Integral complete \(r\)-partite graphs\(^\star\)

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Abstract

A graph is called integral if all the eigenvalues of its adjacency matrix are integers. In this paper, we give a useful sufficient and necessary condition for complete \(r\)-partite graphs to be integral, from which we can construct infinite many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. The discovery of these integral complete \(r\)-partite graphs is a new contribution to the search of such integral graphs. Finally, we propose several basic open problems for further study.

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1. Introduction

We shall consider only simple undirected graphs (i.e. undirected graphs without loops or multiple edges). For a graph \(G\), let \(V(G)\) denote the vertex set and \(E(G)\) the edge set. The characteristic polynomial \(|xI - A|\) of the adjacency matrix \(A\) (or \(A(G)\)) of \(G\) is called the characteristic polynomial of \(G\) and denoted by \(P(G;x)\). The spectrum of \(A(G)\) is also called the spectrum of \(G\).

The notion of integral graphs was first introduced by Harary and Schwenk [10]. A graph \(G\) is called integral if all the zeros of the characteristic polynomial \(P(G;x)\) are integers. In general, the problem of characterizing integral graphs seems to be difficult. Thus it makes sense to restrict our investigations to some interesting families of graphs, for instance, cubic graphs [4,20], complete tripartite graphs [18], graphs with three eigenvalues [17], graphs with maximum degree 4 [2], etc. Trees present another important family of graphs for which the problem has been considered in [5–7,11–16,21,22,24–26]. Other results on integral graphs can be found in [1,8,9,23]. For all other facts on graph spectra (or terminology), see [8,9].

A complete \(r\)-partite graph \(K_{p_1,p_2,\ldots,p_r}\) is a graph with a set \(V = V_1 \cup V_2 \cup \cdots \cup V_r\) of \(p_1 + p_2 + \cdots + p_r(=n)\) vertices, where \(V_i\)'s are nonempty disjoint sets, \(|V_i| = p_i\) for \(1 \leq i \leq r\), such that two vertices in \(V\) are adjacent if and only if they belong to different \(V_i\)'s. An infinite family of integral complete multipartite graphs was constructed in [18], where the author mentioned the general problem on integral complete multipartite graphs. He thought that it is possible that for \(r > 3\) there also exist an infinite number of integral complete \(r\)-partite graphs. But he did not find such integral graphs. Balińska and Simić [2] thought that the general problem seems to be intractable. In this paper, we give a useful sufficient and necessary
condition for complete $r$-partite graphs to be integral, from which we can construct infinite many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. The discovery of these integral complete $r$-partite graphs is a new contribution to the search of such integral graphs. In fact, M. Roitman’s result on the integral complete tripartite graphs is generalized in this paper (see also [18, MR 86a:05089]). Finally, we propose several basic open problems for further study.

2. A sufficient and necessary condition for complete $r$-partite graphs to be integral

In this section, we shall give a useful sufficient and necessary condition for complete $r$-partite graphs to be integral. The following lemma can be found in [9].

Lemma 2.1. For the complete $r$-partite graph $K_{p_1, p_2, \ldots , p_r}$ on $n$ vertices, we have

$$P(K_{p_1, p_2, \ldots , p_r}, x) = x^{n-r} \left(1 - \sum_{i=1}^{r} \frac{p_i}{x + p_i}\right) \prod_{j=1}^{r} (x + p_j).$$

Assume that the number of distinct integers of $p_1, p_2, \ldots , p_r$ is $s$. Without loss of generality, assume that the first $s$ ones are the distinct integers such that $p_1 < p_2 < \cdots < p_s$. Suppose that $a_i$ is the multiplicity of $p_i$ for each $i = 1, 2, \ldots , s$. The complete $r$-partite graph $K_{p_1, p_2, \ldots , p_r}$ is also denoted by $K_{a_1, a_2, \ldots , a_s, p_r}$, where $r = \sum_{j=1}^{r} a_i$ and $|V| = n = \sum_{i=1}^{r} a_i p_i$.

Example 2.2 (See Cvetković et al. [9]). For the complete $r$-partite graph $K_{p_1, p_2, \ldots , p_r} = K_{a_1, a_2, \ldots , a_s, p_r}$, if $s = 2$, $a_1 = a_2 = 1$, then $K_{p_1, p_2}$ is integral if and only if $p_1, p_2$ is a perfect square.

Corollary 2.3. For the complete $r$-partite graph $K_{p_1, p_2, \ldots , p_r} = K_{a_1, a_2, \ldots , a_s, p_r}$ on $n$ vertices, we have

$$P(K_{a_1, a_2, \ldots , a_s, p_r}, x) = x^{n-r} \prod_{i=1}^{s} (x + p_i)^{a_i-1} \left[\prod_{i=1}^{s} (x + p_i) - \sum_{j=1}^{s} a_j p_j \prod_{i=1, i \neq j}^{s} (x + p_i)\right].$$

The following theorem is immediate.

Theorem 2.4. The complete $r$-partite graph $K_{p_1, p_2, \ldots , p_r} = K_{a_1, a_2, \ldots , a_s, p_r}$ is integral if and only if

$$\prod_{i=1}^{s} (x + p_i) - \sum_{j=1}^{s} a_j p_j \prod_{i=1, i \neq j}^{s} (x + p_i) = 0$$

has no other roots but integral ones.

We shall discuss Eq. (1) to get more information. First, we divide both sides of Eq. (1) by $\prod_{i=1}^{n} (x + p_i)$, and obtain

$$\sum_{i=1}^{s} \frac{a_i p_i}{x + p_i} = 1.$$

Let $F(x) = \sum_{i=1}^{n} \left([a_i p_i]/(x + p_i)\right) - 1$. Clearly, $-p_i$’s are not roots of Eq. (1) for $1 \leq i \leq s$. Hence, all solutions of Eq. (1) are the same as those of Eq. (2). Now we consider the roots of $F(x)$ over the set of real numbers. Note that $F(x)$ is discontinuous at each point $p_i$. For $1 \leq i \leq s$, we have that $F(-p_i - 0) = -\infty$, $F(-p_i + 0) = +\infty$, $F(-\infty) = F(+\infty) = -1$, $F'(x) = -\sum_{i=1}^{s} ([a_i p_i]/(x + p_i)^2)$. We deduce that $F(x)$ is strictly monotone decreasing on each of the continuous intervals over the set of real numbers. Using Zero Point Theorem of Mathematical Analysis, we get that $F(x)$ has $s$ distinct real roots. If $-\infty < u_0 < u_{s-1} < \cdots < u_1 < +\infty$ are the roots of $F(x)$, then

$$-p_s < u_0 < -p_{s-1} < u_{s-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty$$

holds.
On the other hand, we note that Eq. (2) can be written as
\[
\frac{a_1 p_1}{x + p_1} + \frac{a_2 p_2}{x + p_2} + \cdots + \frac{a_s p_s}{x + p_s} = 1.
\]
(4)

From the above discussion, we have.

**Theorem 2.5.** The complete r-partite graph \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_s} \) is integral if and only if all the solutions of Eq. (4) are integers. Moreover, there exist integers \( u_1, u_2, \ldots, u_s \) satisfying (3) such that the following linear equation system in \( a_1, a_2, \ldots, a_s \)
\[
\begin{align*}
\frac{a_1 p_1}{u_1 + p_1} + \frac{a_2 p_2}{u_1 + p_2} + \cdots + \frac{a_s p_s}{u_1 + p_s} &= 1, \\
\cdots & \cdots \\
\frac{a_1 p_1}{u_s + p_1} + \frac{a_2 p_2}{u_s + p_2} + \cdots + \frac{a_s p_s}{u_s + p_s} &= 1
\end{align*}
\]
has positive integral solutions \((a_1, a_2, \ldots, a_s)\).

**Theorem 2.6.** The complete r-partite graph \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_s} \) on \( n \) vertices is integral if and only if there exist integers \( u_i \) and positive integers \( p_i \) \((i = 1, 2, \ldots, s)\) such that \(-p_i < u_i < -p_{i-1} < u_{i-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty\) and
\[
\left. a_k = \frac{\prod_{i=1}^{r} (p_k + u_i)}{p_k \prod_{i=1, i \neq k} (p_k - p_i)} \quad (k = 1, 2, \ldots, s) \right.
\]
are positive integers.

**Proof.** From Cauchy’s result on determinants in [3], we know that
\[
\begin{vmatrix}
1 & x_1 + \beta_1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & x_s + \beta_1 & \cdots & 1
\end{vmatrix}
= \frac{\prod_{1 \leq i < j \leq s} (x_j - x_i) (\beta_j - \beta_i)}{\prod_{1 \leq i < j \leq s} (x_i + \beta_j)}.
\]
The determinant of the coefficient matrix \( D \) of the linear equation system (5) is the following:
\[
[D] = \left| \begin{array}{cccc}
p_1 & \cdots & p_s \\
p_1 & \cdots & p_s \\
\vdots & \ddots & \vdots \\
p_1 & \cdots & p_s \\
1 & \cdots & 1 \\
1 & \cdots & 1 \\
\end{array} \right| = \prod_{i=1}^{s} p_i \\
\begin{vmatrix}
p_1 & \cdots & p_s \\
p_1 & \cdots & p_s \\
\vdots & \ddots & \vdots \\
p_1 & \cdots & p_s \\
1 & \cdots & 1 \\
1 & \cdots & 1 \\
\end{vmatrix}
= \frac{\prod_{i=1}^{s} p_i \prod_{1 \leq i < j \leq s} (u_j - u_i) (p_j - p_i)}{\prod_{1 \leq i < j \leq s} (u_i + p_j)} \neq 0,
\]

\[
[D_k] = \lim_{p_k \rightarrow +\infty} |D| = \frac{\prod_{i=1, i \neq k}^{s} p_i \prod_{1 \leq i < j \leq s} (u_j - u_i) (p_j - p_i)}{\prod_{1 \leq i < j \leq s} (u_i + p_j) \prod_{1 \leq i < j \leq s} (p_k - p_i)}
\]
for \( k = 1, 2, \ldots, s \).
By using the well-known Cramer’s Rule to solve the linear equation system (5) in $a_1, a_2, \ldots, a_s$, we obtain that

$$a_k = \frac{\prod_{i=1}^{s}(p_k + u_i)}{p_k \prod_{i \neq k}^{s}(p_k - p_i)} \quad (k = 1, 2, \ldots, s).$$

Because $-p_i < u_i < -p_{i-1} < u_{i-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty$ and $p_i \geq 1$ for $i = 1, 2, \ldots, s$, we can deduce that $a_k > 0$ ($k = 1, 2, \ldots, s$) and $u_1 > 0$.

The remaining part of the theorem can be easily proved from Lemma 2.1 and Theorems 2.4 and 2.5.

**Proof.** It is easy to see that $v$ is a root of $\Phi_{\vec{a}, \vec{b}}(x)$ if and only if $u/q$ is an integral root of $\Phi_{\vec{a}, \vec{b}}(x)$.

**Remark 2.10.** Theorem 2.9 shows it is reasonable to study Eq. (2) only when $(p_1, p_2, \ldots, p_s) = 1$. Let us call such a vector primitive. So, in general, the primitive vectors are the only ones which are of interest.

3. **Integral complete r-partite graphs**

In this section, we shall construct infinite many new classes of integral complete r-partite graphs, different from those of [8,9,18].
The idea of constructing such integral graph is as follows: First, we properly choose positive integers \( p_1, p_2, \ldots, p_s \). Then, we try to find integers \( u_i (i = 1, 2, \ldots, s) \) satisfying (3) such that there are positive integral solutions \( (a_1, a_2, \ldots, a_s) \) for the linear equation system \( (5) \) (or such that all \( a_i \)'s of (6) are positive integers). Finally, we obtain positive integers \( a_1, a_2, \ldots, a_s \) such that all the solutions of Eq. (4) are integers. Thus, we have constructed many new classes of integral graphs \( K_{a_1, p_1, a_2, p_2, \ldots, a_s, p_s} \).

**Example 3.1.** Let \( p_1 = 1, p_2 = 9 \) and \( u_2 = -4 \). If \( u_1 = 72t - 9 \) and \( t \) is a positive integer, then \( K_{p_1, p_2, a_1, a_2} = K_{a_1, p_1, a_2, p_2} \) is integral for infinite many positive integers \( a_1, a_2 \) given by (7) and (8).

**Proof.** From Theorem 2.6, we have that
\[
a_1 = \frac{(p_1 + u_1)(p_1 + u_2)}{p_1(p_1 - p_2)} = \frac{3}{5}(u_1 + 1) \tag{7}
\]
and
\[
a_2 = \frac{(p_2 + u_1)(p_2 + u_2)}{p_2(p_2 - p_1)} = \frac{5}{7}(u_1 + 9). \tag{8}
\]

So, \( K_{a_1, p_1, a_2, p_2} \) is integral if and only if \( a_1 \) and \( a_2 \) are positive integers. From (7) and (8), we get the Diophantine equation
\[
27a_2 - 5a_1 = 15.
\]

From elementary number theory knowledge, all positive integral solutions of Eq. (9) are given by \( a_1 = 27t - 3, a_2 = 5t, \) and \( u_1 = 72t - 9, \) where \( t \) is a positive integer.

Hence, \( K_{p_1, p_2, a_1, a_2} = K_{a_1, p_1, a_2, p_2} \) is integral for the above infinite many integers \( a_1 \) and \( a_2 \). \( \square \)

The following Lemma 3.2 can be found in [19].

**Lemma 3.2.** Let \( a, b \) and \( c \) be integers with \( d = (a, b) \), we have

1. If \( d \nmid c \), then the linear Diophantine equation in two variables
   \[
   ax + by = c
   \]
   does not have integral solutions.

2. If \( d \mid c \), then there are infinite many integral solutions for Eq. (10). Moreover, if \( x = x_0, y = y_0 \) is a particular solution of Eq. (10), then all its solutions are given by
   \[
   x = x_0 + (b/d)t, \quad y = y_0 - (a/d)t,
   \]
   where \( t \) is an integer.

**Theorem 3.3.** For \( s = 2 \), let \( p_1 < p_2 \). Then \( K_{a_1, p_1, a_2, p_2} \) is integral if and only if one of the following two conditions holds:

1. When \( (m, k) = 1 \), let \( p_1 = m, p_2 = m + k, m \geq 1, k \geq 2, 1 \leq q < k \), where \( m, k \) and \( q \) are positive integers. Then, \( a_1 \) and \( a_2 \) must be positive integral solutions for the Diophantine equation
   \[
   q(m + k)a_2 - m(k - q)a_1 = q(k - q). \tag{11}
   \]

2. When \( (m, k) = d \geq 2 \), let \( p_1 = m, p_2 = m + k, m = m_1d, k = k_1d \), \( (m_1, k_1) = 1 \), \( q = q_1d, 1 \leq q_1 < k_1 \), where \( m_1, k_1, q_1 \) and \( d \) are positive integers. Then, \( a_1 \) and \( a_2 \) must be positive integral solutions for the Diophantine equation
   \[
   q_1(m_1 + k_1)a_2 - m_1(k_1 - q_1)a_1 = q_1(k_1 - q_1). \tag{12}
   \]

**Proof.** Because \( p_1 < p_2 \), from Theorem 2.6, we know \( K_{a_1, p_1, a_2, p_2} \) is integral if and only if there exist integers \( u_1, u_2 \) and positive integers \( p_1, p_2 \) satisfying \( -p_2 < u_2 < -p_1 < u_1 < +\infty \) such that \( a_1 \) and \( a_2 \) are positive integers, where
\[
a_k = \frac{\prod_{i=1}^{s}(p_i + u_i)}{\prod_{i=1}^{s}(p_i - p_1)} \quad \text{for } k = 1, 2. \]
Hence, we choose \( p_1 = m, p_2 = m + k, u_2 = -(m + q), m \geq 1, k \geq 2, 1 \leq q < k \), where \( m, k \) and \( q \) are positive integers, and we have
\[
a_1 = \frac{q(m + u_1)}{mk}, \quad a_2 = \frac{(m + k + u_1)(k - q)}{k(m + k)}.
\]
where \( t \) and \( v \) we know that there are infinitely many integral solutions for Eq. (12). Therefore, there are infinitely many positive integral solutions \((a_1, a_2)\) for Eq. (11).

Case 1: When \((m, k) = 1\), we have \((m + k, m) = 1\), and \(d_1|q(k - q)\). Moreover, there are solutions for Eq. (11). From Lemma 3.2 and the condition \((m, k) = 1\), we know that there are infinite many integral solutions for Eq. (11). Therefore, there are infinite many positive integral solutions \((a_1, a_2)\) for Eq. (11).

Case 2: When \((m, k) = d \geq 2\), let \( m = m_1d, k = k_1d_1\) \((m_1, k_1) = 1\), where \(m_1, k_1\) and \( d \) are positive integers. We have \((m_1 + k_1, m_1) = 1\), \(d_1 = (q(m_1 + k_1), m_1d(k_1d - q))\). If \(d_1|q(k - q) = q(k_1d - q)\), then \(d_1|q\). Thus, let \(q = q_1d_1\), \(1 \leq q_1 < k_1\), where \(q_1\) is a positive integer. We can reduce Eq. (11)–(12). Hence, from Lemma 3.2 and the condition \((m_1, k_1) = 1\), we know that there are infinite many integral solutions for Eq. (12). Therefore, there are infinite many positive integral solutions \((a_1, a_2)\) for Eq. (12).

Thus, the theorem is proved. \(\square\)

**Example 3.4.** (1) For \(s = 3\), let \(p_1 = 1, p_2 = 5, p_3 = 9, u_2 = -3\) and \(v_3 = -7\). If \(u_1 = 120t - 105, t\) is a positive integer, then \(K_{p_1, p_2, p_3, p_4} = K_{a_1, a_2, p_1, p_2, p_3, p_4}\) is integral for infinite many positive integers \(a_1, a_2\) and \(a_3\).

(2) For any positive integer \(q\), if \(s = 3\), let \(p_i' = p_iq\) and \(u_i' = u_iq\) for \(i = 1, 2, 3\), where \(p_i, u_i\) and \(a_i\) \((i = 1, 2, 3)\) are the same as those of (1) in Example 3.4, then \(K_{p_1', p_2', p_3', p_4'} = K_{a_1, p_1q, a_2, p_2q, a_3, p_3q}\) is integral, too.

**Proof.** (1) From Theorem 2.6, we have that

\[
\begin{align*}
a_1 & = \frac{1}{3}(u_1 + 1), \\
a_2 & = \frac{1}{2}(u_1 + 5), \\
a_3 & = \frac{1}{2}(u_1 + 9).
\end{align*}
\]

So, \(K_{a_1, p_1, p_2, p_3, p_4}\) is integral if and only if \(a_1, a_2\) and \(a_3\) are positive integers. By (14) and (15), we get the Diophantine equation

\[
6a_3 - 5a_2 = 1.
\]

From elementary number theory knowledge, all the positive integral solutions of Eq. (16) are given by \(a_2 = 6t - 5, a_3 = 5t - 4\), where \(t\) is a positive integer, from (13) and (14), we have \(u_1 = 120t - 105, a_1 = 45t - 39\), where \(t\) is a positive integer.

Hence, when \(p_1 = 1, p_2 = 5, p_3 = 9, a_1 = 45t - 39, a_2 = 6t - 5, a_3 = 5t - 4\), where \(t\) is a positive integer, \(K_{p_1, p_2, p_3, p_4} = K_{a_1, p_1, a_2, p_2, a_3, p_3}\) is integral.

(2) From Theorem 2.9 and (1) of Example 3.4, it is easy to prove \(K_{p_1', p_2', p_3', p_4'} = K_{a_1, p_1q, a_2, p_2q, a_3, p_3q}\) is integral, too. \(\square\)

**Example 3.5.** (1) (See Roitman [18]) For \(s = 3\), let \(a_1 = a_2 = a_3 = 1\), \(p_1 = 4u^2(v^3 + u^3), p_2 = 3u^2v^2(u^3 + 6uvv^2)(-u^3 + 5uvv^2), p_3 = 4v^3(u^3 + v^3)\), where \(u, v\) are positive integers such that \((3 - \sqrt{5})v < u < v\), then \(K_{p_1, p_2, p_3} = K_{a_1, a_2, a_3, p_1, p_2, p_3}\) is integral.

(2) For any positive integer \(q\), if \(s = 3\), let \(a_i\) and \(p_i\) \((i = 1, 2, 3)\) be the same as those of (1) in Example 3.5, then \(K_{p_1', p_2', p_3', p_4'} = K_{a_1, p_1q, a_2, p_2q, a_3, p_3q}\) is integral, too.

**Proof.** (1) (See Roitman [18]) We show first that \(p_1, p_2\) and \(p_3\) are positive integers different each other. In fact, the condition \(0 < (3 - \sqrt{5})v < u < v\) ensures \(-u^2 + 6uv - v^2 > 0, p_1 > 0\) and \(p_1 < p_2\). Assume that \(p_1 = p_2\) and let \(u = du_0, v = dv_0\), where \((u_0, v_0) = 1\). Then,

\[
4u_0^2(u_0^2 + v_0^2) = 3u_0^3v_0^2(34u_0^2v_0^2 - u_0^2 - v_0^2)
\]

and so

\[
u_0^3 + v_0^3 \equiv 0 \pmod{3} \Rightarrow u_0 \equiv v_0 \equiv 0 \pmod{3},
\]

contradicting the fact \((u_0, v_0) = 1\). Therefore \(p_1 \neq p_2\). Similarly \(p_3 \neq p_2\).

By Corollary 2.3, we have

\[
P(K_{a_1, p_1, a_2, p_2, a_3, p_3}, x) = x^{p_1 + p_2 + p_3 - 3} \left[ \prod_{i=1}^{3} (x + p_i) - \sum_{i=1}^{3} p_i \prod_{i=1, i \neq j}^{3} (x + p_i) \right]
\]

\[
= x^{-3 + 4u^2v^3 + 3u^2v^3(-u^2 + 6uv - v^2)} \cdot (x - u_1)(x - u_2)(x - u_3),
\]

where \(u_1 = 24u^2v^3(u^2 + v^3), u_2 = -2uv(u^2 + v^3)(-u^2 + 6uvv^2), u_3 = -2uv(u^2 + v^3)^2(u^2 + 6uvv^2).\)
From Theorem 2.4, we know that \( K_{a_1; p_1 q; p_2 q; p_3} = K_{p_1; p_2; p_3} \) is integral.

(2) From Theorem 2.9 and (1) of Example 3.5, it is easy to prove that \( K_{a_1; p_1 q; p_2 q; p_3 q} = K_{p_1; p_2 q; p_3 q} \) is integral, too.

\[ \Box \]

**Theorem 3.6.** For \( s=3 \), let \( q \) be any positive integer, and let \( p_i \) \((i=1, 2, 3)\) be positive integers in Table 1, \( a_i = a_2 = a_3 = 1 \), and \( u_i \) \((i=1, 2, 3)\) be those of Theorem 2.6. Then \( K_{a_1; p_1 q; p_2 q; p_3 q} = K_{p_1; p_2 q; p_3 q} \) is integral.

**Proof.** It is easy to check the correctness by making use of Theorems 2.4, 2.5 or 2.6 and 2.9.

\[ \Box \]

**Remark 3.7.** An infinite family of integral complete tripartite graphs \( K_{p_1; p_2; p_3} \) was constructed in [18]. In Table 1, by using a computer, we have found 34 solutions \((p_1, p_2, p_3)\), where \( s=3 \) and \( p_1 < p_2 < p_3 \), \( 1 \equiv p_1 \equiv 50 \), \( p_1 + 1 \equiv p_2 \equiv p_1 + 50 \), and \( p_1 + 1 \equiv p_3 \equiv p_2 + 100 \). We shall construct infinite many classes of such integral graphs from Theorems 2.4, 2.5, 2.6 and 2.9. They are different from those in existing literature (see [8, 9, 18]). We believe that it is useful for constructing other integral complete tripartite graphs. When \( s=3 \), \( p_1 < p_2 < p_3 \) and \( a_1 = a_2 = a_3 = 1 \). For any positive integer \( q \), the complete tripartite graph \( K_{5; 6,12 q} \) is integral and

\[
\text{Spec}(K_{5; 6,12 q}) = \begin{pmatrix} -10q & -6q & 0 & 16q \\ 1 & 1 & 25q - 3 & 1 \end{pmatrix}.
\]

If \( q = 1 \), let \( s = 3 \), \( p_1 < p_2 < p_3 \) and \( a_1 = a_2 = a_3 = 1 \), we know that the complete tripartite graph \( K_{5; 6,12} \) is an integral one, the order of which is 25, which is much smaller than those given in [8, 9, 18].

**Remark 3.8.** Theorem 3.6 generates an infinite set of vectors \( \vec{p} = (p_1, p_2, p_3) \) for which (2) has integral solutions only. But there is only finite number of primitive vectors in this infinite set (in general, the primitive vectors are the only ones which are of interest). The infinite series built in [18] gives an infinite series of the primitive solutions. Thus Theorem 3.6 is much weaker than the result of [18]. However, by analyzing Table 1 one can see that all its rows except the row \( \vec{p} = (5, 13, 77) \) have the following property: \( u_1 / p_1 = -2 \) for a suitable \( i \in \{1, 2, 3\} \). This observation gives a hint to a new infinite series of primitive triples \( \vec{p} \), see the following, which is due to a referee.

Let \( u_3 < u_2 < u_1 \) be the roots of \( F(x) = F(\vec{p})(x) = 0 \). Set \( v_3 = -u_3 \), \( v_2 = -u_2 \), \( v_1 = u_1 \). Then \( v_i 's \) are positive integers which satisfy the following conditions:

\[
u_1 = v_2 + v_3,
\]
\[
2 v_2^2 + v_2 v_3 + v_3^2 = p_1 p_2 + p_1 p_3 + p_2 p_3,
\]
\[
(v_2 + v_3) v_2 v_3 = 2 p_1 p_2 p_3.
\]

(17)
Let us look for solutions of (17) such that $v_3 = 2p_i$ for some $i = 1, 2, 3$. Forgetting about ordering of $p_i$’s we may assume that $v_3 = 2p_3$. Then (17) is equivalent to the following:

$$p_1 + p_2 = 2v_3,$$

$$p_1p_2 = (v_2 + v_3)v_2.$$  \hfill (18)

These equations have an integral solutions for $p_1$, $p_2$ if and only if $v_2^2 - (v_2 + v_3)v_2$ is a perfect square, say $m^2$. Then $p_1 = v_3 - m$, $p_2 = v_3 + m$, $p_3 = v_3/2$.

$$v_2^2 - (v_2 + v_3)v_2 = m^2 \iff$$

$$\iff x = \frac{v_3}{m}, \quad y = \frac{v_2}{m}, \quad x^2 - xy - y^2 = 1 \iff$$

$$\iff x = \frac{t^2 + 1}{t^2 + t - 1}, \quad y = t(x - 1), \quad t \in \mathbb{Q}.$$  

We may assume that $m > 0$. It follows from $p_1 = m(x - 1) > 0$ and $v_2 > 0$ that $x > 1$ and $t > 0$. The first inequality is equivalent to

$$\frac{2 - t}{t^2 + t - 1} > 0 \iff \frac{\sqrt{5} - 1}{2} = 0.618.. < t < 2 \text{ or } t < -\frac{\sqrt{5} - 1}{2} = -1.618...$$  

Thus $p_1 = v_3 - m = m(x - 1)$, $p_2 = v_3 + m = m(x + 1)$, $p_3 = v_3/2 = mx/2$ are nonnegative.

Write $t = a/b$ where $(a, b) = 1$ and $a > 0, b > 0$ we obtain

$$\frac{v_3}{m} = x = \frac{a^2 + b^2}{a^2 + ab - b^2}, \quad \frac{v_2}{m} = y = \frac{2ab - a^2}{a^2 + ab - b^2}.$$  

After some routine transformations we obtain

$$p_1 = \frac{2b(2b - a)}{d}, \quad p_2 = \frac{2a(2a + b)}{d}, \quad p_3 = \frac{a^2 + b^2}{d},$$

$$u_1 = \frac{2b(2a + b)}{d}, \quad u_2 = \frac{2a(2b - a)}{d}, \quad u_3 = \frac{2(a^2 + b^2)}{d},$$  \hfill (19)

where $d = (2b(2b - a), 2a(2a + b), a^2 + b^2)$. Note that $d \in \{1, 2, 5, 10\}$.

Take for example $a = 2$, $b = 3$. Then $d = 1$ and

$$p_1 = 24, \quad p_2 = 28, \quad p_3 = 13, \quad u_1 = 42, \quad u_2 = -16, \quad u_3 = -26.$$  

This is one of the triple given Table 1. All triple given in this table except $(5, 12, 77)$ may be obtained from (19).

Note that the above numbering of $p_i$’s may not coincide with one fixed before.

Hence, we obtain the following result.

**Theorem 3.9.** For $s = 3$, let $q$ be any positive integer, and let $p_i$ and $u_i$ ($i = 1, 2, 3$) be positive integers in the above formulae (19), $a_1 = a_2 = a_3 = 1$. Then $K_{a_1, p_1; a_2, p_2; a_3, p_3} = K_{p_1, q; p_2, q; p_3}$ is integral.

**Example 3.10.** For any positive integer $q$, if $s = 3$, let $p_1 = q$, $p_2 = 3q$, $p_3 = 5q$, $u_2 = -2q$, $u_3 = -4q$, then there do not exist positive integers $a_1, a_2, a_3$ such that $K_{a_1, p_1; a_2, p_2; a_3, p_3}$ is integral.

**Proof.** When $s = 3$, $p_1 = q$, $p_2 = 3q$, $p_3 = 5q$, $u_2 = -2q$, $u_3 = -4q$. Suppose that we can construct integral graphs $K_{a_1, p_1; a_2, p_2; a_3, p_3}$. From Theorem 2.6, we know that $K_{a_1, p_1; a_2, p_2; a_3, p_3}$ is integral if and only if there exist integers $u_i$ and positive integers $p_i$ ($i = 1, 2, 3$) satisfying $-p_3 < u_3 < -p_2 < u_2 < -p_1 < u_1 < +\infty$ such that

$$a_k = \prod_{i=1}^{k}(p_i + u_i) \prod_{i=1}^{k-1}p_i$$

($k = 1, 2, 3$) are positive integers.
Hence, we obtain
\[ a_1 = \frac{3}{8q} (u_1 + q), \]
\[ a_2 = \frac{1}{12q} (u_1 + 3q), \]
\[ a_3 = \frac{3}{40q} (u_1 + 5q). \] 

By (21) and (22), we have
\[ 20a_3 - 18a_2 = 3. \] 

From Lemma 3.2, we know that there are no integral solutions for Eq. (23).

Hence, we cannot construct integral graphs \( K_{a_1', p_1 a_2', p_2 a_3'} \).

**Theorem 3.11.** For the complete \( r \)-partite graph \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, p_1 a_2, p_2, \ldots, a_r, p_r} \) on \( n \) vertices, let \( m, s \) and \( q \) be positive integers, and \( s \geq 3 \), then we have

1. If \( p_i = m + 2(i - 1) \) for \( i = 1, 2, \ldots, s \), then there are no integers \( a_i \) (\( i = 1, 2, \ldots, s \)) such that \( K_{a_1, p_1 a_2, p_2 a_3, \ldots, a_r, p_r} \) is an integral graph.
2. If \( p_1' = p_2' = \cdots = p_r' = \lfloor m + 2(i - 1) \rfloor q \) for \( i = 1, 2, \ldots, s \), let \( u_j = -(m + 2j - 3)q \) for \( j = 2, 3, \ldots, s \), then there are no integers \( a_i' \) (\( i = 1, 2, \ldots, s \)) such that \( K_{a_1', p_1 a_2', p_2 a_3', \ldots, a_r', p_r'} \) is an integral graph.

**Proof.** (1) Suppose that we can construct an integral graph \( K_{a_1, p_1 a_2, p_2 a_3, \ldots, a_r, p_r} \). From Theorem 2.6, we know that \( K_{a_1, p_1 a_2, p_2 a_3, \ldots, a_r, p_r} \) is integral if and only if there exist integers \( u_i \) and positive integers \( p_i \) (\( i = 1, 2, \ldots, s \)) satisfying \(-p_i < u_i < -p_{i-1} < u_{i-1} < \cdots < u_2 < -p_1 < u_1 < +\infty \) such that all \( a_k \) (\( k = 1, 2, \ldots, s \)) are positive integers, where

\[ a_k = \frac{\prod_{i=1}^{k} (p_i - u_i)}{\prod_{i=1}^{k} (p_i + u_i)} \] 

for \( k = 1, 2, \ldots, s \). Hence, we can only choose
\[ u_j = -(m + 2j - 3), \quad j = 2, 3, \ldots, s. \]

We obtain
\[ a_{s-1} = \frac{(m + 2s - 4 + u_1) \cdot (2s - 5)!}{2(m + 2s - 4) \cdot (2s - 4)!}, \]
\[ a_s = \frac{(m + 2s - 2 + u_1) \cdot (2s - 3)!}{(m + 2s - 2) \cdot (2s - 2)!}. \]

From (24) and (25), we have
\[ (m + 2s - 2) \cdot (2s - 2)! \cdot a_s - 2(m + 2s - 4)(2s - 3) \cdot (2s - 4)! \cdot a_{s-1} = 2 \cdot (2s - 3)!. \] 

Since \( s \geq 3 \), let \( d = ((m + 2s - 2 + u_1) \cdot (2s - 2)!, 2(m + 2s - 4)(2s - 3) \cdot (2s - 4)!) \), then \( d = 2 \cdot (2s - 4)! \cdot ((m + 2s - 2 + u_1) \cdot (s - 1), (m + 2s - 4)(2s - 3)) \). Thus, \( d \{2 \cdot (2s - 3)!!\} \). From Lemma 3.2, we know that there are no integral solutions \((a_{s-1}, a_s)\) for Eq. (26).

Hence, \( K_{a_1, p_1 a_2, p_2 a_3, \ldots, a_r, p_r} \) cannot be an integral graph.

(2) Suppose that we can construct an integral graph \( K_{a_1', p_1 a_2', p_2 a_3', \ldots, a_r', p_r'} \). From Theorem 2.6, we similarly obtain
\[ a'_{s-1} = \frac{[q(m + 2s - 4) + u'_1] \cdot (2s - 5)!}{2q(m + 2s - 4) \cdot (2s - 4)!}, \]
\[ a'_s = \frac{[q(m + 2s - 2) + u'_1] \cdot (2s - 3)!}{q(m + 2s - 2) \cdot (2s - 2)!}. \]

From (27) and (28), we have
\[ (m + 2s - 2) \cdot (2s - 2)! \cdot a'_s - 2(m + 2s - 4)(2s - 3) \cdot (2s - 4)! \cdot a'_{s-1} = 2 \cdot (2s - 3)!!. \]

From Lemma 3.2, we know that there are no integral solutions \((a'_{s-1}, a'_s)\) for Eq. (29).

Hence, \( K_{a_1', p_1 a_2', p_2 a_3', \ldots, a_r', p_r'} \) cannot be an integral graph.
4. Further discussion

In this paper, we have mainly investigated integral complete \( r \)-partite graph \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_r, p_s} \) on \( n \) vertices. When \( s = 1, 2, 3 \), some results of such integral graphs can be found in [8,9,18] and the present paper. When \( s \geq 4 \), we have not found such integral graphs. We tried to get some general results. Thus, we raise the following questions for further study.

**Question 4.1.** Are there any integral complete \( r \)-partite graphs \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_r, p_s} \) with arbitrarily large \( s \)?

For complete \( r \)-partite graphs \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_r, p_s} \), when \( s = 1, 2, 3 \), let \( a_1 = a_2 = \cdots = a_s = 1 \), some results of such integral graphs can be found in [8,9,18] and the present paper. However, when \( s \geq 4 \), \( a_1 = a_2 = \cdots = a_s = 1 \), we have not found such integral graphs. Hence, we have

**Question 4.2.** Are there any integral complete \( r \)-partite graphs \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_r, p_s} \) with \( a_1 = a_2 = \cdots = a_s = 1 \) when \( s \geq 4 \)?

For complete \( r \)-partite graphs \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_r, p_s} \), we give a sufficient and necessary condition for \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, \ldots, a_r, p_s} \) to be integral. In particular, when \( s = 1, 2 \), we got all parameter solutions for \( K_{a_1, a_2, p_2} \) to be integral graphs in [8,9] and the present paper. When \( s \geq 3 \), we have not got such general good results. Hence, we have

**Question 4.3.** When \( s = 3, 4, 5, \ldots \), can we give a better sufficient and necessary condition for \( K_{a_1, a_2, \ldots, a_r, p_s} \) to be integral?

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