Available online at www.sciencedirect.com
SCIENCE DIRECT
DISCRETE MATHEMATICS
Discrete Mathematics 283 (2004) 231–241
www.elsevier.com/locate/disc

Integral complete r-partite graphs☆

Ligong Wang[a], Xueliang Li[b], Cornelis Hoede[c]

[a]Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, People's Republic of China
[b]Center for Combinatorics, Nankai University, Tianjin, 300071, People's Republic of China
[c]Faculty of Mathematical Science, University of Twente, P.O.Box 217, Enschede 7500 AE, The Netherlands

Received 15 April 2002; received in revised form 4 February 2004; accepted 19 February 2004

Abstract

A graph is called integral if all the eigenvalues of its adjacency matrix are integers. In this paper, we give a useful sufficient and necessary condition for complete r-partite graphs to be integral, from which we can construct infinite many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. The discovery of these integral complete r-partite graphs is a new contribution to the search of such integral graphs. Finally, we propose several basic open problems for further study.

MSC: 05C05; 11D09; 11D41

Keywords: Integral graph; Complete r-partite graph; Diophantine equation; Graph spectrum

1. Introduction

We shall consider only simple undirected graphs (i.e. undirected graphs without loops or multiple edges). For a graph G, let $V(G)$ denote the vertex set and $E(G)$ the edge set. The characteristic polynomial $|xI - A|$ of the adjacency matrix A (or $A(G)$) of G is called the characteristic polynomial of G and denoted by $P(G,x)$. The spectrum of $A(G)$ is also called the spectrum of G.

The notion of integral graphs was first introduced by Harary and Schwenk [10]. A graph G is called integral if all the zeros of the characteristic polynomial $P(G,x)$ are integers. In general, the problem of characterizing integral graphs seems to be difficult. Thus it makes sense to restrict our investigations to some interesting families of graphs, for instance, cubic graphs [4,20], complete tripartite graphs [18], graphs with three eigenvalues [17], graphs with maximum degree 4 [2], etc. Trees present another important family of graphs for which the problem has been considered in [5–7,11–16,21,22, 24–26]. Other results on integral graphs can be found in [1,8,9,23]. For all other facts on graph spectra (or terminology), see [8,9].

A complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$ is a graph with a set $V = V_1 \cup V_2 \cup \cdots \cup V_r$ of $p_1 + p_2 + \cdots + p_r(=n)$ vertices, where V_i's are nonempty disjoint sets, $|V_i| = p_i$ for $1 \leqslant i \leqslant r$, such that two vertices in V are adjacent if and only if they belong to different V_i's. An infinite family of integral complete tripartite graphs was constructed in [18], where the author mentioned the general problem on integral complete multipartite graphs. He thought that it is possible that for $r > 3$ there also exist an infinite number of integral complete r-partite graphs. But he did not find such integral graphs. Balińska and Simić [2] thought that the general problem seems to be intractable. In this paper, we give a useful sufficient and necessary

☆ Supported by National Science Foundation of China, the Fund of "The Developing Program for Outstanding Persons" in NPU and S & T Innovative Foundation for Young Teachers of Northwestern Polytechnical University. The research of this paper was partially done while the first author was visiting the University of Twente during the visit from November 25, 2001 to December 22, 2001.

E-mail addresses: ligongwangnpu@yahoo.com.cn (L. Wang), x.li@eyou.com (X. Li), hoede@math.utwente.nl (C. Hoede).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2004.02.011

condition for complete r-partite graphs to be integral, from which we can construct infinite many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. The discovery of these integral complete r-partite graphs is a new contribution to the search of such integral graphs. In fact, M. Roitman's result on the integral complete tripartite graphs is generalized in this paper (see also [18, MR 86a:05089]). Finally, we propose several basic open problems for further study.

2. A sufficient and necessary condition for complete r-partite graphs to be integral

In this section, we shall give a useful sufficient and necessary condition for complete r-partite graphs to be integral. The following lemma can be found in [9].

Lemma 2.1. *For the complete r-partite graph $K_{p_1,p_2,\dots,p_r}$ on n vertices, we have*

$$P(K_{p_1,p_2,\dots,p_r},x) = x^{n-r}\left(1-\sum_{i=1}^{r}\frac{p_i}{x+p_i}\right)\prod_{j=1}^{r}(x+p_j).$$

Assume that the number of distinct integers of $p_1, p_2, \dots, p_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $p_1 < p_2 < \cdots < p_s$. Suppose that a_i is the multiplicity of p_i for each $i = 1, 2, \dots, s$. The complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{p_1,\dots,p_1,\dots,p_s,\dots,p_s}$ is also denoted by $K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = n = \sum_{i=1}^{s} a_i p_i$.

Example 2.2 (See Cvetković et al. [9]). For the complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$, if $s=2$, $a_1=a_2=1$, then K_{p_1,p_2} is integral if and only if $p_1 p_2$ is a perfect square.

Corollary 2.3. *For the complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ on n vertices, we have*

$$P(K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s},x) = x^{n-r}\prod_{i=1}^{s}(x+p_i)^{a_i-1}\left[\prod_{i=1}^{s}(x+p_i) - \sum_{j=1}^{s} a_j p_j \prod_{i=1,i\neq j}^{s}(x+p_i)\right].$$

The following theorem is immediate.

Theorem 2.4. *The complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ is integral if and only if*

$$\prod_{i=1}^{s}(x+p_i) - \sum_{j=1}^{s} a_j p_j \prod_{i=1,i\neq j}^{s}(x+p_i) = 0 \tag{1}$$

has no other roots but integral ones.

We shall discuss Eq. (1) to get more information. First, we divide both sides of Eq. (1) by $\prod_{i=1}^{s}(x+p_i)$, and obtain

$$\sum_{i=1}^{s}\frac{a_i p_i}{x+p_i} = 1. \tag{2}$$

Let $F(x) = \sum_{i=1}^{s} [(a_i p_i)/(x+p_i)] - 1$. Clearly, $-p_i$'s are not roots of Eq. (1) for $1 \leqslant i \leqslant s$. Hence, all solutions of Eq. (1) are the same as those of Eq. (2). Now we consider the roots of $F(x)$ over the set of real numbers. Note that $F(x)$ is discontinuous at each point p_i. For $1 \leqslant i \leqslant s$, we have that $F(-p_i-0)=-\infty$, $F(-p_i+0)=+\infty$, $F(-\infty)=F(+\infty)=-1$, $F'(x)=-\sum_{i=1}^{s}[(a_i p_i)/(x+p_i)^2]$. We deduce that $F(x)$ is strictly monotone decreasing on each of the continuous intervals over the set of real numbers. Using Zero Point Theorem of Mathematical Analysis, we get that $F(x)$ has s distinct real roots. If $-\infty < u_s < u_{s-1} < \cdots < u_1 < +\infty$ are the roots of $F(x)$, then

$$-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty \tag{3}$$

holds.

On the other hand, we note that Eq. (2) can be written as

$$\frac{a_1 p_1}{x+p_1}+\frac{a_2 p_2}{x+p_2}+\cdots+\frac{a_s p_s}{x+p_s}=1. \tag{4}$$

From the above discussion, we have.

Theorem 2.5. *The complete r-partite graph $K_{p_1,p_2,\dots,p_r}=K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ is integral if and only if all the solutions of Eq. (4) are integers. Moreover, there exist integers $u_1,u_2,\dots,u_s$ satisfying (3) such that the following linear equation system in $a_1,a_2,\dots,a_s$*

$$\begin{cases} \dfrac{a_1 p_1}{u_1+p_1}+\dfrac{a_2 p_2}{u_1+p_2}+\cdots+\dfrac{a_s p_s}{u_1+p_s}=1, \\ \cdots\cdots\cdots \quad \cdots\cdots\cdots \\ \dfrac{a_1 p_1}{u_s+p_1}+\dfrac{a_2 p_2}{u_s+p_2}+\cdots+\dfrac{a_s p_s}{u_s+p_s}=1 \end{cases} \tag{5}$$

has positive integral solutions $(a_1,a_2,\dots,a_s)$.

Theorem 2.6. *The complete r-partite graph $K_{p_1,p_2,\dots,p_r}=K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ on n vertices is integral if and only if there exist integers u_i and positive integers p_i $(i=1,2,\dots,s)$ such that $-p_s<u_s<-p_{s-1}<u_{s-1}<\cdots<-p_2<u_2<-p_1<u_1<+\infty$ and*

$$a_k=\frac{\prod_{i=1}^{s}(p_k+u_i)}{p_k\prod_{i=1,i\neq k}^{s}(p_k-p_i)} \quad (k=1,2,\dots,s) \tag{6}$$

are positive integers.

Proof. From Cauchy's result on determinants in [3], we know that

$$\begin{vmatrix} \dfrac{1}{\alpha_1+\beta_1} & \cdots & \dfrac{1}{\alpha_1+\beta_s} \\ \vdots & \ddots & \vdots \\ \dfrac{1}{\alpha_s+\beta_1} & \cdots & \dfrac{1}{\alpha_s+\beta_s} \end{vmatrix} = \frac{\prod_{1\leqslant i<j\leqslant s}(\alpha_j-\alpha_i)(\beta_j-\beta_i)}{\prod_{1\leqslant i,j\leqslant s}(\alpha_i+\beta_j)}.$$

The determinant of the coefficient matrix D of the linear equation system (5) is the following:

$$|D|=\begin{vmatrix} \dfrac{p_1}{u_1+p_1} & \cdots & \dfrac{p_s}{u_1+p_s} \\ \vdots & \ddots & \vdots \\ \dfrac{p_1}{u_s+p_1} & \cdots & \dfrac{p_s}{u_s+p_s} \end{vmatrix} = \prod_{i=1}^{s} p_i \begin{vmatrix} \dfrac{1}{u_1+p_1} & \cdots & \dfrac{1}{u_1+p_s} \\ \vdots & \ddots & \vdots \\ \dfrac{1}{u_s+p_1} & \cdots & \dfrac{1}{u_s+p_s} \end{vmatrix}$$

$$=\frac{\prod_{i=1}^{s} p_i\prod_{1\leqslant i<j\leqslant s}(u_j-u_i)(p_j-p_i)}{\prod_{1\leqslant i,j\leqslant s}(u_i+p_j)}\neq 0,$$

$$|D_k|=\begin{vmatrix} \dfrac{p_1}{u_1+p_1} & \dfrac{p_2}{u_1+p_2} & \cdots & \dfrac{p_{k-1}}{u_1+p_{k-1}} & 1 & \dfrac{p_{k+1}}{u_1+p_{k+1}} & \cdots & \dfrac{p_{s-1}}{u_1+p_{s-1}} & \dfrac{p_s}{u_1+p_s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{p_1}{u_s+p_1} & \dfrac{p_2}{u_s+p_2} & \cdots & \dfrac{p_{k-1}}{u_s+p_{k-1}} & 1 & \dfrac{p_{k+1}}{u_s+p_{k+1}} & \cdots & \dfrac{p_{s-1}}{u_s+p_{s-1}} & \dfrac{p_s}{u_s+p_s} \end{vmatrix}$$

$$=\lim_{p_k\to+\infty}|D|=\frac{\prod_{i=1,i\neq k}^{s} p_i\prod_{1\leqslant i<j\leqslant s}(u_j-u_i)(p_j-p_i)}{\prod_{1\leqslant i,j\leqslant s,j\neq k}(u_i+p_j)\prod_{i=1,i\neq k}^{s}(p_k-p_i)}$$

for $k=1,2,\dots,s$.

By using the well-known Cramer's Rule to solve the linear equation system (5) in $a_1, a_2, \ldots, a_s$, we obtain that

$$a_k = \frac{\prod_{i=1}^{s}(p_k + u_i)}{p_k \prod_{i=1, i\neq k}^{s}(p_k - p_i)} \qquad (k = 1, 2, \ldots, s).$$

Because $-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty$ and $p_i \geqslant 1$ for $i = 1, 2, \ldots, s$, we can deduce that $a_k > 0$ $(k = 1, 2, \ldots, s)$ and $u_1 > 0$.

The remaining part of the theorem can be easily proved from Lemma 2.1 and Theorems 2.4 and 2.5. □

Corollary 2.7. *If the complete r-partite graph* $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ *is integral, then we have the following results*:

(1) $a_k > 0$ *for* $k = 1, 2, \ldots, s$, *and* $u_1 > 0$.
(2) $\sum_{i=1}^{s} u_i = \sum_{i=1}^{s}(a_i - 1)p_i$.
(3) $\prod_{i=1}^{s} u_i = \prod_{i=1}^{s} p_i(1 - \sum_{i=1}^{s} a_i)$.
(4)

$$\mathrm{Spec}(K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}) = \begin{pmatrix} -p_s & u_s & -p_{s-1} & u_{s-1} & \ldots & -p_2 & u_2 & -p_1 & 0 & u_1 \\ a_s - 1 & 1 & a_{s-1} - 1 & 1 & \ldots & a_2 - 1 & 1 & a_1 - 1 & \sum_{i=1}^{s} a_i(p_i - 1) & 1 \end{pmatrix}.$$

Proof. It is easy to check the correctness from Corollary 2.3. □

The following lemma is due to a referee.

Lemma 2.8. *Denote*

$$\Psi_{\vec{a}, \vec{p}}(x) = \sum_{i=1}^{s} \frac{a_i p_i}{x + p_i},$$

$$\Phi_{\vec{a}, \vec{p}}(x) = \left(\prod_{i=1}^{s}(x + p_i)\right)(1 - \Psi_{\vec{a}, \vec{p}}(x)),$$

where vectors

$$\vec{a} = (a_1, a_2, \ldots, a_s), \qquad \vec{p} = (p_1, p_2, \ldots, p_s) \in \mathbb{Z}^s.$$

Let q be a nonzero integer. Then u is an integral root of $\Phi_{\vec{a}, q\vec{p}}(x)$ *if and only if* u/q *is an integral root of* $\Phi_{\vec{a}, \vec{p}}(x)$.

Proof. It is easy to see that v is a root of $\Phi_{\vec{a}, \vec{p}}(x)$ if and only if qv is a root of $\Phi_{\vec{a}, q\vec{p}}(x)$. Therefore if all the roots of $\Phi_{\vec{a}, \vec{p}}(x)$ are integers, then the roots of $\Phi_{\vec{a}, q\vec{p}}(x)$ are integers as well.

Assume now that all roots of $\Phi_{\vec{a}, q\vec{p}}(x)$ are integral and let v be one of them. Then v/q is a rational root of $\Phi_{\vec{a}, \vec{p}}(x)$. Since $\Phi_{\vec{a}, \vec{p}}(x)$ is a monic polynomial with integral coefficients, its rational roots should be integers. Therefore $v/q \in \mathbb{Z}$. □

From the above lemma we can obtain the following result.

Theorem 2.9. *For any positive integer q, the complete r-partite graph* $K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, \ldots, a_s \cdot p_s q}$ *is integral if and only if the complete r-partite graph* $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ *is integral.*

Remark 2.10. Theorem 2.9 shows that it is reasonable to study Eq. (2) only when $(p_1, p_2, \ldots, p_s) = 1$. Let us call such a vector primitive. So, in general, the primitive vectors are the only ones which are of interest.

3. Integral complete *r*-partite graphs

In this section, we shall construct infinite many new classes of integral complete r-partite graphs, different from those of [8,9,18].

The idea of constructing such integral graph is as follows: First, we properly choose positive integers $p_1, p_2, \ldots, p_s$. Then, we try to find integers u_i $(i = 1, 2, \ldots, s)$ satisfying (3) such that there are positive integral solutions $(a_1, a_2, \ldots, a_s)$ for the linear equation system (5) (or such that all a_k's of (6) are positive integers). Finally, we obtain positive integers $a_1, a_2, \ldots, a_s$ such that all the solutions of Eq. (4) are integers. Thus, we have constructed many new classes of integral graphs $K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$.

Example 3.1. Let $p_1 = 1, p_2 = 9$ and $u_2 = -4$. If $u_1 = 72t - 9$ and t is a positive integer, then $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2}$ is integral for infinite many positive integers a_1, a_2 given by (7) and (8).

Proof. From Theorem 2.6, we have that

$$a_1 = \frac{(p_1 + u_1)(p_1 + u_2)}{p_1(p_1 - p_2)} = \tfrac{3}{8}(u_1 + 1) \tag{7}$$

and

$$a_2 = \frac{(p_2 + u_1)(p_2 + u_2)}{p_2(p_2 - p_1)} = \tfrac{5}{72}(u_1 + 9). \tag{8}$$

So, $K_{a_1 \cdot p_1, a_2 \cdot p_2}$ is integral if and only if a_1 and a_2 are positive integers. From (7) and (8), we get the Diophantine equation

$$27a_2 - 5a_1 = 15. \tag{9}$$

From elementary number theory knowledge, all positive integral solutions of Eq. (9) are given by $a_1 = 27t - 3, a_2 = 5t$, and $u_1 = 72t - 9$, where t is a positive integer.

Hence, $K_{p_1, p_2, p_3, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2}$ is integral for the above infinite many integers a_1 and a_2. □

The following Lemma 3.2 can be found in [19].

Lemma 3.2. *Let a, b and c be integers with $d = (a, b)$, we have*

(1) *If $d \nmid c$, then the linear Diophantine equation in two variables*

$$ax + by = c \tag{10}$$

does not have integral solutions.

(2) *If $d \mid c$, then there are infinite many integral solutions for Eq. (10). Moreover, if $x = x_0, y = y_0$ is a particular solution of Eq. (10), then all its solutions are given by*

$$x = x_0 + (b/d)t, \quad y = y_0 - (a/d)t,$$

where t is an integer.

Theorem 3.3. *For $s = 2$, let $p_1 < p_2$. Then $K_{a_1 \cdot p_1, a_2 \cdot p_2}$ is integral if and only if one of the following two conditions holds*:

(1) *When $(m, k) = 1$, let $p_1 = m, p_2 = m + k$, $m \geqslant 1$, $k \geqslant 2$, $1 \leqslant q < k$, where m, k and q are positive integers. Then, a_1 and a_2 must be positive integral solutions for the Diophantine equation*

$$q(m + k)a_2 - m(k - q)a_1 = q(k - q). \tag{11}$$

(2) *When $(m, k) = d \geqslant 2$, let $p_1 = m, p_2 = m + k$, $m = m_1 d$, $k = k_1 d$, $(m_1, k_1) = 1$, $q = q_1 d$, $1 \leqslant q_1 < k_1$, where m_1, k_1, q_1 and d are positive integers. Then, a_1 and a_2 must be positive integral solutions for the Diophantine equation*

$$q_1(m_1 + k_1)a_2 - m_1(k_1 - q_1)a_1 = q_1(k_1 - q_1). \tag{12}$$

Proof. Because $p_1 < p_2$, from Theorem 2.6, we know $K_{a_1 \cdot p_1, a_2 \cdot p_2}$ is integral if and only if there exist integers u_1, u_2 and positive integers p_1, p_2 satisfying $-p_2 < u_2 < -p_1 < u_1 < +\infty$ such that a_1 and a_2 are positive integers, where $a_k = \frac{\prod_{i=1}^{2}(p_k + u_i)}{p_k \prod_{i=1, i \neq k}^{2}(p_k - p_i)}$ for $k = 1, 2$. Hence, we choose $p_1 = m$, $p_2 = m + k$, $u_2 = -(m + q)$, $m \geqslant 1$, $k \geqslant 2$, $1 \leqslant q < k$, where m, k and q are positive integers, and we have

$$a_1 = \frac{q(m + u_1)}{mk}, \quad a_2 = \frac{(m + k + u_1)(k - q)}{k(m + k)}.$$

Hence, we get Eq. (11). From Lemma 3.2, we know there are solutions for Eq. (11) if and only if $d_1|q(k-q)$, where $d_1=(q(m+k),m(k-q))$.

Now, we discuss the following two cases.

Case 1: When $(m,k)=1$, we have $(m+k,m)=1$, and $d_1|q(k-q)$. Moreover, there are solutions for Eq. (11). From Lemma 3.2 and the condition $(m,k)=1$, we know that there are infinite many integral solutions for Eq. (11). Therefore, there are infinite many positive integral solutions (a_1,a_2) for Eq. (11).

Case 2: When $(m,k)=d\geqslant 2$, let $m=m_1d$, $k=k_1d$, $(m_1,k_1)=1$, where m_1,k_1 and d are positive integers. We have $(m_1+k_1,m_1)=1$, $d_1=(qd(m_1+k_1),m_1d(k_1d-q))$. If $d_1|q(k-q)=q(k_1d-q)$, then $d|q$. Thus, let $q=q_1d$, $1\leqslant q_1<k_1$, where q_1 is a positive integer. We can reduce Eq. (11)–(12). Hence, from Lemma 3.2 and the condition $(m_1,k_1)=1$, we know that there are infinite many integral solutions for Eq. (12). Therefore, there are infinite many positive integral solutions (a_1,a_2) for Eq. (12).

Thus, the theorem is proved. □

Example 3.4. (1) For $s=3$, let $p_1=1, p_2=5, p_3=9, u_2=-3$ and $u_3=-7$. If $u_1=120t-105$, t is a positive integer, then $K_{p_1,p_2,\dots,p_r}=K_{a_1\cdot p_1,a_2\cdot p_2,a_3\cdot p_3}$ is integral for infinite many positive integers a_1,a_2 and a_3.

(2) For any positive integer q, if $s=3$, let $p_i'=p_iq$ and $u_i'=u_iq$ for $i=1,2,3$, where p_i, u_i and a_i $(i=1,2,3)$ are the same as those of (1) in Example 3.4, then $K_{a_1\cdot p_1',a_2\cdot p_2',a_3\cdot p_3'}=K_{a_1\cdot p_1q,a_2\cdot p_2q,a_3\cdot p_3q}$ is integral, too.

Proof. (1) From Theorem 2.6, we have that

$$a_1=\tfrac{3}{8}(u_1+1), \tag{13}$$

$$a_2=\tfrac{1}{20}(u_1+5), \tag{14}$$

$$a_3=\tfrac{1}{24}(u_1+9). \tag{15}$$

So, $K_{a_1\cdot p_1,a_2\cdot p_2,a_3\cdot p_3}$ is integral if and only if a_1, a_2 and a_3 are positive integers. By (14) and (15), we get the Diophantine equation

$$6a_3-5a_2=1. \tag{16}$$

From elementary number theory knowledge, all the positive integral solutions of Eq. (16) are given by $a_2=6t-5, a_3=5t-4$, where t is a positive integer, from (13) and (14), we have $u_1=120t-105$, $a_1=45t-39$, where t is a positive integer.

Hence, when $p_1=1, p_2=5, p_3=9, a_1=45t-39$, $a_2=6t-5, a_3=5t-4$, where t is a positive integer, $K_{p_1,p_2,\dots,p_r}=K_{a_1\cdot p_1,a_2\cdot p_2,a_3\cdot p_3}$ is integral.

(2) From Theorem 2.9 and (1) of Example 3.4, it is easy to prove $K_{p_1',p_2',\dots,p_r'}=K_{a_1\cdot p_1',a_2\cdot p_2',a_3\cdot p_3'}=K_{a_1\cdot p_1q,a_2\cdot p_2q,a_3\cdot p_3q}$ is integral, too. □

Example 3.5. (1) (See Roitman [18]) For $s=3$, let $a_1=a_2=a_3=1$, $p_1=4u^2(u^2+v^2)^3$, $p_2=3u^2v^2(u^2+6uv+v^2)(-u^2+6uv-v^2)$, $p_3=4v^2(u^2+v^2)^3$, where u,v are positive integers such that $(3-\sqrt{8})v<u<v$, then $K_{p_1,p_2,\dots,p_r}=K_{a_1\cdot p_1,a_2\cdot p_2,a_3\cdot p_3}$ is integral.

(2) For any positive integer q, if $s=3$, let a_i and p_i $(i=1,2,3)$ be the same as those of (1) in Example 3.5, then $K_{a_1\cdot p_1q,a_2\cdot p_2q,a_3\cdot p_3q}=K_{p_1q,p_2q,p_3q}$ is integral, too.

Proof. (1) (See Roitman [18]) We show first that p_1,p_2 and p_3 are positive integers different each other. In fact, the condition $0<(3-\sqrt{8})v<u<v$, ensures $-u^2+6uv-v^2>0$, $p_3>0$ and $p_1<p_3$. Assume that $p_1=p_2$ and let $u=du_0$, $v=dv_0$, where $(u_0,v_0)=1$. Then,

$$4u_0^2(u_0^2+v_0^2)^3=3u_0^2v_0^2(34u_0^2v_0^2-u_0^4-v_0^4)$$

and so

$$u_0^2+v_0^2\equiv 0\,(\mathrm{mod}\,3)\quad\Rightarrow\quad u_0\equiv v_0\equiv 0\,(\mathrm{mod}\,3),$$

contradicting the fact $(u_0,v_0)=1$. Therefore $p_1\neq p_2$. Similarly $p_3\neq p_2$.

By Corollary 2.3, we have

$$\begin{aligned}P(K_{a_1\cdot p_1,a_2\cdot p_2,a_3\cdot p_3},x)&=x^{p_1+p_2+p_3-3}\left[\prod_{i=1}^{3}(x+p_i)-\sum_{j=1}^{3}p_j\prod_{i=1,i\neq j}^{3}(x+p_i)\right]\\&=x^{-3+4(u^2+v^2)^4+3u^2v^2(-u^2+6uv-v^2)(u^2+6uv+v^2)}(x-u_1)(x-u_2)(x-u_3),\end{aligned}$$

where $u_1=24u^2v^2(u^2+v^2)^2$, $u_2=-2uv(u^2+v^2)^2(-u^2+6uv-v^2)$, $u_3=-2uv(u^2+v^2)^2(u^2+6uv+v^2)$.

Table 1
Integral complete tripartite graph K_{p_1q,p_2q,p_3q}, where q is a positive integer

p_1	p_2	p_3	u_1	u_2	u_3	p_1	p_2	p_3	u_1	u_2	u_3
3	17	65	39	−5	−34	4	13	48	32	−6	−26
5	8	12	16	−6	−10	5	12	77	40	−7	−33
6	34	130	78	−10	−68	7	13	45	35	−9	−26
8	26	96	64	−12	−52	9	25	91	63	−13	−50
10	16	24	32	−12	−20	12	17	56	48	−14	−34
12	25	88	66	−16	−50	13	24	28	42	−16	−26
14	26	90	70	−18	−52	15	24	36	48	−18	−30
15	37	133	95	−21	−74	16	37	132	96	−22	−74
17	33	35	55	−21	−34	20	32	48	64	−24	−40
21	39	135	105	−27	−78	24	34	112	96	−28	−68
24	41	140	112	−30	−82	25	40	60	80	−30	−50
26	48	56	84	−32	−52	29	36	80	90	−32	−58
29	39	77	91	−33	−58	30	48	72	96	−36	−60
34	66	70	110	−42	−68	35	56	84	112	−42	−70
37	63	85	119	−45	−74	39	72	84	126	−48	−78
40	64	96	128	−48	−80	41	60	104	130	−48	−82
45	72	108	144	−54	−90	50	80	120	160	−60	−100

From Theorem 2.4, we know that $K_{a_1\cdot p_1,a_2\cdot p_2,a_3\cdot p_3} = K_{p_1,p_2,p_3}$ is integral.

(2) From Theorem 2.9 and (1) of Example 3.5, it is easy to prove that $K_{a_1\cdot p_1q,a_2\cdot p_2q,a_3\cdot p_3q} = K_{p_1q,p_2q,p_3q}$ is integral, too. □

Theorem 3.6. *For $s=3$, let q be any positive integer, and let p_i $(i=1,2,3)$ be positive integers in Table 1, $a_1=a_2=a_3=1$, and u_i $(i=1,2,3)$ be those of Theorem 2.6. Then $K_{a_1\cdot p_1q,a_2\cdot p_2q,a_3\cdot p_3q} = K_{p_1q,p_2q,p_3q}$ is integral.*

Proof. It is easy to check the correctness by making use of Theorems 2.4, 2.5 or 2.6 and 2.9. □

Remark 3.7. An infinite family of integral complete tripartite graphs K_{p_1,p_2,p_3} was constructed in [18]. In Table 1, by using a computer, we have found 34 solutions (p_1, p_2, p_3), where $s=3$ and $p_1 < p_2 < p_3$, $1 \leqslant p_1 \leqslant 50$, $p_1+1 \leqslant p_2 \leqslant p_1+50$, and $p_2 + 1 \leqslant p_3 \leqslant p_2 + 100$. We shall construct infinite many classes of such integral graphs from Theorems 2.4, 2.5, 2.6 and 2.9. They are different from those in existing literature (see [8,9,18]). We believe that it is useful for constructing other integral complete tripartite graphs. When $s = 3$, $p_1 < p_2 < p_3$ and $a_1 = a_2 = a_3 = 1$. For any positive integer q, the complete tripartite graph $K_{5q,8q,12q}$ is integral and

$$\mathrm{Spec}(K_{5q,8q,12q}) = \begin{pmatrix} -10q & -6q & 0 & 16q \\ 1 & 1 & 25q-3 & 1 \end{pmatrix}.$$

If $q = 1$, let $s = 3$, $p_1 < p_2 < p_3$ and $a_1 = a_2 = a_3 = 1$, we know that the complete tripartite graph $K_{5,8,12}$ is an integral one, the order of which is 25, which is much smaller than those given in [8,9,18].

Remark 3.8. Theorem 3.6 generates an infinite set of vectors $\vec{p} = (p_1, p_2, p_3)$ for which (2) has integral solutions only. But there is only finite number of primitive vectors in this infinite set (in general, the primitive vectors are the only ones which are of interest). The infinite series built in [18] gives an infinite series of the primitive solutions. Thus Theorem 3.6 is much weaker than the result of [18]. However, by analyzing Table 1 one can see that all its rows except the row $\vec{p} = (5, 13, 77)$ have the following property: $u_3/p_i = -2$ for a suitable $i \in \{1, 2, 3\}$. This observation gives a hint to a new infinite series of primitive triples $\vec{p}$, see the following, which is due to a referee.

Let $u_3 < u_2 < u_1$ be the roots of $F(x) = F_{\vec{p}}(x) = 0$. Set $v_3 = -u_3$, $v_2 = -u_2$, $v_1 = u_1$. Then v_i's are positive integers which satisfy the following conditions:

$$\begin{aligned} &v_1 = v_2 + v_3, \\ &v_2^2 + v_2v_3 + v_3^2 = p_1p_2 + p_1p_3 + p_2p_3, \qquad (17) \\ &(v_2 + v_3)v_2v_3 = 2p_1p_2p_3. \end{aligned}$$

Let us look for solutions of (17) such that $v_3 = 2p_i$ for some $i = 1,2,3$. Forgetting about ordering of p_i's we may assume that $v_3 = 2p_3$. Then (17) is equivalent to the following:

$$p_1 + p_2 = 2v_3,$$

$$p_1 p_2 = (v_2 + v_3)v_2. \tag{18}$$

These equations have an integral solutions for p_1, p_2 if and only if $v_3^2 - (v_2 + v_3)v_2$ is a perfect square, say m^2. Then $p_1 = v_3 - m$, $p_2 = v_3 + m$, $p_3 = v_3/2$.

$$v_3^2 - (v_2 + v_3)v_2 = m^2 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{v_3}{m}, \quad y = \frac{v_2}{m}, \quad x^2 - xy - y^2 = 1 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{t^2 + 1}{t^2 + t - 1}, \quad y = t(x - 1), \quad t \in \mathbb{Q}.$$

We may assume that $m > 0$. It follows from $p_1 = m(x - 1) > 0$ and $v_2 > 0$ that $x > 1$ and $t > 0$. The first inequality is equivalent to

$$\frac{2 - t}{t^2 + t - 1} > 0 \quad \Leftrightarrow \quad \frac{\sqrt{5} - 1}{2} = 0.618.. < t < 2 \text{ or } t < \frac{-\sqrt{5} - 1}{2} = -1.618\ldots .$$

Thus $p_1 = v_3 - m = m(x - 1)$, $p_2 = v_3 + m = m(x + 1)$, $p_3 = v_3/2 = mx/2$ are nonnegative.

Write $t = a/b$ where $(a, b) = 1$ and $a > 0$, $b > 0$ we obtain

$$\frac{v_3}{m} = x = \frac{a^2 + b^2}{a^2 + ab - b^2}, \quad \frac{v_2}{m} = y = \frac{2ab - a^2}{a^2 + ab - b^2}.$$

After some routine transformations we obtain

$$p_1 = \frac{2b(2b - a)}{d}, \quad p_2 = \frac{2a(2a + b)}{d}, \quad p_3 = \frac{a^2 + b^2}{d},$$

$$u_1 = \frac{2b(b + 2a)}{d}, \quad u_2 = -\frac{2a(2b - a)}{d}, \quad u_3 = -\frac{2(a^2 + b^2)}{d}, \tag{19}$$

where $d = (2b(2b - a), 2a(2a + b), a^2 + b^2)$. Note that $d \in \{1, 2, 5, 10\}$.

Take for example $a = 2$, $b = 3$. Then $d = 1$ and

$$p_1 = 24, \quad p_2 = 28, \quad p_3 = 13, \quad u_1 = 42, \quad u_2 = -16, \quad u_3 = -26.$$

This is one of the triple given Table 1. All triple given in this table except $(5, 12, 77)$ may be obtained from (19).

Note that the above numbering of p_i's may not coincide with one fixed before.

Hence, we obtain the following result.

Theorem 3.9. *For $s = 3$, let q be any positive integer, and let p_i and u_i $(i = 1,2,3)$ be positive integers in the above formulae* (19), $a_1 = a_2 = a_3 = 1$. *Then $K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, a_3 \cdot p_3 q} = K_{p_1 q, p_2 q, p_3 q}$ is integral.*

Example 3.10. For any positive integer q, if $s = 3$, let $p_1 = q$, $p_2 = 3q$, $p_3 = 5q$, $u_2 = -2q$, $u_3 = -4q$, then there do not exist positive integers a_1, a_2, a_3 such that $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ is integral.

Proof. When $s = 3$, $p_1 = q$, $p_2 = 3q$, $p_3 = 5q$, $u_2 = -2q$, $u_3 = -4q$. Suppose that we can construct integral graphs $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$. From Theorem 2.6, we know that $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ is integral if and only if there exist integers u_i and positive integers p_i $(i = 1,2,3)$ satisfying $-p_3 < u_3 < -p_2 < u_2 < -p_1 < u_1 < +\infty$ such that $a_k = \frac{\prod_{i=1}^{3}(p_k + u_i)}{p_k \prod_{i=1, i \neq k}^{3}(p_k - p_i)}$ $(k = 1,2,3)$ are positive integers.

Hence, we obtain

$$a_1 = \frac{3}{8q}(u_1 + q), \tag{20}$$

$$a_2 = \frac{1}{12q}(u_1 + 3q), \tag{21}$$

$$a_3 = \frac{3}{40q}(u_1 + 5q). \tag{22}$$

By (21) and (22), we have

$$20a_3 - 18a_2 = 3. \tag{23}$$

From Lemma 3.2, we know that there are no integral solutions for Eq. (23).

Hence, we cannot construct integral graphs $K_{a_1\cdot p_1, a_2\cdot p_2, a_3\cdot p_3}$. $\square$

Theorem 3.11. *For the complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ on n vertices, let m, s and q be positive integers, and $s \geqslant 3$, then we have*

1. *If $p_i = m + 2(i-1)$ for $i = 1,2,\dots,s$, then there are no integers a_i $(i = 1,2,\dots,s)$ such that $K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ is an integral graph.*
2. *If $p_i' = p_i q = [m + 2(i-1)]q$ for $i = 1,2,\dots,s$, let $u_j' = -(m+2j-3)q$ for $j = 2,3,\dots,s$, then there are no integers a_i' $(i = 1,2,\dots,s)$ such that $K_{a_1'\cdot p_1',a_2'\cdot p_2',\dots,a_s'\cdot p_s'}$ is an integral graph.*

Proof. (1) Suppose that we can construct an integral graph $K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$. From Theorem 2.6, we know that $K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ is integral if and only if there exist integers u_i and positive integers p_i $(i = 1,2,\dots,s)$ satisfying $-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < u_2 < -p_1 < u_1 < +\infty$ such that all a_k $(k = 1,2,\dots,s)$ are positive integers, where $a_k = \frac{\prod_{i=1}^{s}(p_k+u_i)}{p_k\prod_{i=1,i\neq k}^{s}(p_k-p_i)}$ for $k = 1,2,\dots,s$. Hence, we can only choose

$$u_j = -(m+2j-3), \quad j = 2,3,\dots,s.$$

We obtain

$$a_{s-1} = \frac{(m+2s-4+u_1)\cdot(2s-5)!!}{2(m+2s-4)\cdot(2s-4)!!}, \tag{24}$$

$$a_s = \frac{(m+2s-2+u_1)\cdot(2s-3)!!}{(m+2s-2)\cdot(2s-2)!!}, \tag{25}$$

From (24) and (25), we have

$$(m+2s-2)\cdot(2s-2)!!\cdot a_s - 2(m+2s-4)(2s-3)\cdot(2s-4)!!\cdot a_{s-1} = 2\cdot(2s-3)!!. \tag{26}$$

Since $s \geqslant 3$, let $d = ((m+2s-2+u_1)\cdot(2s-2)!!, 2(m+2s-4)(2s-3)\cdot(2s-4)!!)$, then $d = 2\cdot(2s-4)!!\cdot((m+2s-2+u_1)(s-1),(m+2s-4)(2s-3))$. Thus, $d\nmid[2\cdot(2s-3)!!]$. From Lemma 3.2, we know that there are no integral solutions (a_{s-1}, a_s) for Eq. (26).

Hence, $K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ cannot be an integral graph.

(2) Suppose that we can construct an integral graph $K_{a_1'\cdot p_1' q,a_2'\cdot p_2',\dots,a_s'\cdot p_s'}$. From Theorem 2.6, we similarly obtain

$$a_{s-1}' = \frac{[q(m+2s-4)+u_1']\cdot(2s-5)!!}{2q(m+2s-4)\cdot(2s-4)!!}, \tag{27}$$

$$a_s' = \frac{[q(m+2s-2)+u_1']\cdot(2s-3)!!}{q(m+2s-2)\cdot(2s-2)!!}, \tag{28}$$

From (27) and (28), we have

$$(m+2s-2)\cdot(2s-2)!!\cdot a_s' - 2(m+2s-4)(2s-3)\cdot(2s-4)!!\cdot a_{s-1}' = 2\cdot(2s-3)!!. \tag{29}$$

From Lemma 3.2, we know that there are no integral solutions (a_{s-1}', a_s') for Eq. (29).

Hence, $K_{a_1'\cdot p_1' q,a_2'\cdot p_2',\dots,a_s'\cdot p_s'}$ cannot be an integral graph. $\square$

4. Further discussion

In this paper, we have mainly investigated integral complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ on n vertices. When $s = 1,2,3$, some results of such integral graphs can be found in [8,9,18] and the present paper. When $s \geqslant 4$, we have not found such integral graphs. We tried to get some general results. Thus, we raise the following questions for further study.

Question 4.1. *Are there any integral complete r-partite graphs* $K_{p_1,\ldots,p_r} = K_{a_1\cdot p_1,\ldots,a_s\cdot p_s}$ *with arbitrarily large* s?

For complete r-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$, when $s = 1,2,3$, let $a_1{=}a_2 = \cdots = a_s = 1$, some results of such integral graphs can be found in [8,9,18] and the present paper. However, when $s \geqslant 4$, $a_1 = a_2 = \cdots = a_s = 1$, we have not found such integral graphs. Hence, we have

Question 4.2. *Are there any integral complete r-partite graphs* $K_{p_1,\ldots,p_r} = K_{a_1\cdot p_1,\ldots,a_s\cdot p_s}$ *with* $a_1 = a_2 = \cdots = a_s = 1$ *when* $s \geqslant 4$?

For complete r-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$, we give a sufficient and necessary condition for $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ to be integral. In particular, when $s = 1,2$, we got all parameter solutions for $K_{a_1\cdot p_1,a_2\cdot p_2}$ to be integral graphs in [8,9] and the present paper. When $s \geqslant 3$, we have not got such general good results. Hence, we have

Question 4.3. *When* $s = 3,4,5,\ldots,$ *can we give a better sufficient and necessary condition for* $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ *to be integral*?

Acknowledgements

The authors are very grateful to the referees for many detailed comments and suggestions, which are very helpful for improving the presentation of the manuscript. Lemma 2.8 and Theorem 3.9 are due to an anonymous referee, to whom we would like to express our special thanks.

References

[1] K.T. Balińska, D. Cvetković, Z. Radosavljević, S.K. Simić, D. Stevanović, A survey on integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. 13 (2002) 42–65.
[2] K.T. Balińska, S.K. Simić, The nonregular, bipartite, integral graphs with maximum degree 4. Part I: basic properties, Discrete Math. 236 (2001) 13–24.
[3] P. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New York, Berlin, Heidelberg, 1995, pp. 106–107.
[4] F.C. Bussemaker, D. Cvetković, There are exactly 13 connected cubic integral graphs, Publ. Elektrotech. Fak. Ser. Mat. Fiz. 544 (1976) 43–48.
[5] Z.F. Cao, On the integral trees of diameter R when $3 \leqslant R \leqslant 6$, J. Heilongjiang Univ. 2 (1–3) (1988) 95.
[6] Z.F. Cao, Some new classes of integral trees with diameter 5 or 6, J. Systems Sci. Math. Sci. 11 (1) (1991) 20–26.
[7] M. Capobianco, S. Maurer, D. McCarthy, J. Molluzzo, A collection of open problems, Ann. New York Aca. Sci. (1980) 582–583.
[8] D. Cvetković, M. Doob, I. Gutman, A. Torgašev, Recent Results in the Theory of Graph Spectra, North-Holland, Amsterdam, 1988.
[9] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs—Theory and Application, Academic Press, New York, Francisco, London, 1980.
[10] F. Harary, A.J. Schwenk, Which graphs have integral spectra? in: R. Bari, F. Harary (Eds.), Graphs and Combinatorics, Lecture Notes in Mathematics, Vol. 406, Springer, Berlin, 1974, pp. 45–51.
[11] P. Híc, On balanced integral trees of diameter 6, CO-MAT-TECH '97, 5. vedecká konferencia s medzinárodnou úćasťou, Trnava 14–15 (1997) 125–129.
[12] P. Híc, R. Nedela, Balanced integral trees, Math. Slovaca 48 (5) (1998) 429–445.
[13] M.S. Li, W.S. Yang, J.B. Wang, Notes on the spectra of trees with small diameters, J. Changsha Railway Univ. 18 (2) (2000) 84–87.
[14] X.L. Li, G.N. Lin, On the problem of integral trees, Chinese Sci. Bull. 32 (11) (1987) 813–816 (in Chinese) (Chinese Sci. Bull. 33 (10) (1988) 802–806 (in English)).
[15] X.L. Li, L.G. Wang, Integral trees—a survey, Chinese J. Eng. Math. 17 (Suppl) (2000) 91–93, 96.
[16] R.Y. Liu, Integral trees of diameter 5, J. Systems Sci. Math. Sci. 8 (4) (1988) 357–360.
[17] M. Muzychuk, M. Klin, On graphs with three eigenvalues, Discrete Math. 189 (1998) 191–207.

[18] M. Roitman, An infinite family of integral graphs, Discrete Math. 52 (2–3) (1984) 313–315.
[19] K.H. Rosen, Elementary Number Theory and Its Applications, Addison-Wesley Publishing Company, London, Amsterdam, 1984, pp. 87–90.
[20] A.J. Schwenk, Exactly thirteen connected cubic graphs have integral spectra, in: Y. Alavi, D.R. Lick (Eds.), Lecture Notes in Mathematics Series A, Vol. 642, Springer, Berlin, 1978, pp. 516–533.
[21] L.G. Wang, X.L. Li, Some new classes of integral trees with diameters 4 and 6, Australasian J. Combin. 21 (2000) 237–243.
[22] L.G. Wang, X.L. Li, X.J. Yao, Integral trees with diameters 4, 6 and 8, Australasian J. Combin. 25 (2002) 29–44.
[23] L.G. Wang, X.L. Li, S.G. Zhang, Construction of integral graphs, Appl. Math. J. Chinese Univ. Ser. B 15 (3) (2000) 239–246.
[24] L.G. Wang, X.L. Li, S.G. Zhang, Some new families of integral trees with diameters 4 and 6, Electron. Notes Discrete Math. 8 (2001) (or L.G. Wang, X.L. Li, S.G. Zhang, Families of integral trees with diameters 4, 6 and 8, Discrete Appl. Math. 136 (2–3) (2004) 349–362).
[25] M. Watanabe, A.J. Schwenk, Integral startlike trees, J. Aust. Math. Soc. Ser. A 28 (1979) 120–128.
[26] P.Z. Yuan, Integral trees of diameter 4, J. Systems Sci. Math. Sci. 18 (2) (1998) 177–181.