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# Inertia effects in rheometrical flow systems Part 3: Some energy considerations with respect to the flow field in the balance rheometer

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With 3 figures

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*Summary*

Following up a previous paper by one of the present authors on the flow field in the balance rheometer, inertia effects being included, in this paper some energy considerations with respect to this flow field are presented. It is shown that in a frame rotating with the same angular velocity as the hemispheres the power supplied by these hemispheres equals the rate of energy dissipation in the sample, i.e. in this coordinate system there is no "stress power paradox". Further it is shown that the "elastic" couple for a Newtonian liquid, appearing in the calculations, stems from the extra kinetic energy caused by the deviation of the actual flow field from the flow field that appears when inertia effects are ignored.

*Zusammenfassung*

Als Fortsetzung des früheren Beitrages eines der hier genannten Autoren über das Strömungsfeld in einem Képès-Rheometer unter Berücksichtigung der Flüssigkeitsträgheit werden in diesem Beitrag einige Energiebetrachtungen angestellt. Es wird gezeigt, daß in einem Koordinatensystem, das mit gleicher Winkelgeschwindigkeit wie die Halbkugeln rotiert, die durch diese Halbkugeln zugeführte Leistung der in der Probe dissipierten Leistung gleich ist, d. h. daß in diesem Koordinatensystem das sogenannte Spannungsenergieparadox nicht vorliegt. Es wird weiter gezeigt, daß das bei einer newtonschen Flüssigkeit auftretende „elastische" Drehmoment seinen Ursprung in der zusätzlichen kinetischen Energie hat, die der Abweichung des tatsächlichen Strömungsfeldes von dem unter Vernachlässigung der Flüssigkeitsträgheit berechneten Strömungsfeld entspricht.

*Key words*

Balance rheometer, inertia effects, stress power paradox

*Notations*

| | |
|---|---|
| $a$ | distance between rotation axes of orthogonal rheometer |
| $b, c$ | positive numbers (see eq. [29]) |
| $f_n(z)$ | $j_n(z)$, $y_n(z)$, $h_n^{(1)}(z)$ or $h_n^{(2)}(z)$ |
| $h$ | distance between discs of orthogonal rheometer |
| $h_n^{(1)}(z) = j_n(z) + iy_n(z)$, $h_n^{(2)}(z) = j_n(z) - iy_n(z)$ | spherical Bessel functions of the third kind and order $n$ |
| $i$ | $\sqrt{-1}$ |
| $j_n(z)$ | spherical Bessel function of the first kind and order $n$ |
| $k$ | $-(\rho\omega/2\eta)^{1/2}$ |
| $n$ | integer |
| $p_1$ | $j_1(\alpha r_2)y_1(\alpha r_1) - j_1(\alpha r_1)y_1(\alpha r_2)$ |
| $q_1$ | $\sum_{i=1}^{2} j_1(\alpha r_i)$ |
| $r$ | spherical polar coordinate |
| $r_1(r_2)$ | radius of inner (outer) hemisphere |
| $s_1$ | $\sum_{i=1}^{2} y_1(\alpha r_i)$ |
| $t$ | time |
| $u_r$, $u_\theta$, $u_\phi$ | physical components of displacement |
| $x$, $z$ | variables |
| $x$, $y$, $z$ | cartesian coordinates in eq. [1] |
| $y_n(z)$ | spherical Bessel function of the second kind and order $n$ |
| $E_{kin}$ | kinetic energy |
| $E_s$ | stored energy |
| $F$ | force |
| $F_n(z)$ | $J_n(z)$, $Y_n(z)$, $H_n^{(1)}(z)$ or $H_n^{(2)}(z)$ |
| $G^* = G' + iG''$ | complex shear modulus |
| $H_n^{(1)}(z) = J_n(z) + iY_n(z)$, $H_n^{(2)}(z) = J_n(z) - iY_n(z)$ | Bessel functions of the third kind and order $n$ |
| Im … | imaginary part of … |
| $J_n(z)$ | Bessel function of the first kind and order $n$ |
| $N$ | $\frac{\alpha}{p_1} \sum_{i=1}^{2} r_i^2 \{j_0(\alpha r_i) \cdot s_1 - y_0(\alpha r_i) \cdot q_1\}$ |
| $\Delta P$ | energy supplied to the sample during one cycle |
| Re … | real part of … |
| $S$ | strain |

| | |
|---|---|
| $U$ | $\frac{1}{p_1} \sum_{i=1}^{2} r_i^2 \{y_2(\alpha r_i) \cdot q_1 - j_2(\alpha r_i) \cdot s_1\}$ |
| $\Delta W$ | energy dissipated in the sample during one cycle |
| $Y_n$ | Bessel function of the second kind and order $n$ |
| $\alpha$ | $\omega(\rho/G^*)^{1/2}$-complex shear wave factor |
| $\gamma$ | $\varepsilon/(r_1 + r_2)$ |
| $\delta$ | loss angle |
| $\varepsilon$ | angle between rotation axes of balance rheometer |
| $\eta$ | viscosity |
| $\theta$ | spherical polar coordinate |
| $\chi$ | order of cylinder function (see eq. [24]) |
| $\Lambda$ | linear combination of spherical Bessel functions of first, second, and third kind, in which the coefficients are independent of the argument and the order |
| $\mu$, $\nu$ | constants (see eq. [24]) |
| $\Pi$ | see $\Lambda$ |
| $\rho$ | density |
| $\tau$ | shear stress |
| $\phi$ | spherical polar coordinate |
| $\Phi$, $\Psi$ | cylinder functions |
| $\omega$ | angular velocity |

## 1. Introduction

In 1969 Abbott and Walters (1) presented an exact solution of the Navier-Stokes equations for the flow between E(ccentric) R(otating) D(iscs). Their solution implies that each plane $z$ = constant (see fig. 1) moves as if rigid with angular velocity $\omega$ about a point, but the locus of these points as $z$ varies is not a straight line joining the centres of the two discs. In deriving the equation of the locus of rotation centres a particular choice for two appearing integration constants had to be made. This choice was made in such a way as to rule out the presence of a pressure driven flow, and at the same time ensure that the flow field is symmetric with respect to the plane $z = h/2$. In a later paper Goldstein and Schowalter (2), using a different expansion procedure, arrived at the same result. These authors, too, excluded the presence of a pressure driven flow.

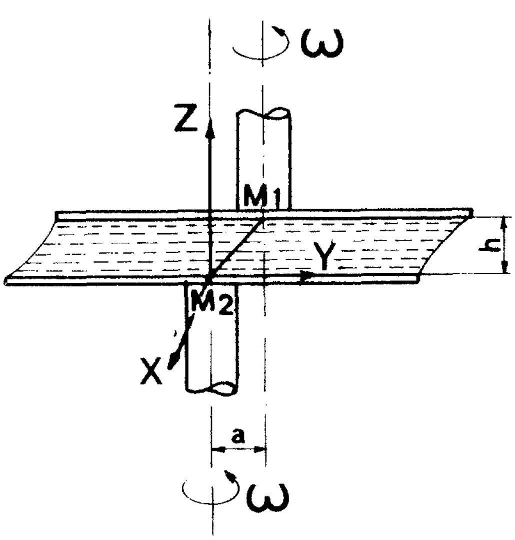


Fig. 1. Geometry of the orthogonal rheometer

In a paper by one of the present authors (3), to be referred to as Part 1 in this paper, a simple but powerful procedure was introduced to calculate the flow field in ERD flow. In this procedure use is made of the fact that in a frame rotating with the same speed as the discs the deformation can be described as a superposition of two harmonic simple shear deformations, polarized perpendicularly to each other, having the same amplitude $a$, but showing a phase difference of $\pi/2$ radians. To ensure that, using this procedure, the flow field is symmetric with respect to the plane $z = h/2$ and no pressure driven flow occurs, it appeared necessary to impose symmetric boundary conditions on both composing deformations, see figure 2.

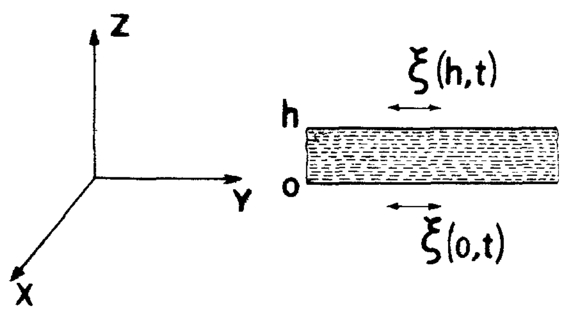


Fig. 2. Description of the motion of the fluid in a rotating coordinate system
$\xi(h, t) = (a/2) \exp i\omega t$; $\xi(0, t) = -(a/2) \exp i\omega t$

In Part 1 it was shown that for the flow between ERD

(i) for a Newtonian liquid $F_y \neq 0$. In the elementary treatment of ERD flow, where inertia effects are neglected, $F_y$ is proportional to the storage modulus $G'(\omega)$ of the sample. Since for a Newtonian liquid $G'(\omega) = 0$, the stored energy, connected with $F_y$, must stem from a different source. It was shown that this stored energy originates from the "extra" kinetic energy due to the fact that planes $z$ = constant rotate about points which do not coincide with their centres;

(ii) in the rotating frame, mentioned above, the rate of energy dissipation in the sample equals the power supplied by the disc forces. So, in this frame there is no "stress power paradox" (4, 5).

In a subsequent paper of the same author (6), to be referred to as Part 2 in this paper, the same procedure for calculating the flow field was applied to the balance rheometer.

It is the aim of the present paper to show that conclusions analogous to (i) and (ii) also apply to the balance rheometer. Unfortunately, however, in Part 2 no symmetric boundary conditions were applied in calculating the flow field, although it was shown that the ERD rheometer is a limiting case of the balance rheometer. In order to make the treatments for both rheometers consistent the relevant equations for the latter rheometer with symmetric boundary conditions are deduced in section 3.

## 2. Short outline of the procedure

Figure 3 shows the geometry of the balance rheometer. The fluid is contained between two concentric hemispheres. In operation the inner hemisphere is rotated with constant angular velocity $\omega$ about a vertical axis. The outer hemisphere, the axis of which is supposed to have a frictionless bearing, rotates with (nearly) the same angular velocity about an axis which makes a small angle $\varepsilon$ with the vertical one.

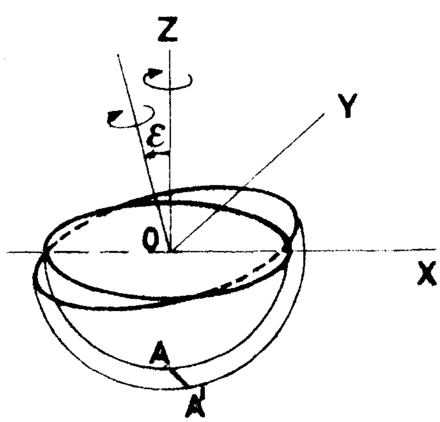


Fig. 3. Geometry of the balance rheometer

In Part 2 it was shown that in a coordinate system fixed to the inner hemisphere and oriented as shown in figure 3 the deformation of the sample can be described as due to a superposition of two harmonic oscillations of the outer hemisphere, one about the $x$-axis, the other one about the $z$-axis. Both oscillations have the same angular amplitude $\varepsilon$, but show a phase difference of $\pi/2$ radians. The resulting deformation of the sample is therefore due to a rotation of the outer hemisphere over an angle $\varepsilon$ about a horizontal axis, the direction of which turns counter-clockwise with the angular velocity $\omega$. In case of a viscoelastic material this results in couples on the hemispheres, the direction of which also turns counter-clockwise with the angular velocity $\omega$, but advances the deformation by a certain angle. As the coordinate system itself turns clockwise with the angular velocity $\omega$, the couples are fixed in the laboratory frame. For the determination of the couples exerted by the fluid on the hemispheres it suffices therefore to calculate the amplitude of the couple exerted on the hemispheres by one of the composing harmonic oscillations, see Part 1. As to the couples caused by inertial forces a similar reasoning applies.

## 3. Calculation of the flow field

As suggested by the geometry, spherical polar coordinates $r$, $\theta$, and $\phi$, defined by

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \qquad [1]$$

are used. To ensure that, in the procedure outlined before, the displacement of the sample has symmetric boundary conditions, the angular amplitudes of both hemispheres are chosen as

$$\left.\begin{aligned} \varepsilon_1 &= -\frac{\varepsilon r_2}{r_1 + r_2} \equiv -\gamma r_2 \quad \text{for} \quad r = r_1 \\ \varepsilon_2 &= \frac{\varepsilon r_1}{r_1 + r_2} \equiv \gamma r_1 \quad \text{for} \quad r = r_2 \end{aligned}\right\}, \qquad [2]$$

where $\varepsilon = \varepsilon_2 - \varepsilon_1$ is the angle between the axes of both hemispheres. This leads to the following boundary conditions for $u_\phi$:

$$\left.\begin{aligned} u_\phi &= -\gamma\sin\theta\exp i\omega t \quad \text{for} \quad r = r_1 \\ u_\phi &= \gamma\sin\theta\exp i\omega t \quad \text{for} \quad r = r_2 \end{aligned}\right\}. \qquad [3]$$

For the particulars of the calculation of $u_\phi$ the reader is referred to Part 2. The result is

$$u_\phi = \frac{\gamma \sin\theta \{j_1(\alpha r)\cdot s_1 - y_1(\alpha r)\cdot q_1\}}{p_1} \exp i\omega t\,. \quad [4]$$

Moreover

$$u_\theta = u_r = 0\,. \quad [5]$$

## 4. Couples on the hemispheres

The shear stress in the sample is found from

$$\tau_{r\phi} = 2G^* s_{r\phi}\,, \quad [6]$$

where

$$S_{r\phi} = \frac{1}{2}\left\{\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right\} = \frac{1}{2}\left\{\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right\}. \quad [7]$$

From [4] it follows:

$$\frac{\partial u_\phi}{\partial r} = \frac{\gamma\alpha\sin\theta\{j_1'(\alpha r)\cdot s_1 - y_1'(\alpha r)\cdot q_1\}}{p_1} \cdot \exp i\omega t\,. \quad [8]$$

Now

$$\left.\begin{aligned} j_1'(z) &= j_0(z) - \tfrac{2}{z} j_1(z) \\ y_1'(z) &= y_0(z) - \tfrac{2}{z} y_1(z) \end{aligned}\right\}. \quad [9]$$

Substitution of [9] into [8] leads to

$$\frac{\partial u_\phi}{\partial r} = -\left[\frac{\frac{2\alpha\gamma}{z}\sin\theta\{j_1(\alpha r)\cdot s_1 - y_1(\alpha r)\cdot q_1\}}{p_1} + \frac{\alpha\gamma\sin\theta\{j_0(\alpha r)\cdot s_1 - y_0(\alpha r)\cdot q_1\}}{p_1}\right] \exp i\omega t. \quad [10]$$

With the aid of the recurrence relations

$$\left.\begin{aligned} j_{n-1}(z) + j_{n+1}(z) &= (2n+1)z^{-1} j_n(z) \\ y_{n-1}(z) + y_{n+1}(z) &= (2n+1)z^{-1} y_n(z) \end{aligned}\right\} \quad [11]$$

it is readily shown that

$$\frac{\partial u_\phi}{\partial r} = \left[\frac{\frac{\alpha\gamma}{z}\sin\theta\{j_1(\alpha r)\cdot s_1 - y_1(\alpha r)\cdot q_1\}}{p_1} + \frac{\alpha\gamma\sin\theta\{y_2(\alpha r)\cdot q_1 - j_2(\alpha r)\cdot s_1\}}{p_1}\right] \exp i\omega t. \quad [12]$$

From [7], [4], and [12] it follows:

$$2S_{r\phi} = \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} = \frac{\alpha\gamma\sin\theta\{y_2(\alpha r)\cdot q_1 - j_2(\alpha r)\cdot s_1\}}{p_1}\exp i\omega t\,. \quad [13]$$

From [13] and [6] the (complex) amplitude $\hat{\tau}_{r\phi}$ of the shear stress is found:

$$\hat{\tau}_{r\phi} = \frac{G^*\alpha\gamma\sin\theta\{y_2(\alpha r)\cdot q_1 - j_2(\alpha r)\cdot s_1\}}{p_1}\,, \quad [14]$$

from which the (complex) amplitude of the $z$-component of the couple on the inner hemisphere, $\hat{M}_{1,z}$, is calculated according to

$$\hat{M}_{1,z} = r_1^3 \int_0^{\pi}\int_0^{-\pi} \hat{\tau}_{r\phi}\sin^2\theta\, d\theta\, d\phi = \frac{4\pi\gamma^2\alpha G^* r_1^4\{y_2(\alpha r_1)\cdot q_1 - j_2(\alpha r_1)\cdot s_1\}}{3p_1}\,. \quad [15]$$

The couple on the outer hemisphere is found from [15] by substituting $r_2$ for $r_1$.

## 5. Energy considerations

### *5.1. Stored energy in a linear viscoelastic fluid*

The stored energy, $E_s$, is found from

$$\begin{aligned} E_s &= -\frac{1}{2}\mathrm{Re}\sum_{i=1}^{2}\hat{M}_{i,z}\varepsilon_i \\ &= -\mathrm{Re}\sum_{i=1}^{2}\frac{2\pi G^*\gamma\alpha}{3p_1}\varepsilon_i r_i^3 \cdot \{y_2(\alpha r_i)\cdot q_1 - j_2(\alpha r_i)\cdot s_1\} \\ &= -\frac{2\pi\gamma^2\alpha G^*}{3}\mathrm{Re}\sum_{i=1}^{2} \cdot \frac{r_i^2\{y_2(\alpha r_i)\cdot q_1 - j_2(\alpha r_i)\cdot s_1\}}{p_1} \\ &\equiv -\frac{2\pi\gamma^2\alpha G^*}{3}\mathrm{Re}\, U\,, \end{aligned} \quad [16]$$

where it should be noted that $\alpha G^* = \rho\omega^2$ is a real quantity. For a Newtonian liquid

$$G^* = iG''. \quad [17]$$

Substitution of [17] into [16] leads to

$$E_s = \frac{2\pi\gamma^2 G''}{3} \operatorname{Im} \alpha U. \qquad [18]$$

From [18] it is learned that even for a Newtonian liquid energy is stored in the sample. Since no elastic energy can be stored in this case, $E_s$ in [18] must be solely determined by the (extra) kinetic energy of the sample.

### 5.2. Stored kinetic energy in a Newtonian liquid

The kinetic energy, $E_{\text{kin}}$, is found from the contributions of both harmonic oscillations, which leads to (compare Part 1, eq. [44]):

$$E_{\text{kin}} = \int_{r_1}^{r_2} \int_{\pi/2}^{\pi} \int_{0}^{2\pi} \frac{1}{2} \rho \left\{ \frac{\partial}{\partial t} |\hat{u}_\phi| \cos \omega t \right\}^2 \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi + \int_{r_1}^{r_2} \int_{\pi/2}^{\pi} \int_{0}^{2\pi} \frac{1}{2} \rho \left\{ \frac{\partial}{\partial t} |\hat{u}_\phi| \sin \omega t \right\}^2 \cdot r \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{2} \rho \omega^2 \int_{r_1}^{r_2} \int_{\pi/2}^{\pi} \int_{0}^{2\pi} |\hat{u}_\phi|^2 r^2 \sin\theta \, dr\, d\theta\, d\phi. \qquad [19]$$

Now

$$|\hat{u}_\phi|^2 = \hat{u}_\phi \cdot \hat{u}_\phi^*; \qquad [20]$$

$$\hat{u}_\phi^* = \frac{\gamma \sin\theta \{ j_1^*(\alpha r) \cdot s_1^* - y_1^*(\alpha r) \cdot q_1^* \}}{p_1^*}. \qquad [21]$$

Further it holds:

$$\left. \begin{aligned} j_n^*(z) &= j_n(z^*) \\ y_n^*(z) &= y_n(z^*) \end{aligned} \right\}. \qquad [22]$$

So [19] takes the form

$$E_{\text{kin}} = \frac{1}{2} \rho \omega^2 \gamma^2 \int_{\pi/2}^{\pi} \sin^3\theta\, d\theta \int_0^{2\pi} d\phi \cdot \int_{r_1}^{r_2} r^2 \frac{\{j_1(\alpha r)\cdot s_1 - y_1(\alpha r) \cdot q_1\}\{j_1(\alpha^* r) \cdot s_1^* - y_1(\alpha^* r)\cdot q_1^*\}}{p_1 p_1^*} dr$$

$$= -\frac{2\pi\rho\omega^2\gamma^2}{3} \cdot \int_{r_1}^{r_2} r^2 \frac{\{j_1(\alpha r)\cdot s_1 - y_1(\alpha r) \cdot q_1\}\{j_1(\alpha^* r) \cdot s_1^* - y_1(\alpha^* r)\cdot q_1^*\}}{p_1 p_1^*} dr. \qquad [23]$$

To solve the integral in [23] consider the relation valid for cylinder functions*) of the same order (7):

$$\int^{z} x \phi_\chi(\mu x)\, \Psi_\chi(\nu x)\, dx = \frac{\nu \Phi_\chi(\mu z)\, \Psi_{\chi-1}(\nu z) - \mu z\, \Phi_{\chi-1}(\mu z)\, \Psi_\chi(\nu z)}{\mu^2 - \nu^2}. \qquad [24]$$

Between spherical Bessel functions and "ordinary" Bessel functions the following relations hold:

$$f_n(z) = \left( \frac{\pi}{2z} \right)^{1/2} F_{n+(1/2)}(z), \qquad [25]$$

where $f_n(z)$ denotes $j_n(z)$, $y_n(z)$, $h_n^{(1)}(z)$ or $h_n^{(2)}(z)$, and $F_{n+(1/2)}(z)$ denotes $J_{n+(1/2)}(z)$, $Y_{n+(1/2)}(z)$, $H^{(1)}_{n+(1/2)}(z)$ or $H^{(2)}_{n+(1/2)}(z)$.

From [25] and [24], with $\chi = n$, it is found

$$\int^{r} r^2 \Pi_n(\mu r) \Lambda_n(\nu r)\, dr = \frac{r^2\{\nu \Pi_n(\mu r) \Lambda_{n-1}(\nu r) - \mu \Pi_{n-1}(\mu r) \Lambda_n(\nu r)\}}{\mu^2 - \nu^2}, \qquad [26]$$

where $\Pi$ and $\Lambda$ are linear combinations of spherical Bessel functions with the same constrictions as hold for cylinder functions. [26] is the analogue of [24] for spherical Bessel functions with $\chi = n$.

From [23] and [26] it follows:

$$E_{\text{kin}} = -\frac{2\pi\rho\omega^2\gamma^2}{3(\alpha^2 - \alpha^{*2})} [N^* - N], \qquad [27]$$

where

$$N = \frac{\alpha}{p_1} \sum_{i=1}^{2} r_i^2 \{ j_0(\alpha r_i) \cdot s_1 - y_0(\alpha r_i) \cdot q_1 \}. \qquad [28]$$

Suppose

$$N = b + ci. \qquad [29]$$

Then

$$N^* - N = -2ci. \qquad [30]$$

*) Linear combination of Bessel functions of first, second, and third kind, in which the coefficients are independent of the argument and the order.

With the aid of [11] the following relation is deduced:

$$j_0(\alpha r) \cdot s_1 - y_0(\alpha r) \cdot q_1 = \frac{3}{z}\{j_1(\alpha r) \cdot s_1 - y_1(\alpha r) \cdot q_1\} + \{y_2(\alpha r) \cdot q_1 - j_2(\alpha r) \cdot s_1\}. \quad [31]$$

From [28] and [31] it then follows:

$$N = 3(r_2 - r_1) + \frac{\alpha}{p_1} \sum_{i=1}^{2} r_i^2 \{y_2(\alpha r_i) \cdot q_1 - j_2(\alpha r_i) \cdot s_1\} = 3(r_2 - r_1) + \alpha U. \quad [32]$$

For a Newtonian liquid

$$\alpha^2 - \alpha^{*2} = -4ik^2, \quad [33]$$

$$\rho\omega^2 = 2G''k^2. \quad [34]$$

From eqs. [27–30] and [32–34] it finally follows:

$$E_{\text{kin}} = \frac{2\pi\gamma^2 G''}{3} c = \frac{2\pi\gamma^2 G''}{3} \operatorname{Im} \alpha U. \quad [35]$$

Comparing [35] with [18] it is seen that for a Newtonian liquid the stored energy equals the extra kinetic energy.

The extra kinetic energy is due to the fact that the locus of rotation centres of surfaces with $r$ = constant, inertia effects taken into account, deviates from that locus if inertia effects are ignored.

### 5.3. *Energy dissipation in a linear viscoelastic fluid*

The energy, $\Delta W$, dissipated in the sample during one cycle amounts to

$$\Delta W = 2 \int_{r_1}^{r_2} \int_{\pi/2}^{\pi} \int_{0}^{2\pi} \int_{0}^{2\pi/\omega} \operatorname{Re}(\hat{\tau}_{r\phi}) \cdot \operatorname{Re}\left(\frac{2\partial \hat{S}_{r\phi}}{\partial t}\right) \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi \, dt. \quad [36]$$

From [6] it follows:

$$\hat{\tau}_{r\phi} = 2G^* \hat{S}_{r\phi} = \frac{G'}{\cos\delta} \hat{S}_{r\phi} \exp i\delta. \quad [37]$$

With

$$\hat{S}_{r\phi} = |\hat{S}_{r\phi}| \exp i\psi \quad [38]$$

eq. [36] turns into

$$\Delta W = -\frac{2G'\omega}{\cos\delta} \int_{r_1}^{r_2} \int_{\pi/2}^{\pi} \int_{0}^{2\pi} \int_{0}^{2\pi/\omega} |2\hat{S}_{r\phi}|^2 r^2 \sin\theta \cdot \cos(\omega t + \psi + \delta) \cdot \sin(\omega t + \psi) dr\, d\theta\, d\phi\, dt$$
$$= -\frac{2\pi G' \sin\delta}{\cos\delta} \int_{r_1}^{r_2} \int_{\pi/2}^{\pi} \int_{0}^{2\pi} |2\hat{S}_{r\phi}|^2 \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi. \quad [39]$$

From [13] and [39] if follows:

$$\Delta W = -2\pi G' \tan\delta \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} \sin^3\theta \, d\theta \cdot \int_{r_1}^{r_2} r^2 \left| \frac{\alpha\gamma\{y_2(\alpha r) \cdot q_1 - j_2(\alpha r) \cdot s_1\}}{p_1} \right|^2 dr$$
$$= -\frac{8\pi G'\gamma^2 \alpha\alpha^* \tan\delta}{3} \int_{r_1}^{r_2} r^2 \cdot \left| \frac{\{y_2(\alpha r) \cdot q_1 - j_2(\alpha r) \cdot s_1\}}{p_1} \right|^2 dr. \quad [40]$$

Now

$$\int_{r_1}^{r_2} r^2 \left| \frac{\{y_2(\alpha r) \cdot q_1 - j_2(\alpha r) \cdot s_1\}}{p_1} \right|^2 dr = \int_{r_1}^{r_2} r^2 \frac{\{y_2(\alpha r) \cdot q_1 - j_2(\alpha r) \cdot s_1\}\{y_2(\alpha^* r) \cdot q_1^* - j_2(\alpha^* r) \cdot s_1^*\}}{p_1 p_1^*} dr. \quad [41]$$

From [40], [41], and [26] it is found:

$$\Delta W = -\frac{8\pi^2 G'\gamma^2 \alpha\alpha^* \tan\delta}{3(\alpha^2 - \alpha^{*2})} [\alpha U^* - \alpha^* U]. \quad [42]$$

For a linear viscoelastic fluid

$$\alpha^2 - \alpha^{*2} = -4i\alpha' \tan\delta/2, \quad [43]$$

$$\alpha\alpha^* = \frac{\alpha'^2}{\cos^2\delta/2}. \quad [44]$$

So

$$\Delta W = \frac{4\pi^2 G'\gamma^2}{3i\cos\delta} [\alpha U^* - \alpha^* U] = -\frac{8\pi^2 G'\gamma^2}{3\cos\delta} \operatorname{Im} \alpha^* U. \quad [45]$$

The energy, $\Delta P$, supplied to the sample by the hemisphere couples during one cycle equals

$$\Delta P = -2\pi \operatorname{Im} \sum_{i=1}^{2} \hat{M}_i \varepsilon_i = -\operatorname{Im} \frac{8\pi^2 G^* \gamma^2 \alpha}{3} U$$

$$= -\operatorname{Im} \frac{8\pi^2 G^* \gamma^2 \alpha}{3\alpha^*} \alpha^* U . \qquad [46]$$

Now

$$G^* \alpha/\alpha^* = \frac{G'}{\cos\delta} \cdot \exp i\delta \cdot \exp -i\delta$$

$$= G'/\cos\delta . \qquad [47]$$

So

$$\Delta P = -\frac{8\pi^2 G' \gamma^2}{3\cos\delta} \operatorname{Im} \alpha^* U . \qquad [48]$$

From [48] and [45] it can be concluded that in a rotating frame the energy dissipated in the sample is supplied by the hemisphere couples.

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