (C_h, d_h, C_g, d_g), and thus the feasible set, such that the resulting perturbed problem satisfies conditions (a)–(c). Following the lines of the proof of Theorem 6.18, p.121 in [29], we can prove that for almost all (A, b) for the associated perturbed problem the feasible points where LICQ fails are g.c. points of Type 4 or 5 and the g.c. points where $LICQ$ holds are of Type 1, 2 or 3. The perturbation result is now a consequence of the Fubini theorem applied to the set of perturbations $(C_h, d_h, C_g, d_g) \times (A, b)$.

Based on this result and with the help of partitions of the unity, the density of I_k now follows using standard arguments, see [29] for details. Finally, the set $I := \cap_{k=3}^{\infty} I_k$ gives us the desired generic set. □

We now discuss two special instances of variational inequalities, the VIP with box constraints VIP(Φ, $[0, 1]^n$) and the NLCP. An NLCP is a problem of the form:

$$\text{find } y \in \mathbb{R}^n \text{ such that: } y \geq 0,\ \Phi(y) \geq 0,\ \Phi(y)^T y = 0$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a given function. It is easily seen that NLCP can equivalently be written as VIP$(\Phi, \mathbb{R}^n_+)$:

$$\text{find } y \in \mathbb{R}^n_+ \text{ such that: } \Phi(y)^T(z-y) \geq 0 \quad \forall z \in \mathbb{R}^n_+.$$

For numerical reasons, it is preferable to deal with compact feasible sets. We, therefore, include an additional constraint $\|y\|^2 \leq p$ with some large $p > 0$. For these two special cases, it is convenient to choose an embedding which parameterizes $\Phi(y)$ but leaves the constraints unchanged. So, we choose the embedding VIP(t): for $t \in [0, 1]$ find $y \in Y$ such that

$$(t\Phi(y) + (1-t)(y-y_0))^T(z-y) \geq 0 \quad \forall z \in Y$$

$$\text{with} \quad \begin{array}{ll} Y := \left\{y \in \mathbb{R}^n_+ \mid \|y\|^2 \leq p\right\} & \text{in the case of NLCP} \\ Y := \{y \in [0,1]^n\} & \text{in the case of box constraints} \end{array}$$

where (the starting point) y_0 is some interior point of Y. It easily is verified that the LICQ constraint qualification holds in both feasible sets. A (modified) analysis shows that generically (w.r.t. $\Phi(y)$) the parametric problems VIP(t) are regular for $t \in (0, 1)$. Since LICQ holds, only gc-points of Types 1–3 can occur.

In both special cases, one can also chose the simpler embedding $t\Phi(y) + (1-t)c$ with $c \in \mathbb{R}^n$ (see [27] for details). In this case, the solution y_0 of VIP(t) at $t = 0$ will always be a boundary point of Y; so, we will start with $J_0 \neq \emptyset$. Moreover, if we chose $c > 0$ the initial solution will be $y_0 = 0$ with $J_0 = \{1, \ldots, n\}$ and y_0 is a gc-point of Type 1.

For a more precise discussion of the numerical aspects of the embedding method applied to non-linear programs, we refer to [15,30]. In these papers also illustrative numerical examples are given.

4.2. *Penalty embedding for VIP*

Penalty embeddings for solving optimization problems have been developed in Dentcheva et al. [14,25], Gollmer et al. [17] and in Gómez, [26]. Their main advantage is that, as we shall see, generically gc-points of Type 5 can be excluded. These embeddings can be adapted to our VIP problem as follows. Given a problem VIP(Φ, h, g), we define the parametric problem VIP$^2(\Phi, h, g; t)$ by:

$$\begin{pmatrix} t\Phi(y) + (1-t)y \\ v \\ w - w_0 \end{pmatrix}^T \begin{pmatrix} z_y - y \\ z_v - v \\ z_w - w \end{pmatrix} \geq 0, \quad \forall z \in Y^2(t)$$

$$Y^2(t) := \left\{ (y, v, w) \in \mathbb{R}^{n+m+s} \;\middle|\; \begin{array}{ll} th_i(y) + (1-t)v_i = 0, & i = 1, \ldots, m, \\ tg_j(y) + (1-t)w_j \geq 0, & j = 1, \ldots, s, \\ \|(y, v, w)\|^2 \leq p. & \end{array} \right\}$$

Here, $p \gg 1$ is a chosen number and $w_0 \in \mathbb{R}^s_{++}$, that is w_0 is a vector of $\mathbb{R}^s$ whose components are strictly positive.

Note that this problem VIP(t) depends on the additional variables $(v, w) \in \mathbb{R}^{m+s}$. As expected, $(y, v, w) = (0, 0, w_0)$ is a non-degenerate solution of the starting problem VIP$^2(\Phi, h, g; 0)$.

With respect to the solvability of the parametric problem, the following sufficient condition can be used.

Proposition 6 *Suppose that* Φ *is a monotone operator* $h_1, \dots, h_m$*, a linear function and* $-g_1, \dots, -g_s$ *are convex. Then there exists a solution of* $VIP^2(\Phi, h, g; t)$ *for all* $t \in [0, 1)$

Proof Follows the same ideas of Proposition 4. □

For the penalty embedding the following genericity property holds.

Proposition 7 *The set of functions* $(\Phi, h, g) \in C^3(\mathbb{R}^n, \mathbb{R}^{n+m+s})$*, such that* $VIP^2(\Phi, h, g; t) \in \mathcal{F}|_{t\in(0,1)}$ *holds, is generic wrt.* C^3_S*-topology in* $C^3(\mathbb{R}^n, \mathbb{R}^{n+m+s})$*.*

Moreover, for any generic problem $VIP^2(\Phi, h, g; t) \in \mathcal{F}|_{t\in(0,1)}$ *gc-points of Type 5 are excluded.*

Proof The proof of this statement can be done by following the line of the proof of Proposition 5; see [27] for some more details. In this embedding, the number $n+m+s$ of variables is always greater than or equal to the number $m+s+1$ of constraints. Therefore, generically the gc-points of Type 5 are excluded. □

Applied to NLCP, by adding a compactification constraint, we obtain:

$$(t\Phi(y) + (1-t)y)^T(z_y - y) + (w - w_0)^T(z_w - w) \geq 0, \quad \forall z \in Y^2(t), \tag{11}$$

where $w_0 \in \mathbb{R}^n_{++}$ and

$$Y^2(t) = \left\{(y, w) \in \mathbb{R}^{n+n} \mid ty + (1-t)w \geq 0, \ \|y, w\|^2 \leq p.\right\}$$

In this case, LICQ is satisfied, even when the compactification constraint is active.

Taking $w_0 \in \mathbb{R}^{2n}_{++}$, the box constrained VIP can analogously be embedded as follows:

$$(t\Phi(y) + (1-t)y)^T(z_y - y) + (w - w_0)^T(z_w - w) \geq 0 \quad \forall z \in Y^2(t) \tag{12}$$

where $w = (w_1, w_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and

$$Y^2(t) = \left\{ (y, w) \in \mathbb{R}^{n+n} \;\middle|\; \begin{array}{r} ty + (1-t)w_1 \geq 0, \\ ty + (1-t)w_2 \leq 1, \\ \|(y, w)\|^2 \leq p \end{array} \right\}$$

Remark 2 For both special cases again, genericity results wrt. the function $\Phi(y)$ can be obtained. That is, for a generic set of functions Φ the g.c. points of VIP defined in (11) and (12) are of Type 1, 2, 3 or 4. The proof follows the same steps as the proof of Proposition 5.

However, we want to emphasize that for solving NLCP and box-constrained VIP, the proposed standard embedding, is preferable to the penalty embedding. The reason is that LICQ always holds for the standard embedding because of the simple structure of Y. Note that for the case of box-constrained VIP the condition LICQ may fail even in the penalty approach (see [27] for an example).

Finally, we wish to make the following remark. If the VIP represents the variational formulation of an optimization problem, then the standard embedding of Section 4.1 coincides with the variational reformulation of the parametric optimization problem obtained by the modified standard embedding.

For the penalty embedding the same correspondence holds.

5. Conclusions

In this article, we studied two types of embeddings for solving VIP problems, the standard and the penalty embedding. The generic behaviour of these embeddings have been analysed. In particular, the special cases of box-constrained VIP and of the NLCP problem have been considered.

So, the theoretical basis for implementing the approach has been set. Indeed, for the generic singularities, the local structure of the set of solutions has been established. Moreover, continuation strategies can be applied for solving, locally, the non-linear systems which describe the set of solutions.

The program package PAFO has been developed to numerically perform the path-following strategy for solving parametric optimization problems. For more details, see [13]. PAFO starts with a solution (y_0, t_0). Then, the systems of equations whose solutions fulfil the necessary optimality conditions is constructed and solved by a predictor-corrector scheme. So, locally around (y_0, t_0), a discretization of the path of solutions of the problem is obtained. We want to point out that PAFO constructs the system of equations as a black-box. So, parametric VIP cannot be solved with this package. A practical implementation of a path-following method for solving parametric VIP is a matter of future research. However, we can expect that the difficulties appearing in the solution of one-parametric optimization models will also take place in the VIP framework.

So, as for optimization problems, in the embeddings for VIP problems, it may happen that $t = 1$ is not attained and a good approximation to the solution of the original problem is not computed. However, if the set Y is convex and Φ is monotone, a better behaviour is expected.

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