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Integral Equations and Operator Theory

Stability Analysis in Continuous and Discrete Time, using the Cayley Transform

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Abstract. For semigroups and for bounded operators we introduce the new notion of Bergman distance. Systems with a finite Bergman distance share the same stability properties, and the Bergman distance is preserved under the Cayley transform. This way, we get stability results in continuous and discrete time. As an example, we show that bounded perturbations lead to pairs of semigroups with finite Bergman distance. This is extended to a class of Desch–Schappacher perturbations.

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1. Introduction

Consider the linear differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{1.1}$$

with the state x in the separable Hilbert space X and A the infinitesimal generator of the strongly continuous semigroup $(e^{At})_{t\geq 0}$. A standard way of numerically solving this differential equation is the Crank–Nicolson method [7]. In this method the differential equation (1.1) is replaced by the difference equation

$$x_d(n+1) = \left(I + \frac{\Delta A}{2}\right)\left(I - \frac{\Delta A}{2}\right)^{-1} x_d(n), \quad x_d(0) = x_0, \tag{1.2}$$

where Δ is the time step. Since we look at the stability properties of the semigroup, we can choose Δ freely. For symplicity we take $\Delta = 2$. The operator $(I + A)(I - A)^{-1}$ is known as the Cayley transform of A, and we denote it by A_d.

A natural question is whether the solution $x_d(n) = A_d^n x_0$ of (1.2) is a good approximation of the solution $e^{At}x_0$ of (1.1). We will not consider this question, but concentrate on the stability properties of both equations. If X is finite-dimensional, and thus A a matrix, then it is well-known that both equations share the same stability properties, i.e. A has all his eigenvalues in the open/closed left-half plane if and only if A_d has all its eigenvalues in the open/closed unit circle. This property on the eigenvalues hold for the operators A and A_d as well. However, for infinite dimensional spaces this tells little about the stability of the solutions. The central question in this paper is the following.

If we know that the semigroup is strongly stable, so for all $x_0 \in X$, $e^{At}x_0 \to 0$, as $t \to \infty$, what can be said about the solutions of the difference equation (1.2), and hence about $A_d^n x_0$ for $n \to \infty$?

It is well known that if $(e^{At})_{t\geq 0}$ is a contraction semigroup, that is $\|e^{At}\| \leq 1$, then $\|A_d\| \leq 1$ and thus $\|A_d^n\| \leq 1$, for all $n \geq 0$, for a detailed proof see e.g. [8, Theorem 3.4.9], although the result is much older. If $(e^{At})_{t\geq 0}$ is a bounded analytic semigroup, then $\|A_d^n\| \leq M_2$, for all $n \geq 0$, see [5]. Thus, in these cases the solutions of (1.2) are bounded. If additionally, the semigroup is strongly stable, then $(A_d^n)_{n\geq 0}$ is strongly stable as well, see [5].

We extend the class of semigroups which behave nicely with respect to the Cayley transform, by introducing the new notion of *Bergman distance*. We say that two semigroups, $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$, have a *finite Bergman distance* if the following two inequalities are satisfied for all $x_0 \in X$:

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t}\, dt < \infty, \tag{1.3}$$

$$\int_0^\infty \|(e^{A^*t} - e^{\tilde{A}^*t})x_0\|^2 \frac{1}{t}\, dt < \infty. \tag{1.4}$$

Note that the measure $t^{-1}\, dt$ is the invariant measure for the multiplication group $\mathbb{R}^+$. The space $L^2_1(\mathbb{R}^+)$ with this measure is isometrically isomorphic to the unweighted Bergman space $\mathcal{A}^2(\Pi^+)$, see [3, Theorem 1]. Thus two semigroups have finite Bergman distance, if $(e^{At} - e^{\tilde{A}t})x_0$ and $(e^{A^*t} - e^{\tilde{A}^*t})x_0$ are in the Bergman space for all $x_0 \in X$.

In Sect. 6, we investigate which pair of generators have finite Bergman distance. Among others, we show that if A and $\tilde{A}$ generate exponentially stable semigroups, and if $A - \tilde{A}$ is bounded, then they have a finite Bergman distance.

For the sequences of bounded operators, $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$, we say that they have a *finite Bergman distance* if the following two inequalities are satisfied for all $x_0 \in X$:

$$\sum_{k=1}^\infty \frac{1}{k} \left\| A_d^k x_0 - \tilde{A}_d^k x_0 \right\|^2 < \infty,$$

$$\sum_{k=1}^\infty \frac{1}{k} \left\| A_d^{*k} x_0 - \tilde{A}_d^{*k} x_0 \right\|^2 < \infty.$$

One of our main results is, that the Cayley transform conserves the Bergman distance. That is, the following equality holds for all $x_0 \in X$:

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t}\, dt = \sum_{k=1}^\infty \frac{1}{k} \left\| \left(A_d^k - \tilde{A}_d^k \right) x_0 \right\|^2 . \tag{1.5}$$

We prove this equality in Sect. 4.

Furthermore, operators with finite Bergman distance have similar stability properties. In Sect. 2, we show this for the continuous-time case and in Sect. 3 we examine the discrete-time case.

If $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have a finite Bergman distance and $(A_d^n)_{n\geq 0}$ is *strongly stable*, i.e. $A_d^n x \to 0$ as $n \to \infty$, then also $(\tilde{A}_d^n)_{n\geq 0}$ is strongly stable. Combining this with Eq. (1.5), leads to the following theorem:

Theorem 1.1. *Let $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance. Then $(A_d^n)_{n\geq 0}$ is strongly stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is strongly stable.*

Furthermore, the other implication also holds. Thus, if $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have a finite Bergman distance, then $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have similar stability properties. We prove this in Sect. 5.

2. Stability in Continuous Time

The finite Bergman distance devides semigroups into classes. In this section we show that within these classes of semigroups the stability properties are the same.

First, we define what we mean by stability of semigroups.

Definition 2.1. The C_0-semigroup $(e^{At})_{t\geq 0}$ is *bounded* if there exists a constant $M \geq 1$ such that

$$\|e^{At}\| \leq M, \quad \text{for all } t \geq 0.$$

The C_0-semigroup $(e^{At})_{t\geq 0}$ is *exponentially stable* if there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\|e^{At}\| \leq Me^{-\omega t}, \quad \text{for all } t \geq 0.$$

The C_0-semigroup $(e^{At})_{t\geq 0}$ is *strongly stable* if for all $x_0 \in X$,

$$e^{At}x_0 \to 0, \quad \text{as } t \to \infty.$$

Van Casteren, [1], gave the following characterisation of bounded and strongly stable semigroups.

Lemma 2.2. *The semigroup $(e^{At})_{t\geq 0}$ is bounded if and only if there exists a M such that for all $t \geq 0$, and all $x_0 \in X$,*

$$\frac{1}{t}\int_0^t \|e^{As}x_0\|^2\, ds \leq M\|x_0\|^2 \quad \text{and} \quad \frac{1}{t}\int_0^t \|e^{A^*s}x_0\|^2\, ds \leq M\|x_0\|^2 \tag{2.1}$$

with M independent of t and x_0.

Furthermore, if $(e^{At})_{t\geq 0}$ *is bounded and for all* x_0

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|e^{As}x_0\|^2\, ds = 0, \tag{2.2}$$

then $(e^{At})_{t\geq 0}$ *is strongly stable.*

With Lemma 2.2, we can show that two semigroups with a finite Bergman distance, have the same stability properties.

Theorem 2.3. *Let* $(e^{At})_{t\geq 0}$ *and* $(e^{\tilde{A}t})_{t\geq 0}$ *have a finite Bergman distance. Then*

1. $(e^{At})_{t\geq 0}$ *is bounded if and only if* $(e^{\tilde{A}t})_{t\geq 0}$ *is bounded,*
2. $(e^{At})_{t\geq 0}$ *is exponentially stable if and only if* $(e^{\tilde{A}t})_{t\geq 0}$ *is exponentially stable,*
3. $(e^{At})_{t\geq 0}$ *is strongly stable if and only if* $(e^{\tilde{A}t})_{t\geq 0}$ *is strongly stable.*

Proof. We prove the boundedness or stability of $(e^{At})_{t\geq 0}$, given the boundedness or stability of $(e^{\tilde{A}t})_{t\geq 0}$. By symmetry, the other implication then also holds. We begin with item 1.

1. For all $t > 0$ and $x_0 \in X$, the following inequalities hold:

$$\begin{aligned}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2\, ds &\leq \frac{1}{t}\int_0^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\, ds + \frac{1}{t}\int_0^t 2\|e^{\tilde{A}s}x_0\|^2\, ds\\ &\leq 2\int_0^t \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\, ds + 2\sup_t \|e^{\tilde{A}t}\|^2\|x_0\|^2\\ &\leq M_1\|x_0\|^2,\end{aligned}$$

where we have used (1.3) and the boundedness of $(e^{\tilde{A}t})_{t\geq 0}$. Similarly, we obtain the dual result. Hence by Lemma 2.2, we conclude that $(e^{At})_{t\geq 0}$ is bounded.

2. For $t > 1$, we have for all $x_0 \in X$

$$\begin{aligned}\int_1^t \frac{1}{s}\|e^{As}x_0\|^2\, ds &\leq 2\int_1^t \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\, ds + 2\int_1^t \frac{1}{s}\|e^{\tilde{A}s}x_0\|^2\, ds\\ &\leq M_2\|x_0\|^2,\end{aligned} \tag{2.3}$$

where we have used the finite Bergman distance and the exponential stability of $(e^{\tilde{A}t})_{t\geq 0}$.

The exponential stability of $(e^{\tilde{A}t})_{t\geq 0}$ trivially implies that $(e^{\tilde{A}t})_{t\geq 0}$ is bounded. By item 1., we have that $(e^{At})_{t\geq 0}$ is bounded as well. Combining this with (2.3), we find

$$\begin{aligned}\ln(t)\|e^{At}x_0\|^2 &= \int_1^t \frac{1}{s}\|e^{At}x_0\|^2\,ds \\ &\leq \int_1^t \frac{1}{s}\|e^{A(t-s)}\|^2\|e^{As}x_0\|^2\,ds \\ &\leq M_1 \int_1^t \frac{1}{s}\|e^{As}x_0\|^2\,ds \leq M_1 M_2\|x_0\|^2.\end{aligned}$$

So for $t > 1$ we have that

$$\|e^{At}\|^2 \leq \frac{M_1 M_2}{\ln(t)}.$$

Since for large t this will be less one, we have that $(e^{At})_{t\geq 0}$ is exponentially stable.

3. Since $\int_0^\infty \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\,ds < \infty$, for every $\varepsilon > 0$, there exists a t_ε such that $\int_{t_\varepsilon}^\infty \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\,ds < \varepsilon$. For $x_0 \in X$, there holds

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2\,ds &\leq \lim_{t\to\infty}\frac{1}{t}\int_0^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\,ds \\ &\quad + \lim_{t\to\infty}\frac{1}{t}\int_0^t 2\|e^{\tilde{A}s}x_0\|^2\,ds.\end{aligned}$$

Using (2.2), we have that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2\,ds &\leq \lim_{t\to\infty}\frac{1}{t}\int_0^{t_\varepsilon} 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\,ds \\ &\quad + \lim_{t\to\infty}\frac{1}{t}\int_{t_\varepsilon}^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 + 0 \\ &\leq \lim_{t\to\infty}\frac{t_\varepsilon}{t}\int_0^{t_\varepsilon} \frac{2}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2\,ds \\ &\quad + \lim_{t\to\infty}\int_{t_\varepsilon}^t \frac{2}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 \leq 0 + 2\varepsilon.\end{aligned}$$

Since this holds for all $\varepsilon > 0$, we have shown that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|e^{As}x_0\|^2\, ds = 0,$$

and so $(e^{At})_{t\geq 0}$ is strongly stable by Lemma 2.2. □

3. Stability in Discrete Time

The discrete-time case is similar to the continuous-time case, the finite Bergman distance also creates classes of sequences of bounded operators. Elements within a class share the same stability properties.

First, we define what we mean by stability in discrete time.

Definition 3.1. The operator sequence $(A_d^n)_{n\geq 0}$ is *bounded* if there exists a constant $M \geq 1$ such that

$$\|A_d^n\| \leq M, \quad \text{for all } n \geq 0.$$

The operator sequence $(A_d^n)_{n\geq 0}$ is *power stable* if there exist constants $M \geq 1$ and $\gamma \in (0,1)$ such that

$$\|A_d^n\| \leq M\gamma^n, \quad \text{for all } n \geq 0.$$

The operator sequence $(A_d^n)_{n\geq 0}$ is *strongly stable* if for all $x_0 \in X$,

$$A_d^n x_0 \to 0, \quad \text{as } n \to \infty.$$

Now, we recall a result by Van Casteren [1], and next we show the stability properties are preserved by the finite Bergman distance.

Lemma 3.2. *The operator sequence $(A_d^n)_{n\geq 0}$ is power stable, if and only if there exists a M such that*

$$\frac{1}{N}\sum_{k=1}^{N} \left\|A_d^k x_0\right\|^2 \leq M\|x_0\|^2 \quad \text{and} \quad \frac{1}{N}\sum_{k=1}^{N} \left\|A_d^{*k} x_0\right\|^2 \leq M\|x_0\|^2. \tag{3.1}$$

with M independent of N and x_0. Furthermore, if $(A_d^n)_{n\geq 0}$ is power stable, then $(A_d^n)_{n\geq 0}$ is strongly stable if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \|A_d x_0\|^2 = 0, \tag{3.2}$$

for all $x_0 \in X$.

Theorem 3.3. *Let $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have finite Bergman distance. Then the following assertions hold*:

1. *$(A_d^n)_{n\geq 0}$ is bounded if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is bounded,*
2. *$(A_d^n)_{n\geq 0}$ is power stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is power stable,*
3. *$(A_d^n)_{n\geq 0}$ is strongly stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is strongly stable.*

Proof. We prove the boundedness or stability of $(A_d^n)_{n\geq 0}$, given the boundedness or stability of $(\tilde{A}_d^n)_{n\geq 0}$. By symmetry, the other implication then also holds. The proofs are similar to the ones in the continuous time.

1. Using Eq. (3.1) and the power stability of $(\tilde{A}_d^n)_{n\geq 0}$, we find for all $x_0 \in X$

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N}\left\|A_d^k x_0\right\|^2 &\leq \frac{1}{N}\sum_{k=1}^{N} 2\left\|A_d^k x_0 - \tilde{A}_d^k x_0\right\|^2 + \frac{1}{N}\sum_{k=1}^{N} 2\left\|\tilde{A}_d^k x_0\right\|^2 \\ &\leq 2\sum_{k=1}^{N}\frac{1}{k}\left\|A_d^k x_0 - \tilde{A}_d^k x_0\right\|^2 + 2\sup_k\left\|\tilde{A}_d^k x_0\right\|^2 \\ &\leq M_1\|x_0\|^2.\end{aligned}$$

Similarly, we obtain the dual result. By Lemma 3.2, $(A_d^n)_{n\geq 0}$ is power stable.

2. We have

$$\begin{aligned}\sum_{k=1}^{N}\frac{1}{k}\left\|A_d^k x_0\right\|^2 &\leq 2\sum_{k=1}^{N}\frac{1}{k}\left\|A_d^k x_0 - \tilde{A}_d^k x_0\right\|^2 + 2\sum_{k=1}^{N}\frac{1}{k}\left\|\tilde{A}_d^k x_0\right\|^2 \\ &\leq M_2\|x_0\|^2, \end{aligned} \tag{3.3}$$

where we have used the finite Bergman distance and the power stability of $(\tilde{A}_d^n)_{n\geq 0}$.

The power stability of $(\tilde{A}_d^n)_{n\geq 0}$ implies that $(\tilde{A}_d^n)_{n\geq 0}$ is bounded, so by item 1. $(A_d^n)_{n\geq 0}$ is bounded as well. Combining this with equation (3.3):

$$\begin{aligned}\ln(n+1)\|A_d^n x_0\|^2 &\leq \sum_{k=1}^{n}\frac{1}{k}\left\|A_d^n x_0\right\|^2 \\ &\leq \sum_{k=1}^{n}\frac{1}{k}\|A_d^{n-k}\|^2\left\|A_d^k x_0\right\|^2 \\ &\leq M_1\sum_{k=1}^{n}\frac{1}{k}\left\|A_d^k x_0\right\|^2 \leq M_1 M_2\|x_0\|^2.\end{aligned}$$

So we have that

$$\|A_d^n\|^2 \leq \frac{M_1 M_2}{\ln(n+1)}.$$

Since for large n this will be less than one, we have that $(A_d^n)_{n\geq 0}$ is power stable.

3. Since $\sum_{k=1}^{\infty}\frac{1}{k}\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 < \infty$, for every $\varepsilon > 0$, there exists a n_ε such that $\sum_{k=n_\varepsilon}^{\infty}\frac{1}{k}\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 < \varepsilon$. Using (3.2) we find

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\|A_d^k x_0\right\|^2 &\leq \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} 2\left\|A_d^k x_0 - \tilde{A}_d^k x_0\right\|^2 \\ &\quad + \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} 2\left\|\tilde{A}_d^k x_0\right\|^2\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n_\varepsilon - 1} 2 \left\| A_d^k x_0 - \tilde{A}_d^k x_0 \right\|^2$$

$$+ \lim_{n\to\infty} \frac{1}{n} \sum_{k=n_\varepsilon}^{n} 2 \left\| A_d^k x_0 - \tilde{A}_d^k x_0 \right\|^2 + 0$$

$$\leq 0 + \lim_{n\to\infty} \sum_{k=n_\varepsilon}^{n} \frac{2}{k} \left\| A_d^k x_0 - \tilde{A}_d^k x_0 \right\|^2 \leq 2\varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have shown that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left\| A_d^k x_0 \right\|^2 = 0,$$

and so $(A_d^n)_{n\geq 0}$ is strongly stable. □

4. Equivalence of the Bergman Distances

In the previous sections we have derived properties of operators with finite Bergman distance. In this section, we show that the Cayley transform preserves Bergman distances. First, we define the inner product space H.

Definition 4.1. Let H denote the space of Lebesgue measurable functions f from $[0, \infty)$ to the Hilbert space X such that:

$$\int_0^\infty \|f(t)\|_X^2 t \, dt < \infty.$$

On H we define the following inner product:

$$\langle f, g\rangle_H = \int_0^\infty \langle f(t), g(t)\rangle_X t \, dt. \tag{4.1}$$

The following result is easy to see.

Lemma 4.2. *The inner product space H defined in Definition 4.1 is a Hilbert space.*

To create an orthonormal basis for this Hilbert space, we use the generalised Laguerre polynomials $L_n^{(1)}(t)$ [9, p. 99]. These are defined by

$$L_{n-1}^{(1)}(2t) = \sum_{k=0}^{n-1} \binom{n}{n-k-1} \frac{(-2t)^k}{k!}, \qquad \text{for } n \geq 1 \text{ and } t \in [0, \infty). \tag{4.2}$$

Lemma 4.3. *Let H be the Hilbert space defined by Definition 4.1 and let $\{e_m\}_{m\in\mathbb{N}}$ be an orthonormal basis of X. The vectors $\varphi_{n,m}$ defined by:*

$$\varphi_{n,m}(t) = \frac{q_n(t)}{\sqrt{n}} e_m, \quad n, m \geq 1, \tag{4.3}$$

with

$$q_n(t) = -2e^{-t} L^{(1)}_{n-1}(2t), \tag{4.4}$$

form an orthonormal basis in H.

Proof. We begin by showing that the sequence $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is orthonormal in H. Using Eq. (4.1), we find:

$$\begin{aligned}
\langle \varphi_{n,m}, \varphi_{\nu,\mu}\rangle_H &= \int_0^\infty \left\langle \frac{-2e^{-t}L^{(1)}_{n-1}(2t)}{\sqrt{n}} e_m, \frac{-2e^{-t}L^{(1)}_{\nu-1}(2t)}{\sqrt{\nu}} e_\mu \right\rangle_X t\,dt \\
&= \frac{4}{\sqrt{n}\sqrt{\nu}} \int_0^\infty e^{-2t} t L^{(1)}_{n-1}(2t) L^{(1)}_{\nu-1}(2t)\,dt\, \langle e_m, e_\mu\rangle_X \\
&= \frac{1}{\sqrt{n}\sqrt{\nu}} \int_0^\infty e^{-\tau} \tau L^{(1)}_{n-1}(\tau) L^{(1)}_{\nu-1}(\tau)\,d\tau\, \langle e_m, e_\mu\rangle_X \\
&= \frac{1}{\sqrt{n}\sqrt{\nu}} \Gamma(2) \binom{n}{n-1} \delta_{(n-1)(\nu-1)} \delta_{m\mu} = \delta_{n\nu}\delta_{m\mu},
\end{aligned}$$

where we use the orthogonality of the Laguerre polynomials, see [9, p. 99].

Next we show that the sequence $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is maximal in H. If h is orthogonal to every $\varphi_{n,m}$, then for all n and $m \geq 1$:

$$\langle \varphi_{n,m}, h\rangle_H = \int_0^\infty L^{(1)}_{n-1}(2t) \frac{-2e^{-t}}{\sqrt{n}} \langle e_m, h(t)\rangle_X t\,dt = 0.$$

From the maximality of $\{L^{(1)}_{n-1}(2t)e^{-t}t\}_{n\geq 1}$ in $L^2(0,\infty)$, see [9, p. 107], we conclude that for all $m \geq 1$,

$$\langle e_m, h(t)\rangle_X = 0 \quad \text{almost everywhere.}$$

This, combined with the maximality of $\{e_m\}_{m\in\mathbb{N}}$ in X, leads to the conclusion that the Lebesgue measurable function $h(t) = 0$ almost everywhere. So $h = 0$ in H and $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is maximal. □

Lemma 4.3 gives us the following Parseval equality:

$$\|f\|_H^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle f, \varphi_{n,m}\rangle_H|^2. \tag{4.5}$$

We use the Laguerre polynomials to write the Cayley transform as an integral.

Lemma 4.4. *Let q_n be defined by Eq. (4.4), let A generate a C_0-semigroup and let A_d be the Cayley transform of A. Then,*

$$\int_0^\infty q_n(t) e^{At} x_0\,dt = (-1)^n A_d^n x_0 - x_0, \quad x_0 \in X. \tag{4.6}$$

Proof. We rewrite $q_n(t)$ as follows:

$$\begin{aligned} q_n(t) &= -2e^{-t} L_{n-1}^{(1)}(2t) \\ &= -2e^{-t} \sum_{k=0}^{n-1} \binom{n}{n-k-1} \frac{(-2t)^k}{k!} \\ &= 2 \sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!(k+1)!k!} (-1)^{k+1} (2t)^k e^{-t} \\ &= 2 \sum_{\ell=1}^{n} \frac{n!}{(n-\ell)!\ell!(\ell-1)!} (-1)^{\ell} (2t)^{\ell-1} e^{-t} \\ &= \sum_{\ell=1}^{n} \binom{n}{\ell} (-2)^{\ell} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t}, \end{aligned}$$

where we introduce $\ell = k+1$ in the fourth equality sign.

We insert this into the left-hand side of Eq. (4.6) and using,

$$(A-I)^{-\ell} x_0 = (-1)^{\ell} R(1,A)^{\ell} x_0 = \int_0^{\infty} (-1)^{\ell} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t} e^{At} x_0 \, dt,$$

see [4, p. 57], gives:

$$\begin{aligned} \int_0^{\infty} q_n(t) e^{At} x_0 \, dt &= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_0^{\infty} (-2)^{\ell} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t} e^{At} x_0 \, dt \\ &= \sum_{\ell=1}^{n} \binom{n}{\ell} 2^{\ell} (A-I)^{-\ell} x_0 \\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} 2^{\ell} (A-I)^{-\ell} x_0 - x_0 \\ &= \left(I + 2(A-I)^{-1}\right)^n x_0 - x_0 \\ &= (-1)^n A_d^n x_0 - x_0. \end{aligned}$$

Thus Eq. (4.6) holds. □

The following theorem shows that the Cayley transform preserves the Bergman distances.

Theorem 4.5. *Let A and $\tilde{A}$ generate a C_0-semigroup and let A_d and $\tilde{A}_d$ be their Cayley transforms, then $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have finite Bergman distance if and only if $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have finite Bergman distance.*

Furthermore, for all $x_0 \in X$

$$\int_0^{\infty} \|(e^{At} - e^{\tilde{A}t}) x_0\|_X^2 \frac{1}{t} \, dt = \sum_{n=1}^{\infty} \frac{1}{n} \left\| (A_d^n - \tilde{A}_d^n) x_0 \right\|_X^2. \quad (4.7)$$

Proof. First, we write the left-hand side of (4.7) as a norm in H, see Definition 4.1. Next, we apply the Parseval identity of H, see Eq. (4.5):

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|_X^2 \frac{1}{t}\, dt = \int_0^\infty \left\| \frac{(e^{At} - e^{\tilde{A}t})x_0}{t} \right\|_X^2 t\, dt$$

$$= \left\| \frac{(e^{At} - e^{\tilde{A}t})x_0}{t} \right\|_H^2$$

$$= \sum_{n=1}^\infty \sum_{m=1}^\infty \left| \left\langle \frac{(e^{At} - e^{\tilde{A}t})x_0}{t}, \varphi_{n,m} \right\rangle_H \right|^2 .$$

Zooming in on the inner product, and applying Eq. (4.3) and Lemma 4.4, we find

$$\left\langle \frac{(e^{At} - e^{\tilde{A}t})x_0}{t}, \varphi_{n,m} \right\rangle_H = \int_0^\infty \left\langle \frac{(e^{At} - e^{\tilde{A}t})x_0}{t}, \frac{q_n(t)}{\sqrt{n}} e_m \right\rangle_X t\, dt$$

$$= \frac{1}{\sqrt{n}} \int_0^\infty \left\langle q_n(t)(e^{At} - e^{\tilde{A}t})x_0, e_m \right\rangle_X dt$$

$$= \frac{1}{\sqrt{n}} \left\langle \int_0^\infty q_n(t)(e^{At} - e^{\tilde{A}t})x_0\, dt, e_m \right\rangle_X$$

$$= \frac{1}{\sqrt{n}} \left\langle \left((-1)^n A_d^n - (-1)^n \tilde{A}_d^n\right) x_0, e_m \right\rangle_X .$$

$$= \frac{(-1)^n}{\sqrt{n}} \left\langle \left(A_d^n - \tilde{A}_d^n\right) x_0, e_m \right\rangle_X .$$

We zoom out again and use the Parseval equation of X for the orthonormal basis $\{e_m\}_{m\in\mathbb{N}}$.

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|_X^2 \frac{1}{t}\, dt = \sum_{n=1}^\infty \sum_{m=1}^\infty \left| \frac{(-1)^n}{\sqrt{n}} \left\langle \left(A_d^n - \tilde{A}_d^n\right) x_0, e_m \right\rangle_X \right|^2$$

$$= \sum_{n=1}^\infty \frac{1}{n} \sum_{m=1}^\infty \left| \left\langle \left(A_d^n - \tilde{A}_d^n\right) x_0, e_m \right\rangle_X \right|^2$$

$$= \sum_{n=1}^\infty \frac{1}{n} \left\| \left(A_d^n - \tilde{A}_d^n\right) x_0 \right\|_X^2 .$$

Thus Eq. (4.7) holds. □

5. Proof of the Main Result

In this section, we return to Theorem 1.1. With the results from Sects. 2, 3, and 4, we are able to prove it. First, we reformulate Theorem 1.1 as follows:

Theorem 5.1. *Let* $(e^{At})_{t\geq 0}$ *and* $(e^{\tilde{A}t})_{t\geq 0}$ *be* C_0*-semigroups and let* A_d *and* $\tilde{A}_d$ *denote the Cayley transforms of* A *and* $\tilde{A}$*. Then*

1. *if* $(e^{At})_{t\geq 0}$ *and* $(e^{\tilde{A}t})_{t\geq 0}$ *have a finite Bergman distance*:
   $(A_d^n)_{n\geq 0}$ *is strongly stable if and only if* $(\tilde{A}_d^n)_{n\geq 0}$ *is strongly stable.*
2. *if* $(A_d^n)_{n\geq 0}$ *and* $(\tilde{A}_d^n)_{n\geq 0}$ *have a finite Bergman distance*:
   $(e^{At})_{t\geq 0}$ *is strongly stable if and only if* $(e^{\tilde{A}t})_{t\geq 0}$ *is strongly stable.*

Proof. We begin by recalling that from Theorem 4.5, we know that $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance if and only if $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have a finite Bergman distance.

So to prove item 1. the argument goes as follows. The finite Bergman distance of $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ implies the finite Bergman distance between $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$. Using the third item of Theorem 3.3, we conclude that $(A_d^n)_{n\geq 0}$ is strongly stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is strongly stable.

The second item is proved similarly. □

Now, we return to the central question in this paper: If we know that the semigroup $(e^{At})_{t\geq 0}$ is strongly stable, what can be said about $(A_d^n)_{n\geq 0}$? Or what can be said about sequences $(\tilde{A}_d^n)_{n\geq 0}$ at a finite Bergman distance of $(A_d^n)_{n\geq 0}$.

Before answering this question, we first recall the following result by Guo and Zwart [5, Theorem 4.3].

Lemma 5.2. *Let* $(e^{At})_{t\geq 0}$ *be a* C_0*-semigroup and let* A_d *denote the Cayley transform of* A*. If* $(e^{At})_{t\geq 0}$ *and* $(A_d^n)_{n\geq 0}$ *are bounded, and* $(e^{At})_{t\geq 0}$ *is strongly stable, then* $(A_d^n)_{n\geq 0}$ *is strongly stable.*

Hence, if we combine this lemma with Theorem 5.1, we find that if $(e^{At})_{t\geq 0}$ and $(A_d^n)_{n\geq 0}$ are bounded, then the strong stability of $(e^{At})_{t\geq 0}$ implies the strong stability of $(e^{\tilde{A}t})_{t\geq 0}$, $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$, provided the two semigroups or the two discrete operators have finite Bergman distance.

6. Applications

In this section we present some examples of semigroups with a bounded Bergman distance.

Lemma 6.1. *Let* A *and* $\tilde{A}$ *generate exponentially stable semigroups and let* $A-\tilde{A}$ *be bounded, then* $(e^{At})_{t\geq 0}$ *and* $(e^{\tilde{A}t})_{t\geq 0}$ *have a finite Bergman distance.*

Proof. Let M_1, M_2, ω_1 and ω_2 be positive constants s.t. $\|e^{At}\| \leq M_1 e^{-\omega_1 t}$ and $\|e^{\tilde{A}t}\| \leq M_2 e^{-\omega_2 t}$, respectively. We show that these semigroups satisfy Eq. (1.3) by cutting the time interval $[0, \infty)$ into two parts, and showing, for each part, that the integral is finite.

The first time interval is from 0 to 1. We use the variation of constant formula $e^{At}x_0 = e^{\tilde{A}t}x_0 + \int_0^t e^{As}(A-\tilde{A})e^{\tilde{A}(t-s)}x_0\,ds$.

$$\begin{aligned}\int_0^1 \left\|\left(e^{At}-e^{\tilde{A}t}\right)x_0\right\|^2 \frac{1}{t}\,dt &= \int_0^1 \left\|\int_0^t e^{As}(A-\tilde{A})e^{\tilde{A}(t-s)}x_0\,ds\right\|^2 \frac{1}{t}\,dt \\ &\le \int_0^1 \left(\int_0^t \|e^{As}(A-\tilde{A})e^{\tilde{A}(t-s)}x_0\|\,ds\right)^2 \frac{1}{t}\,dt \\ &\le \int_0^1 \left(\int_0^t M_1\|A-\tilde{A}\|M_2\|x_0\|\,ds\right)^2 \frac{1}{t}\,dt \\ &\le \int_0^1 \left(tM_1M_2\|A-\tilde{A}\|\|x_0\|\right)^2 \frac{1}{t}\,dt \\ &\le M_1^2M_2^2\|A-\tilde{A}\|^2\|x_0\|^2 \int_0^1 t\,dt < \infty.\end{aligned}$$

This holds for all $x_0 \in X$.

The second time interval is from 1 to ∞.

$$\begin{aligned}\int_1^\infty \|\left(e^{At}-e^{\tilde{A}t}\right)x_0\|^2 \frac{1}{t}\,dt &\le \int_1^\infty \|\left(e^{At}-e^{\tilde{A}t}\right)x_0\|^2\,dt \\ &\le \int_1^\infty 2\|e^{At}x_0\|^2 + 2\|e^{\tilde{A}t}x_0\|^2\,dt \\ &\le \frac{M_1^2}{\omega_1}e^{-2\omega_1}\|x_0\|^2 + \frac{M_2^2}{\omega_2}e^{-2\omega_2}\|x_0\|^2 < \infty.\end{aligned}$$

This holds for all $x_0 \in X$. Hence Eq. (1.3) holds.

The proof for the adjoint operators goes the same, and hence, we conclude the proof. □

Next, we apply the previous lemma to the linear quadratic optimal control problem.

Lemma 6.2. *Let A generate an exponentially stable contraction semigroup, and let B be bounded. By Π we denote the stabilizing solution of the algebraic Riccati equation, corresponding to the optimal control problem*

$$\min_u \int_0^\infty \|x(t)\|^2 + \|u(t)\|^2\,dt,$$

see [2, Chapter 6]*. Then the Cayley transform of $A - BB^*\Pi$ is strongly stable.*

Proof. By Lemma 6.1, the semigroups $(e^{At})_{t\ge0}$ and $(e^{(A-BB^*\Pi)t})_{t\ge0}$ have a finite Bergman distance. Since $(e^{At})_{t\ge0}$ is a contraction semigroup, each operator A_d^n has norm less than or equal to one. It is strongly stable a well, since $(e^{At})_{t\geq 0}$ is exponentially stable. Theorem 1.1 proves the assertion. □

In the next example, we show that a subset of the class of Desch–Schappacher perturbations leads to pairs of semigroups with finite Bergman distance. First, we introduce the class of Desch–Schappacher perturbations, see Engel and Nagel [4, Section III.3.a]. We start by defining $\mathcal{X}_{t_0}$ as the space of all strongly continuous, $\mathcal{L}(X)$-valued functions,

$$\mathcal{X}_{t_0} = C\left([0,t_0], \mathcal{L}_s(X)\right), \quad \text{with the norm } \|F\|_\infty = \sup_{r\in[0,t_0]} \|F(r)\|_{\mathcal{L}(X)}.$$

Note that $\mathcal{X}_{t_0}$ is a Banach space. For the C_0-semigroup $(e^{At})_{t\geq 0}$ and the operator $B \in \mathcal{L}(X, X_{-1})$ from X to the extrapolation space $X_{-1} = D(A^*)$ we define the abstract Volterra operator V_B on the space $\mathcal{X}_{t_0}$ by

$$(V_B F)(t) = \int_0^t e^{A_{-1}(t-r)} BF(r)\, dr, \quad \text{for all } t \in [0,t_0] \text{ and } F \in \mathcal{X}_{t_0}.$$

Note that we use the extended semigroup on X_{-1} in this definition. The class of Desch–Schappacher perturbations is defined by

$$\mathcal{S}_{t_0}^{DS} = \{B \in \mathcal{L}(X, X_{-1}) \mid V_B \in \mathcal{L}(\mathcal{X}_{t_0}),\ \|V_B\| < 1\}. \tag{6.1}$$

If we restrict the class of Desch–Schappacher perturbations by two extra conditions, then a perturbation B in this restricted class leads to a finite Bergman distance. The perturbation is denoted by $(A_{-1}+B)_X$ which is defined as follows: $D((A_{-1}+B)_X) = \{x \in X | A_{-1}x + Bx \in X\}$ and for $x \in D((A_{-1}+B)_X)$ $(A_{-1}+B)_X x = A_{-1}x + Bx$.

Lemma 6.3. *Let A be the infinitesimal generator of an exponentially stable semigroup and let $B \in \mathcal{S}_{t_0}^{DS}$. If, for some $M > 1$ and $\alpha > 0$*

$$\|(V_B)\|_{\mathcal{L}(\mathcal{X}_t)} \leq M t^\alpha, \quad \textit{for } t \in (0, t_0),$$

and, for some $q \in (0,1)$

$$\|R(\lambda, A_{-1})B\| \leq q, \quad \textit{for all } \lambda \in \mathbb{C}^+, \tag{6.2}$$

then the semigroups generated by A and $(A_{-1}+B)_X$ have a finite Bergman distance.

Proof. First, we define $\tilde{A} = (A_{-1}+B)_X$. It follows from Eq. (6.2), that the semigroup generated by $\tilde{A}$ is exponentially stable, see [6, Proposition 5.8].

Now, the proof is similar to the proof of Lemma 6.2. Let M_1, M_2, ω_1 and ω_2 be positive constants such that $\|e^{At}\| \leq M_1 e^{-\omega_1 t}$ and $\|e^{\tilde{A}t}\| \leq M_2 e^{-\omega_2 t}$, respectively. We show that these semigroups satisfy Eq. (1.3) by cutting the time interval $[0,\infty)$ into two parts, and showing, for each part, that the integral is finite.

The first time interval is from 0 to t_0. We use the variation of constant formula.

$$\begin{aligned}\int_0^{t_0} \left\| \left(e^{At} - e^{\tilde{A}t}\right) x_0 \right\|^2 \frac{1}{t}\, dt &= \int_0^{t_0} \left\| \int_0^t e^{A_{-1}(t-s)}(A-\tilde{A})e^{\tilde{A}s}\, ds x_0 \right\|^2 \frac{1}{t}\, dt \\ &= \int_0^{t_0} \|(V_B e^{\tilde{A}\cdot})(t)x_0\|^2 \frac{1}{t}\, dt \\ &\leq \int_0^{t_0} M^2 t^{2\alpha-1} M_2^2 \|x_0\|^2\, dt \\ &= \frac{M^2 M_2^2}{2\alpha} t_0^{2\alpha} \|x_0\|^2 < \infty. \end{aligned} \tag{6.3}$$

This holds for all $x_0 \in X$.

For the adjoint operators we make the following observation:

$$\begin{aligned}\left\| \left(e^{\tilde{A}^* t} - e^{A^* t}\right) x_0 \right\| &= \sup_{\|y_0\|=1} \left|\langle y_0, (e^{\tilde{A}^* t} - e^{A^* t})x_0\rangle\right| \\ &= \sup_{\|y_0\|=1} \left|\langle (e^{\tilde{A}t} - e^{At})y_0, x_0\rangle\right| \\ &= \sup_{\|y_0\|=1} \left|\left\langle \int_0^t e^{A(t-s)}(A-\tilde{A})e^{\tilde{A}s}\, ds\, y_0, x_0 \right\rangle\right| \\ &= \sup_{\|y_0\|=1} \left|\left\langle V_B(e^{\tilde{A}\cdot})y_0, x_0 \right\rangle\right| \\ &\leq \sup_{\|y_0\|=1} M t^\alpha M_2 \|y_0\| \|x_0\|. \\ &= M t^\alpha M_2 \|x_0\|. \end{aligned}$$

Using this inequality, we find similar to (6.3), that

$$\int_0^{t_0} \left\| \left(e^{A^* t} - e^{\tilde{A}^* t}\right) x_0 \right\|^2 \frac{1}{t}\, dt \leq \frac{M^2 M_2^2}{2\alpha} t_0^{2\alpha} \|x_0\|^2.$$

The second time interval is from t_0 to ∞. The proof for this interval is similar to the second part of the proof of Lemma 6.1, and is therefor ommitted.

Concluding, we see that the semigroups generated by A and $(A_{-1}+B)_X$ have a finite Bergman distance. □

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