

LINEARIZATION OF DYNAMIC EQUATIONS OF FLEXIBLE MECHANISMS—A FINITE ELEMENT APPROACH

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SUMMARY

A finite element based method is presented for evaluation of linearized dynamic equations of flexible mechanisms about a nominal trajectory. The coefficient matrices of the linearized equations of motion are evaluated as explicit analytical expressions involving mixed sets of generalized co-ordinates of the mechanism with rigid links and deformation mode co-ordinates that characterize deformation of flexible link elements. This task is accomplished by employing the general framework of the geometric transfer function formalism. The proposed method is general in nature and can be applied to spatial mechanisms and manipulators having revolute and prismatic joints. The method also permits investigation of the dynamics of flexible rotors and spinning shafts. Application of the theory is illustrated through a detailed model development of a four-bar mechanism and the analysis of bending vibrations of two single link mechanisms in which the link is considered as a rotating flexible arm or as an unsymmetrical rotating shaft, respectively. The algorithm for the calculation of the matrix coefficients is directly amenable to numerical computation and has been incorporated into the linearization module of the computer program SPACAR.¹

1. INTRODUCTION

In recent publications^{2–4} a finite element based method has been presented for analysing the dynamic behaviour of spatial mechanisms and manipulators with flexible links. The method involves a non-linear finite element formulation representing deformation modes in the description of strain, stress and associated stiffness of the elements. An algorithm has been presented for numerical determination of the geometric transfer functions of multi-degree of freedom mechanisms. These functions describe the configuration and deformation state of the mechanism in terms of their degrees of freedom. The degrees of freedom can be defined using a combination of generalized co-ordinates of the mechanism with rigid links and deformation mode co-ordinates describing vibrations of flexible link elements. The inertia properties of the elements are described using either the lumped- or consistent-mass formulation. With the aid of the first- and second-order geometric transfer functions, the equations of motion are derived in terms of the degrees of freedom.

The object of this paper is to linearize the equations of motion about a nominal trajectory. The linearized equations are of interest from both analysis and control point of view. For analysis, they enable us to study the stability of highly complex mechanisms. From the point of view of manipulator control the linearized equations provide a basis for developing of reduced-order linearized models suitable for control system design. The demand for linearized models, which

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can be updated during the simulation, requires linearization procedures that can be implemented into computer programs.

Several methods have been proposed to obtain linearized dynamic mechanism models. In Reference 5, the Q -matrix formulation and Bejczy's theorem⁶ are used to linearize symbolically the Lagrangian dynamic robot model about a nominal trajectory. In Reference 7, linearized dynamic models of active spatial mechanisms are obtained using vectorial algebraic recursive relations. In the above methods the system members are considered as rigid bodies. Another approach to linearization is to use numerical differentiation procedures⁸ which, however, are not very accurate and the results obtained are not in a form suitable for physical interpretation.

In the present paper the geometric transfer function formalism is applied to develop an algorithm that analytically evaluates the coefficient matrices of the linearized equations of motion about any point along the nominal trajectory. The nominal trajectory determines the position, velocity and acceleration for the mechanism with the restriction that all flexible deformations of the links are suppressed. This paper is organized as follows. Section 2 defines some finite element notions and briefly presents the geometric transfer function formulation. In Section 3 the equations of motion are formulated. In Section 4 the linearized equations of motion are then derived in second-order form. The matrix coefficients of the linearized equations are identified and their functional dependences of the coefficients on the nominal positions, velocities and accelerations are outlined. Finally in Section 5 a series of illustrative examples are discussed to demonstrate the capabilities of the proposed finite element method for generation of linearized dynamic mechanism models. Concluding remarks are advanced in Section 6.

2. GEOMETRIC TRANSFER FUNCTIONS

Characteristic for the present finite element approach to mechanism analysis is that both links and joints are considered as specific finite elements. The links may be modelled by one or more beam elements that may be rigid or deformable depending on whether the flexibility is expected to play a role in the dynamic analysis. The joints may be modelled by coupling elements such as the cylindrical hinge element and the slider-truss element. The location of each element is specified by a vector $\mathbf{x}^k \in X^k$ of nodal co-ordinates, some of which may be Cartesian co-ordinates (x_i^k) of the end nodes, while others describe the orientation of orthogonal triads, rigidly attached at the element nodes. For numerical determination of angular orientation we use Euler parameters (λ_i^k).⁹ The superscript k is added to show that a specific element k is considered. We call X^k the configuration space of the element k . With respect to some reference configuration of the element, the instantaneous values of the nodal co-ordinates determine a fixed number of deformation modes for the element. The number of deformation modes is equal to the number of nodal co-ordinates minus the number of degrees of freedom of the element as a rigid body. The deformation modes are specified by a vector of deformation mode co-ordinates $\mathbf{e}^k \in E^k$, some of which are associated with large relative displacements and rotations (e_i^k) between the element nodes, while others describe small elastic deformations of the element and will be denoted by (e_j^k). We call E^k the deformation space of the element k . The deformation mode co-ordinates (e_i^k) are expressed as non-linear functions D_i^k of the nodal co-ordinates (x_i^k). In Appendix I explicit expressions are presented for the deformation functions of the slider-truss and the spatial beam element. These elements are selected in this paper to be used wherever the theory will be illustrated at the element level. For a detailed description of other elements (e.g. cylindrical hinge element), the reader is referred to References 2 and 3.

A kinematic mechanism model can be build up with finite elements by letting them have nodal points in common. In this way the configuration spaces of the individual elements can be

regarded as subspaces of the mechanism configuration space X , that is

$$X = \sum_k X^k \tag{1}$$

In the same way the element deformation spaces can be regarded as subspaces of the space E of deformation mode co-ordinates for the entire mechanism. Since deformation mode co-ordinates (e_i^k) are related only to the element k , E is the direct sum of the spaces E^k , that is

$$E = \bigoplus_k E^k \tag{2}$$

The spaces X and E can now be split in subspaces in accordance with the constraint conditions and the choice of the generalized co-ordinates. We have

$$X = X^o \oplus X^c \oplus X^m, \quad \text{and} \quad E = E^o \oplus E^m \oplus E^c \tag{3}$$

where the superscripts o, c and m denote the space of invariant, dependent and independent (or generalized) co-ordinates respectively. The problem now formulated for the kinematical analysis is the determination of the nodal co-ordinates and deformation mode co-ordinates for given values of the generalized co-ordinates (x_i^m, e_i^m). Hence determine the maps

$$\mathbf{F}^x: X^m \times E^m \rightarrow X, \quad \text{or} \quad \mathbf{x} = \mathbf{F}^x(\mathbf{x}^m, \mathbf{e}^m) \tag{4}$$

$$\mathbf{F}^e: X^m \times E^m \rightarrow E, \quad \text{or} \quad \mathbf{e} = \mathbf{F}^e(\mathbf{x}^m, \mathbf{e}^m) \tag{5}$$

The maps \mathbf{F}^x and \mathbf{F}^e are called the geometric transfer functions of the mechanism; they express the configuration and deformation state as explicit functions of the set of generalized co-ordinates. The velocity vectors $\dot{\mathbf{x}}$ and $\dot{\mathbf{e}}$ can be calculated from equations (4) and (5) as

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{F}^x}{\partial \mathbf{x}^m} \dot{\mathbf{x}}^m + \frac{\partial \mathbf{F}^x}{\partial \mathbf{e}^m} \dot{\mathbf{e}}^m, \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{DF}^x \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m) \tag{6}$$

$$\dot{\mathbf{e}} = \frac{\partial \mathbf{F}^e}{\partial \mathbf{x}^m} \dot{\mathbf{x}}^m + \frac{\partial \mathbf{F}^e}{\partial \mathbf{e}^m} \dot{\mathbf{e}}^m, \quad \text{or} \quad \dot{\mathbf{e}} = \mathbf{DF}^e \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m) \tag{7}$$

where (\cdot) denotes differentiation with respect to time. The derivative maps \mathbf{DF}^x and \mathbf{DF}^e are called the first-order geometric transfer functions. Again differentiating with respect to time yields the accelerations

$$\ddot{\mathbf{x}} = (\mathbf{D}^2 \mathbf{F}^x \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m) + \mathbf{DF}^x \cdot (\ddot{\mathbf{x}}^m, \ddot{\mathbf{e}}^m) \tag{8}$$

$$\ddot{\mathbf{e}} = (\mathbf{D}^2 \mathbf{F}^e \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m) + \mathbf{DF}^e \cdot (\ddot{\mathbf{x}}^m, \ddot{\mathbf{e}}^m) \tag{9}$$

where $\mathbf{D}^2 \mathbf{F}^x$ and $\mathbf{D}^2 \mathbf{F}^e$ are the second-order geometric transfer functions. Detailed calculations of the first- and second-order geometric transfer functions are given in References 2 and 3. In Appendix II the algorithm for the third-order geometric transfer functions is presented. The latter are key ingredients in the derivation of the linearized equations of motion.

3. EQUATIONS OF MOTION

By means of the first- and second-order geometric transfer functions, the equations of motion are formulated in terms of the degrees of freedom, thereby eliminating the constraint forces associated with the rigid link motion of the mechanism. In view of the different treatment of the translational

and angular velocities in the formulation of the equations of motion it is useful to split the configuration space X of mechanism nodal co-ordinates in subspaces according to

$$X = X^x \oplus X^\lambda, \quad \mathbf{x} \in X^x, \quad \boldsymbol{\lambda} \in X^\lambda \quad (10)$$

where X^x is the space of Cartesian co-ordinates (x_i) and X^λ the space of Euler parameters (λ_i). The corresponding geometric transfer functions are \mathbf{F}^x and \mathbf{F}^λ , respectively. Let \mathbf{M} be the mechanism mass matrix obtained by adding of the lumped- and consistent-mass matrices (see Appendix III), i.e.

$$\mathbf{M} = \sum_k (\mathbf{M}_l^k + \mathbf{M}_c^k) \quad (11)$$

where the summation includes all finite elements. Furthermore, the inertia tensors \mathbf{J} and \mathbf{L} can be obtained in a similar way:

$$\mathbf{J}^x = \sum_k (\mathbf{J}_c^x)^k, \quad \mathbf{J}^\lambda = \sum_k \mathbf{J}_l^\lambda, \quad \mathbf{J}^e = \sum_k (\mathbf{J}_c^e)^k \quad (12a)$$

and

$$\mathbf{L}^x = \sum_k (\mathbf{L}_c^x)^k, \quad \mathbf{L}^e = \sum_k (\mathbf{L}_c^e)^k \quad (12b)$$

where \mathbf{J}_l^k and \mathbf{J}_c^k , \mathbf{L}_c^k represent rotational inertia tensors associated with the lumped- and consistent-inertia formulation of the element k , respectively, see Appendix III. With the notation, $\mathbf{DF}^T = [\mathbf{DF}^{xT}, \mathbf{DF}^{\lambda T}, \mathbf{DF}^{eT}]$ the equations of motion can be expressed in matrix form^{2,3}

$$\begin{aligned} [\mathbf{DF}^T \mathbf{MDF}] \begin{bmatrix} \ddot{\mathbf{x}}^m \\ \ddot{\mathbf{e}}^m \end{bmatrix} &= [\mathbf{DF}^{xT}, \mathbf{DF}^{\lambda T}, \mathbf{DF}^{eT}] \begin{bmatrix} \mathbf{f}^x - (\mathbf{J}^x \cdot \dot{\boldsymbol{\lambda}}) \cdot \dot{\boldsymbol{\lambda}} - (\mathbf{L}^x \cdot \dot{\boldsymbol{\varepsilon}}) \cdot \dot{\boldsymbol{\lambda}} \\ \mathbf{f}^\lambda - (\mathbf{J}^\lambda \cdot \dot{\boldsymbol{\lambda}}) \cdot \dot{\boldsymbol{\lambda}} \\ -\boldsymbol{\sigma} - (\mathbf{J}^e \cdot \dot{\boldsymbol{\lambda}}) \cdot \dot{\boldsymbol{\lambda}} - (\mathbf{L}^e \cdot \dot{\boldsymbol{\varepsilon}}) \cdot \dot{\boldsymbol{\lambda}} \end{bmatrix} \\ &\quad - \mathbf{DF}^T \mathbf{M} \cdot (\mathbf{D}^2 \mathbf{F} \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m) \end{aligned} \quad (13)$$

Here, $[\mathbf{DF}^T \mathbf{MDF}]$ denotes the system mass matrix, \mathbf{f} the vector of externally applied nodal forces and $\boldsymbol{\sigma}$ the stress vector which describes the loading state of the elements constituting the mechanism. The stresses of the flexible elements are characterized by Hooke's law as defined in equation (50). The force vector consists of a part \mathbf{f}^x acting in the sense of the translational velocities (\dot{x}_i) and a part \mathbf{f}^λ , representing the moment components associated with the time derivatives of the Euler parameters ($\dot{\lambda}_i$). The equations of motion form a non-linear system of ordinary differential equations of second-order and describe the general case of coupled rigid link motion and small elastic deformation.

4. LINEARIZED EQUATIONS OF MOTION

Given the non-linear equations of motion in equation (13), consider small perturbations around the nominal trajectory $(\mathbf{x}_0^m, \dot{\mathbf{x}}_0^m, \ddot{\mathbf{x}}_0^m)$ and $(\mathbf{e}_0^m, \dot{\mathbf{e}}_0^m, \ddot{\mathbf{e}}_0^m)$ such that the actual variables are of the form

$$\mathbf{x}^m = \mathbf{x}_0^m + \delta \mathbf{x}^m, \quad \mathbf{e}^m = \begin{bmatrix} e^m \\ \boldsymbol{\varepsilon}^m \end{bmatrix} = \begin{bmatrix} e_0^m \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \delta e^m \\ \delta \boldsymbol{\varepsilon}^m \end{bmatrix} \quad (14a)$$

$$\dot{\mathbf{x}}^m = \dot{\mathbf{x}}_0^m + \delta \dot{\mathbf{x}}^m, \quad \dot{\mathbf{e}}^m = \begin{bmatrix} \dot{e}^m \\ \dot{\boldsymbol{\varepsilon}}^m \end{bmatrix} = \begin{bmatrix} \dot{e}_0^m \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \delta \dot{e}^m \\ \delta \dot{\boldsymbol{\varepsilon}}^m \end{bmatrix} \quad (14b)$$

$$\ddot{\mathbf{x}}^m = \ddot{\mathbf{x}}_0^m + \delta\ddot{\mathbf{x}}^m, \quad \ddot{\mathbf{e}}^m = \begin{bmatrix} \ddot{e}^m \\ \ddot{\boldsymbol{\varepsilon}}^m \end{bmatrix} = \begin{bmatrix} \ddot{e}_0^m \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \delta\ddot{e}^m \\ \delta\ddot{\boldsymbol{\varepsilon}}^m \end{bmatrix} \quad (14c)$$

$$\mathbf{f} = \mathbf{f}_0 + \delta\mathbf{f}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \delta\boldsymbol{\sigma} \quad (14d)$$

The nominal values of the flexible deformation mode co-ordinates $(e_i^m)_0$, $(\dot{e}_i^m)_0$, $(\ddot{e}_i^m)_0$ are assumed to be zero. Throughout this paper, a variable subscripted with 0 is evaluated along the nominal trajectory. The perturbation of a variable is denoted by δ . Expanding the first- and second-order geometric transfer functions \mathbf{DF} and $\mathbf{D}^2\mathbf{F}$ in their Taylor series expansions and disregarding second and higher order terms, results in the approximations

$$\mathbf{DF} = \mathbf{DF}_0 + \mathbf{D}^2\mathbf{F}_0 \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) \quad (15)$$

$$\mathbf{D}^2\mathbf{F} = \mathbf{D}^2\mathbf{F}_0 + \mathbf{D}^3\mathbf{F}_0 \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) \quad (16)$$

where $\mathbf{D}^3\mathbf{F}_0$ is the third-order geometric transfer function evaluated about a point along the nominal trajectory $(\mathbf{x}_0^m, \mathbf{e}_0^m)$, see Appendix II. When the mass matrix \mathbf{M} defined in equation (11) is expanded in a similar way, we have

$$\begin{bmatrix} \mathbf{M}^{xx} & \mathbf{M}^{x\lambda} & \mathbf{M}^{xe} \\ \mathbf{M}^{\lambda x} & \mathbf{M}^{\lambda\lambda} & \mathbf{M}^{\lambda e} \\ \mathbf{M}^{ex} & \mathbf{M}^{e\lambda} & \mathbf{M}^{ee} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_0^{xx} & \mathbf{0} & \mathbf{M}_0^{xe} \\ \mathbf{0} & \mathbf{M}_0^{\lambda\lambda} & \mathbf{0} \\ \mathbf{M}_0^{ex} & \mathbf{0} & \mathbf{M}_0^{ee} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \vdots \mathbf{D}^e\mathbf{M}_0^{x\lambda}\mathbf{DF}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) \vdots \mathbf{D}^\lambda\mathbf{M}_0^{xe}\mathbf{DF}_0^\lambda \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) \\ \mathbf{D}^e\mathbf{M}_0^{\lambda x}\mathbf{DF}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) & \vdots \mathbf{D}^\lambda\mathbf{M}_0^{\lambda\lambda}\mathbf{DF}_0^\lambda \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) \vdots \mathbf{D}^e\mathbf{M}_0^{\lambda e}\mathbf{DF}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) \\ \mathbf{D}^\lambda\mathbf{M}_0^{ex}\mathbf{DF}_0^\lambda \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) & \vdots \mathbf{D}^e\mathbf{M}_0^{e\lambda}\mathbf{DF}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m) \vdots \mathbf{0} \end{bmatrix} \quad (17)$$

The partitioned matrices $\mathbf{M}_0^{x\lambda}$, $\mathbf{M}_0^{\lambda x}$ and $\mathbf{M}_0^{\lambda e}$, $\mathbf{M}_0^{e\lambda}$ are zero matrices because they depend linearly on the flexible deformation mode co-ordinates (ε_i) , see Appendix III. The differentiation operators \mathbf{D}^e and \mathbf{D}^λ working on $\mathbf{M}^{x\lambda}$, \mathbf{M}^{xe} and $\mathbf{M}^{\lambda e}$ represent partial differentiation with respect to the deformation mode co-ordinates (ε_i) and the Euler parameters (λ_i) respectively. Next the velocity dependent inertia vectors $(\mathbf{J} \cdot \dot{\boldsymbol{\lambda}}) \cdot \dot{\boldsymbol{\lambda}}$ and $(\mathbf{L} \cdot \dot{\boldsymbol{\varepsilon}}) \cdot \dot{\boldsymbol{\varepsilon}}$ in equation (13) are expanded. For the lumped inertia vector we have

$$\begin{aligned} (\mathbf{J} \cdot \dot{\boldsymbol{\lambda}}) \cdot \dot{\boldsymbol{\lambda}} &= (\mathbf{J}_0^\lambda \cdot \dot{\boldsymbol{\lambda}}_0) \cdot \dot{\boldsymbol{\lambda}}_0 + ((\mathbf{D}^\lambda\mathbf{J}_0^\lambda\mathbf{DF}_0^\lambda \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m)) \cdot \dot{\boldsymbol{\lambda}}_0) \cdot \dot{\boldsymbol{\lambda}}_0 \\ &\quad + 2((\mathbf{J}_0^\lambda \cdot (\mathbf{D}^2\mathbf{F}_0^\lambda \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m))) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot \dot{\boldsymbol{\lambda}}_0 \\ &\quad + 2((\mathbf{J}_0^\lambda \cdot (\mathbf{DF}_0^\lambda \cdot (\delta\dot{\mathbf{x}}^m, \delta\dot{\mathbf{e}}^m))) \cdot \dot{\boldsymbol{\lambda}}_0 \end{aligned} \quad (18)$$

Expanding the inertia vectors associated with the consistent inertia formulation yields

$$\begin{bmatrix} (\mathbf{J}^x \cdot \dot{\boldsymbol{\lambda}}) \cdot \dot{\boldsymbol{\lambda}} \\ (\mathbf{J}^e \cdot \dot{\boldsymbol{\lambda}}) \cdot \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} ((\mathbf{D}^e\mathbf{J}_0^x\mathbf{DF}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m)) \cdot \dot{\boldsymbol{\lambda}}_0) \cdot \dot{\boldsymbol{\lambda}}_0 \\ ((\mathbf{D}^e\mathbf{J}_0^e\mathbf{DF}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m)) \cdot \dot{\boldsymbol{\lambda}}_0) \cdot \dot{\boldsymbol{\lambda}}_0 \end{bmatrix} \quad (19)$$

and

$$\begin{aligned} \begin{bmatrix} (\mathbf{L}^x \cdot \dot{\boldsymbol{\varepsilon}}) \cdot \dot{\boldsymbol{\varepsilon}} \\ (\mathbf{L}^e \cdot \dot{\boldsymbol{\varepsilon}}) \cdot \dot{\boldsymbol{\varepsilon}} \end{bmatrix} &= \begin{bmatrix} (\mathbf{L}_0^x \cdot (\mathbf{D}^2\mathbf{F}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot \dot{\boldsymbol{\lambda}}_0 \\ (\mathbf{L}_0^e \cdot (\mathbf{D}^2\mathbf{F}_0^e \cdot (\delta\mathbf{x}^m, \delta\mathbf{e}^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot \dot{\boldsymbol{\lambda}}_0 \end{bmatrix} \\ &\quad + \begin{bmatrix} (\mathbf{L}_0^x \cdot (\mathbf{DF}_0^e \cdot (\delta\dot{\mathbf{x}}^m, \delta\dot{\mathbf{e}}^m)) \cdot \dot{\boldsymbol{\lambda}}_0 \\ (\mathbf{L}_0^e \cdot (\mathbf{DF}_0^e \cdot (\delta\dot{\mathbf{x}}^m, \delta\dot{\mathbf{e}}^m)) \cdot \dot{\boldsymbol{\lambda}}_0 \end{bmatrix} \end{aligned} \quad (20)$$

The tensors \mathbf{J}_0^x and \mathbf{J}_0^e are zero tensors because they depend linearly on the flexible deformation

mode co-ordinates (ε_i). Substituting equations (15)–(20) in equation (13) and retaining only first-order approximations of the coefficients, we obtain

$$[\mathbf{M}_0] \begin{bmatrix} \delta \ddot{\mathbf{x}}^m \\ \delta \ddot{\mathbf{e}}^m \end{bmatrix} + [\mathbf{C}_0] \begin{bmatrix} \delta \dot{\mathbf{x}}^m \\ \delta \dot{\mathbf{e}}^m \end{bmatrix} + [\mathbf{K}_0 + \mathbf{G}_0] \begin{bmatrix} \delta \mathbf{x}^m \\ \delta \mathbf{e}^m \end{bmatrix} = [\mathbf{DF}^T] \begin{bmatrix} \delta \mathbf{f} \\ -\delta \sigma \end{bmatrix} \quad (21)$$

These are the fully linearized equations of motion about the nominal trajectory in second-order form. The coefficient matrices may be identified as follows:

(a) System mass matrix \mathbf{M}_0

$$\mathbf{M}_0 = [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \begin{bmatrix} \mathbf{M}_0^{xx} & \mathbf{0} & \mathbf{M}_0^{xe} \\ \mathbf{0} & \mathbf{M}_0^{\lambda\lambda} & \mathbf{0} \\ \mathbf{M}_0^{ex} & \mathbf{0} & \mathbf{M}_0^{ee} \end{bmatrix} \begin{bmatrix} \mathbf{DF}_0^x \\ \mathbf{DF}_0^\lambda \\ \mathbf{DF}_0^e \end{bmatrix} \quad (22)$$

where \mathbf{DF}_0^x , \mathbf{DF}_0^λ and \mathbf{DF}_0^e are the first-order geometric transfer functions, evaluated about a point along the nominal trajectory. Since the inertia forces are linear in the accelerations, the matrix \mathbf{M}_0 is the system mass matrix which appears in the non-linear equations of motion (13). \mathbf{M}_0 is a symmetric and positive definite matrix and thus non-singular.

(b) Geometrically non-linear damping matrix \mathbf{C}_0

$$\mathbf{C}_0 = [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \left[2 \begin{bmatrix} \mathbf{M}_0^{xx} & \mathbf{0} & \mathbf{M}_0^{xe} \\ \mathbf{0} & \mathbf{M}_0^{\lambda\lambda} & \mathbf{0} \\ \mathbf{M}_0^{ex} & \mathbf{0} & \mathbf{M}_0^{ee} \end{bmatrix} \begin{bmatrix} \mathbf{D}^2 \mathbf{F}_0^x \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ \mathbf{D}^2 \mathbf{F}_0^\lambda \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ \mathbf{D}^2 \mathbf{F}_0^e \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \end{bmatrix} \right. \\ \left. + \begin{bmatrix} (\mathbf{L}_0^x \cdot \mathbf{DF}_0^e) \cdot \dot{\lambda}_0 \\ 2(\mathbf{J}_0^\lambda \cdot \mathbf{DF}_0^\lambda) \cdot \dot{\lambda}_0 \\ (\mathbf{L}_0^e \cdot \mathbf{DF}_0^e) \cdot \dot{\lambda}_0 \end{bmatrix} \right] \quad (23)$$

where $\mathbf{D}^2 \mathbf{F}_0^x$, $\mathbf{D}^2 \mathbf{F}_0^\lambda$ and $\mathbf{D}^2 \mathbf{F}_0^e$ are second-order geometric transfer functions evaluated at the nominal trajectory. The matrix \mathbf{C}_0 is non-symmetric, and is sometimes referred to as a gyroscopic matrix. This, however, is only the case for real gyroscopic systems where the matrix \mathbf{C}_0 is anti-symmetric.¹⁰

(c) System stiffness matrix \mathbf{K}_0

$$\mathbf{K}_0 = \mathbf{DF}_0^{eT} \mathbf{S} \mathbf{DF}_0^e \quad (24)$$

where \mathbf{S} is a symmetric matrix obtained by addition of the element stiffness matrices \mathbf{S}^k , defined in equation (50), i.e.

$$\mathbf{S} = \sum_k \mathbf{S}^k \quad (25)$$

(d) Geometrically non-linear stiffness matrix \mathbf{G}_0

$$\mathbf{G}_0 = \mathbf{G}_0^F + \mathbf{G}_0^N \quad (26)$$

where

$$\mathbf{G}_0^F = [-\mathbf{D}^2 \mathbf{F}_0^{xT}, -\mathbf{D}^2 \mathbf{F}_0^{\lambda T}, \mathbf{D}^2 \mathbf{F}_0^{eT}] \begin{bmatrix} \mathbf{f}_0^x \\ \mathbf{f}_0^\lambda \\ \boldsymbol{\sigma}_0 \end{bmatrix} \quad (27)$$

represents the sensitivity of the nodal force vector \mathbf{f} and the stress vector $\boldsymbol{\sigma}$ to variations in the generalized co-ordinates $(\mathbf{x}_i^m, \mathbf{e}^m)$ and may be viewed as the static part of the geometrically non-linear stiffness matrix. The dynamic part \mathbf{G}_0^N is defined as

$$\begin{aligned}
 \mathbf{G}_0^N = & [\mathbf{D}^2\mathbf{F}_0^{xT}, \mathbf{D}^2\mathbf{F}_0^{\lambda T}, \mathbf{D}^2\mathbf{F}_0^{eT}] \begin{bmatrix} \mathbf{M}_0^{xx} & \mathbf{0} & \mathbf{M}_0^{xe} \\ \mathbf{0} & \mathbf{M}_0^{\lambda\lambda} & \mathbf{0} \\ \mathbf{M}_0^{ex} & \mathbf{0} & \mathbf{M}_0^{ee} \end{bmatrix} \begin{bmatrix} \mathbf{DF}_0^x \\ \mathbf{DF}_0^\lambda \\ \mathbf{DF}_0^e \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_0^m \\ \dot{\mathbf{e}}_0^m \end{bmatrix} \\
 & + [\mathbf{D}^2\mathbf{F}_0^{xT}, \mathbf{D}^2\mathbf{F}_0^{\lambda T}, \mathbf{D}^2\mathbf{F}_0^{eT}] \begin{bmatrix} \mathbf{M}_0^{xx} & \mathbf{0} & \mathbf{M}_0^{xe} \\ \mathbf{0} & \mathbf{M}_0^{\lambda\lambda} & \mathbf{0} \\ \mathbf{M}_0^{ex} & \mathbf{0} & \mathbf{M}_0^{ee} \end{bmatrix} \begin{bmatrix} (\mathbf{D}^2\mathbf{F}_0^x \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ (\mathbf{D}^2\mathbf{F}_0^\lambda \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ (\mathbf{D}^2\mathbf{F}_0^e \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \end{bmatrix} \\
 & + [\mathbf{D}^2\mathbf{F}_0^{xT}, \mathbf{D}^2\mathbf{F}_0^{\lambda T}, \mathbf{D}^2\mathbf{F}_0^{eT}] \begin{bmatrix} \mathbf{0} \\ (\mathbf{J}_0^\lambda \cdot \dot{\lambda}_0) \cdot \dot{\lambda}_0 \\ \mathbf{0} \end{bmatrix} \\
 & + [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{D}^e\mathbf{M}_0^{x\lambda}\mathbf{DF}_0^e & \vdots & \mathbf{D}^\lambda\mathbf{M}_0^{xe}\mathbf{DF}_0^\lambda \\ \mathbf{D}^e\mathbf{M}_0^{\lambda x}\mathbf{DF}_0^e & \vdots & \mathbf{D}^\lambda\mathbf{M}_0^{\lambda\lambda}\mathbf{DF}_0^\lambda & \vdots & \mathbf{D}^e\mathbf{M}_0^{\lambda e}\mathbf{DF}_0^e \\ \mathbf{D}^\lambda\mathbf{M}_0^{ex}\mathbf{DF}_0^\lambda & \vdots & \mathbf{D}^e\mathbf{M}_0^{e\lambda}\mathbf{DF}_0^e & \vdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{DF}_0^x \\ \mathbf{DF}_0^\lambda \\ \mathbf{DF}_0^e \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_0^m \\ \dot{\mathbf{e}}_0^m \end{bmatrix} \\
 & + [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \begin{bmatrix} \mathbf{M}_0^{xx} & \mathbf{0} & \mathbf{M}_0^{xe} \\ \mathbf{0} & \mathbf{M}_0^{\lambda\lambda} & \mathbf{0} \\ \mathbf{M}_0^{ex} & \mathbf{0} & \mathbf{M}_0^{ee} \end{bmatrix} \begin{bmatrix} \mathbf{D}^2\mathbf{F}_0^x \\ \mathbf{D}^2\mathbf{F}_0^\lambda \\ \mathbf{D}^2\mathbf{F}_0^e \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_0^m \\ \dot{\mathbf{e}}_0^m \end{bmatrix} \\
 & + [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{D}^e\mathbf{M}_0^{x\lambda}\mathbf{DF}_0^e & \vdots & \mathbf{D}^\lambda\mathbf{M}_0^{xe}\mathbf{DF}_0^\lambda \\ \mathbf{D}^e\mathbf{M}_0^{\lambda x}\mathbf{DF}_0^e & \vdots & \mathbf{D}^\lambda\mathbf{M}_0^{\lambda\lambda}\mathbf{DF}_0^\lambda & \vdots & \mathbf{D}^e\mathbf{M}_0^{\lambda e}\mathbf{DF}_0^e \\ \mathbf{D}^\lambda\mathbf{M}_0^{ex}\mathbf{DF}_0^\lambda & \vdots & \mathbf{D}^e\mathbf{M}_0^{e\lambda}\mathbf{DF}_0^e & \vdots & \mathbf{0} \end{bmatrix} \\
 & \quad \times \begin{bmatrix} (\mathbf{D}^2\mathbf{F}_0^x \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ (\mathbf{D}^2\mathbf{F}_0^\lambda \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ (\mathbf{D}^2\mathbf{F}_0^e \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \end{bmatrix} \\
 & + [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \begin{bmatrix} \mathbf{M}_0^{xx} & \mathbf{0} & \mathbf{M}_0^{xe} \\ \mathbf{0} & \mathbf{M}_0^{\lambda\lambda} & \mathbf{0} \\ \mathbf{M}_0^{ex} & \mathbf{0} & \mathbf{M}_0^{ee} \end{bmatrix} \begin{bmatrix} (\mathbf{D}^3\mathbf{F}_0^x \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ (\mathbf{D}^3\mathbf{F}_0^\lambda \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \\ (\mathbf{D}^3\mathbf{F}_0^e \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m)) \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m) \end{bmatrix} \\
 & + [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \begin{bmatrix} (\mathbf{L}_0 \cdot (\mathbf{D}^2\mathbf{F}_0^e \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m))) \cdot \dot{\lambda}_0 \\ 2(\mathbf{J}_0^\lambda \cdot (\mathbf{D}^2\mathbf{F}_0^\lambda \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m))) \cdot \dot{\lambda}_0 \\ (\mathbf{L}_0^e \cdot (\mathbf{D}^2\mathbf{F}_0^e \cdot (\dot{\mathbf{x}}_0^m, \dot{\mathbf{e}}_0^m))) \cdot \dot{\lambda}_0 \end{bmatrix} \\
 & + [\mathbf{DF}_0^{xT}, \mathbf{DF}_0^{\lambda T}, \mathbf{DF}_0^{eT}] \begin{bmatrix} ((\mathbf{D}^e\mathbf{J}_0^x\mathbf{DF}_0^e) \cdot \dot{\lambda}_0) \cdot \dot{\lambda}_0 \\ ((\mathbf{D}^\lambda\mathbf{J}_0^\lambda\mathbf{DF}_0^\lambda) \cdot \dot{\lambda}_0) \cdot \dot{\lambda}_0 \\ ((\mathbf{D}^e\mathbf{J}_0^e\mathbf{DF}_0^e) \cdot \dot{\lambda}_0) \cdot \dot{\lambda}_0 \end{bmatrix} \tag{28}
 \end{aligned}$$

Generally the matrix G_0^N is a non-symmetric matrix representing the sensitivity of the inertial forces to perturbations of the generalized co-ordinates (x_i^m, e_i^m) . The determination of the third-order geometric transfer function values in equation (28) is very time consuming. In Appendix II an efficient algorithm is derived for the calculation of the quadratic velocity terms containing the third-order geometric transfer functions $D^3F_0^x$, $D^3F_0^l$ and $D^3F_0^e$ without explicitly determining these functions.

5. ILLUSTRATIVE EXAMPLES

5.1. Four-bar mechanism

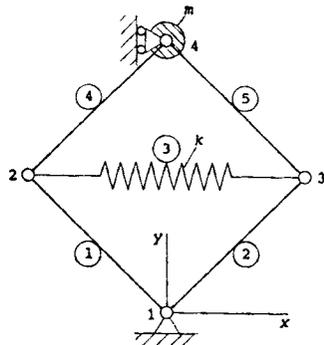
As an illustration of the theory, the linearized equations of motion of the four-bar mechanism, shown in Figure 1, will be derived. The mechanism is modelled by four rigid truss elements, denoted by 1, 2, 4 and 5, which are joined together at their nodal points to form a rhombus. The truss element 3 represents a spring with stiffness k . A concentrated mass m is attached at node 4. The four bars of the mechanism are set at right angles to one another in the nominal configuration. Then the nominal velocities and nominal accelerations are defined by \dot{e}_0^3 and \ddot{e}_0^3 respectively, where the elongation e^3 of the spring has been chosen as the generalized co-ordinate. Having defined the nominal configuration, the first-order, second-order and third-order geometric transfer functions can be calculated. With the first- and second-order functions the differential equations of motion can be generated. Following the method described in this paper one obtains the matrix coefficients of the linearized equation of motion of the four-bar mechanism at the nominal configuration presented in Figure 1. The essential steps and intermediate results are summarized in what follows.

$$\begin{aligned}
 \mathbf{e}^T &= [\mathbf{e}^{oT}, \mathbf{e}^{mT}] = [\varepsilon^1 \ \varepsilon^2 \ \varepsilon^4 \ \varepsilon^5 \ ; \ e^3] \\
 \boldsymbol{\sigma}^T &= [\boldsymbol{\sigma}^{oT}, \boldsymbol{\sigma}^{mT}] = [\sigma^1 \ \sigma^2 \ \sigma^4 \ \sigma^5 \ ; \ \sigma^3] \\
 S &= \text{diag}[- \ - \ - \ - \ ; \ k]
 \end{aligned}
 \tag{29}$$

$$\mathbf{D}\mathbf{F}^{eT} = [0 \ 0 \ 0 \ 0 \ ; \ 1]
 \tag{30}$$

$$\mathbf{x}^T = [\mathbf{x}^{oT}, \mathbf{x}^{cT}] = [x^1 \ y^1 \ x^4 \ ; \ x^2 \ y^2 \ x^3 \ y^3 \ y^4]$$

$$\mathbf{f}^T = [\mathbf{f}^{oT}, \mathbf{f}^{cT}] = [f_{x1} \ f_{y1} \ f_{x4} \ ; \ 0 \ 0 \ 0 \ 0 \ -mg]$$



Nominal configuration:

$$\begin{aligned}
 x_0^2 &= -\frac{1}{2}\sqrt{2}, & x_0^3 &= \frac{1}{2}\sqrt{2}, & x_0^4 &= 0 \\
 y_0^2 &= \frac{1}{2}\sqrt{2}, & y_0^3 &= \frac{1}{2}\sqrt{2}, & y_0^4 &= \sqrt{2}, \\
 e_0^3 &= mg/k
 \end{aligned}$$

Figure 1. Four-bar mechanism

$$\mathbf{M} = \text{diag}[0 \ 0 \ m \ ; \ 0 \ 0 \ 0 \ 0 \ m] \tag{31}$$

$$\mathbf{DF}^{xT} = [0 \ 0 \ 0 \ ; \ -\frac{1}{2} \ -\frac{1}{2} \ \frac{1}{2} \ -\frac{1}{2} \ -1]$$

$$\mathbf{D}^2\mathbf{F}^{xT} = [0 \ 0 \ 0 \ ; \ 0 \ -\frac{1}{2}\sqrt{2} \ 0 \ -\frac{1}{2}\sqrt{2} \ -\sqrt{2}]$$

$$\mathbf{D}^3\mathbf{F}^{xT} = [0 \ 0 \ 0 \ ; \ 0 \ -\frac{3}{2} \ 0 \ -\frac{3}{2} \ -3] \tag{32}$$

Nominal acceleration:

$$\ddot{e}_0^3 = -\sqrt{2}(\dot{e}_0^3)^2 - \frac{k}{m}e_0^3 + g \tag{33}$$

Coefficient matrices of the linearized equation of motion:

$$\mathbf{M}_0 = \mathbf{DF}_0^{xT}\mathbf{MDF}_0^x = m$$

$$\mathbf{C}_0 = \mathbf{DF}_0^{xT}\mathbf{MD}^2\mathbf{F}_0^x \cdot \dot{e}_0^3 = \sqrt{2}m\dot{e}_0^3$$

$$\mathbf{K}_0 = \mathbf{DF}_0^{eT}\mathbf{SDF}_0^e = k$$

$$\mathbf{G}_0^F = -\mathbf{D}^2\mathbf{F}_0^{xT}\mathbf{f} = -\sqrt{2}mg$$

$$\begin{aligned} \mathbf{G}_0^N &= (\mathbf{D}^2\mathbf{F}_0^{xT}\mathbf{MDF}_0^x + \mathbf{DF}_0^{xT}\mathbf{MD}^2\mathbf{F}_0^x)\ddot{e}_0^3 + \mathbf{D}^2\mathbf{F}_0^{xT}\mathbf{MD}^2\mathbf{F}_0^x \cdot (\dot{e}_0^3)^2 + \mathbf{DF}_0^{xT}\mathbf{MD}^3\mathbf{F}_0^x \cdot (\dot{e}_0^3)^2 \\ &= 2m\sqrt{2}\ddot{e}_0^3 + 2m(\dot{e}_0^3)^2 + 3m(\dot{e}_0^3)^2 \end{aligned} \tag{34}$$

Linearized equation of motion:

$$\delta\ddot{e}^3 + 2\sqrt{2}\dot{e}_0^3\delta\dot{e}^3 + \left(\frac{k}{m} - \sqrt{2}g + 2\sqrt{2}\ddot{e}_0^3 + 5(\dot{e}_0^3)^2\right)\delta e^3 = 0 \tag{35}$$

In case of small vibrations about a stable equilibrium position ($\ddot{e}_0^3 = \dot{e}_0^3 = 0$, $e_0^3 = mg/k$) the linearized equation becomes

$$\delta\ddot{e}^3 + \left(\frac{k}{m} - \sqrt{2}g\right)\delta e^3 = 0 \tag{36}$$

and when $k \gg mg$, it reduces to the well known form

$$\delta\ddot{e}^3 + \frac{k}{m}\delta e^3 = 0 \tag{37}$$

5.2. Bending vibrations of a rotating flexible arm

As a second example we study the bending vibrations of a flexible arm attached to a rotating hub. The configuration investigated in this example is illustrated in Figure 2. The arm has a length $l = 0.3$ m and is attached to a massless hub, of radius $r = 0$, which rotates at a constant angular speed Ω about a fixed axis in space. The arm has a uniform circular cross section with a diameter of 0.006 m and is made of steel, having a density of 7.87×10^3 kg/m³ and an elastic modulus of 0.2×10^{12} N/m². The hub and the arm are modelled by a cylindrical hinge element and four spatial beam elements respectively, as shown in Figure 2. The linearized equations of motion governing the free transverse vibrations of the elastic arm are given by

$$\begin{bmatrix} m_0^{ee} & m_0^{e\varepsilon} \\ m_0^{e\varepsilon} & M_0^{e\varepsilon} \end{bmatrix} \begin{bmatrix} \delta\dot{e}^m \\ \delta\dot{\varepsilon}^m \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_0 + \mathbf{G}_0^N \end{bmatrix} \begin{bmatrix} \delta e^m \\ \delta \varepsilon^m \end{bmatrix} = \mathbf{0} \tag{38}$$

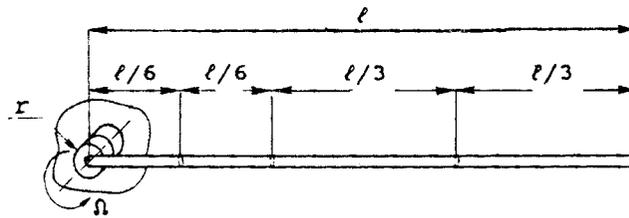


Figure 2. Elastic arm attached to a massless hub, rotating with a constant angular speed Ω about an axis fixed in space

in which e^m is the relative rotation angle of the hinge. ϵ^m is the vector of flexible generalized co-ordinates; m_0^{ee} is the total moment of inertia of the arm with respect to the axis of rotation, \mathbf{m}^{ee} represents the dynamic coupling vector between the gross rigid motion and the flexible motion of the arm and \mathbf{M}_0^{ee} , \mathbf{K}_0 are symmetric matrices determining the cantilevered bending frequencies of the arm. The dynamic stiffness matrix \mathbf{G}_0^N is symmetric for this problem and may be viewed as an additional centrifugal-stiffness matrix. The components of \mathbf{G}_0^N depend quadratically on the angular speed Ω . The associated frequency equation is given by

$$\det(-\omega_i^2 \mathbf{M}_0 + \mathbf{K}_0 + \mathbf{G}_0^N) = 0 \quad (39)$$

where the quantities ω_i are the natural frequencies of the system. Because \mathbf{K}_0 and \mathbf{G}_0^N are semidefinite, the frequency equation admits one zero eigenfrequency ω_1 associated with the rigid body rotation of the arm. The problem under consideration has been solved analytically for the case of zero bending stiffness, $\mathbf{K}_0 = \mathbf{0}$ (cord-condition), by Meijaard.¹¹ The natural frequencies $\omega_{c,i}$ as functions of the hub angular rate Ω are then determined by the equations

$$\frac{(\omega_{c,i})^2}{\Omega^2} = 2i^2 - i - 1, \quad i = 1, 2, \dots, \infty \quad (40)$$

for vibrations within the plane of rotation, and

$$\frac{(\omega_{c,i})^2}{\Omega^2} = 2i^2 - i, \quad i = 1, 2, \dots, \infty \quad (41)$$

for vibrations perpendicular to the plane of rotation.

Figure 3 shows the resulting frequencies for the first two bending modes as functions of the angular speed Ω . The numerically obtained frequencies $\omega_{c,2}$ and $\omega_{c,3}$ of the cord agree with the analytically obtained frequencies of equations (40) and (41). The results are substantially affected by the quadratic terms in the flexible deformation mode co-ordinates (ϵ_i^k) in the expression for the longitudinal deformation ϵ_i^k of the beam elements, see (49a). Neglecting these terms leads to deviations in magnitude of about 30 per cent from the analytical solution of the chord vibrations.

5.3. Bending vibrations of an unsymmetrical rotating shaft

The third example is dealing with the bending vibrations of a uniform shaft having unequal flexural rigidities in the principal directions of its cross section. The shaft is modelled as a simply supported beam and is assumed to be infinitely stiff in torsion. The rotation speed Ω of the shaft is introduced by prescribing the time derivative of the relative rotation angle of the hinge element fixed in the middle of the shaft (see Figure 4).

The free vibrations of the rotating shaft can be described in terms of the flexible deformation mode co-ordinates by the homogeneous equations of motion

$$\mathbf{M} \delta \ddot{\epsilon}^m + \mathbf{C}_0 \delta \dot{\epsilon}^m + (\mathbf{K}_0 + \mathbf{G}_0^N) \delta \epsilon^m = \mathbf{0} \quad (42)$$

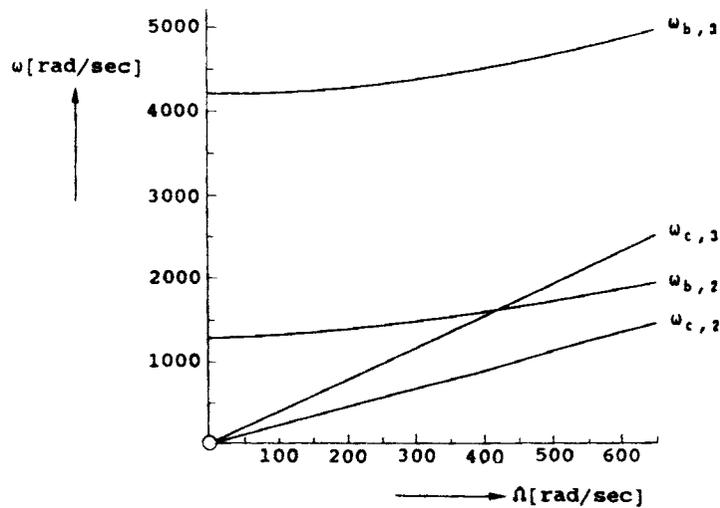


Figure 3. First- and second in-plane bending frequencies ω_b and ω_c of the arm and the cord respectively as functions of the angular speed Ω

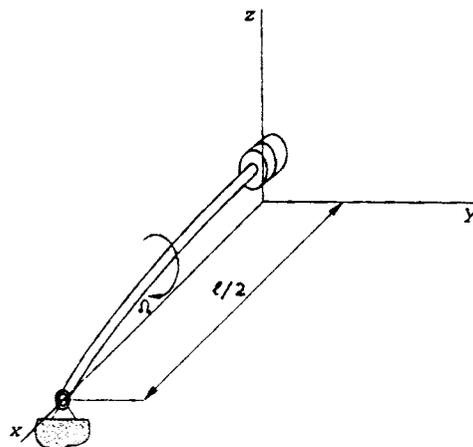


Figure 4. Simply supported shaft rotating with angular speed Ω

where

$$C_0 = 2\Omega(PM_0) \tag{43}$$

and

$$G_0^N = -\Omega^2 M_0 \tag{44}$$

P is a permutation matrix, i.e. the components of matrix (PM_0) are simply a reordering of the components of M_0 . The matrices M_0 , K_0 and G_0^N are symmetric and positive definite. C_0 is a skew-symmetric matrix representing gyroscopic terms. It can be proved that the corresponding eigenvalue problem admits either real or imaginary natural frequencies.¹² It should be noticed that these frequencies, called the natural frequencies of whirl, are calculated with respect to the

rotating co-ordinate system fixed at the shaft. The problem of shaft whirling has been studied extensively by many authors. Kellenberger¹³ presented an analytically obtained frequency equation for the simply supported shaft in the non-dimensional form as

$$\frac{\bar{\omega}}{\omega_i} = \sqrt{\left\{ 1 + \left(\frac{\Omega}{\omega_i}\right)^2 \pm 2\sqrt{\left[\left(\frac{\Omega}{\omega_i}\right)^2 + \left(\frac{v}{4}\right)^2\right]} \right\}} \quad (45)$$

where

$$v = \frac{I_{\bar{y}} - I_{\bar{z}}}{\bar{I}}, \quad \text{with} \quad \bar{I} = \frac{I_{\bar{y}} + I_{\bar{z}}}{2} \quad (46)$$

and

$$\omega_i = (i\pi)^2 \sqrt{\left(\frac{E\bar{I}}{ml^4}\right)}, \quad i = 1, 2, 3, \dots, \quad (47)$$

are the natural frequencies of transverse vibration for a shaft having a symmetrical cross section with bending stiffness $E\bar{I}$, length l and a mass density m per unit length. The frequency equation (45) holds for all natural frequencies of the shaft. In Figure 5, the natural frequencies of whirl $\bar{\omega}$ are plotted as function of the shaft speed Ω for $v = 0, 0.5, 1.0, 1.5$ and 1.8 respectively.

The numerically obtained natural frequencies of whirl of the first vibration mode ($i = 1$) agree perfectly with the analytical results of equation (45) for the case where the shaft is divided in two equal beam elements. If $\Omega = 0$, then $\bar{\omega} = \omega_i \sqrt{1 \pm v/2}$, that is, $\omega_i \sqrt{1 + v/2} = \omega_{\bar{y}}$ and $\omega_i \sqrt{1 - v/2} = \omega_{\bar{z}}$. For stability of the free vibration, the shaft speed Ω must lie outside the interval $\omega_{\bar{z}} < \Omega < \omega_{\bar{y}}$. The width of the unstable region $\omega_{\bar{y}} - \omega_{\bar{z}}$, where $\omega_{\bar{y}}$ and $\omega_{\bar{z}}$ are the upper and lower boundaries of the unstable region, increases with the magnitude of v . The unstable region can be eliminated by simultaneous effects of unsymmetrical stiffness of the shaft and

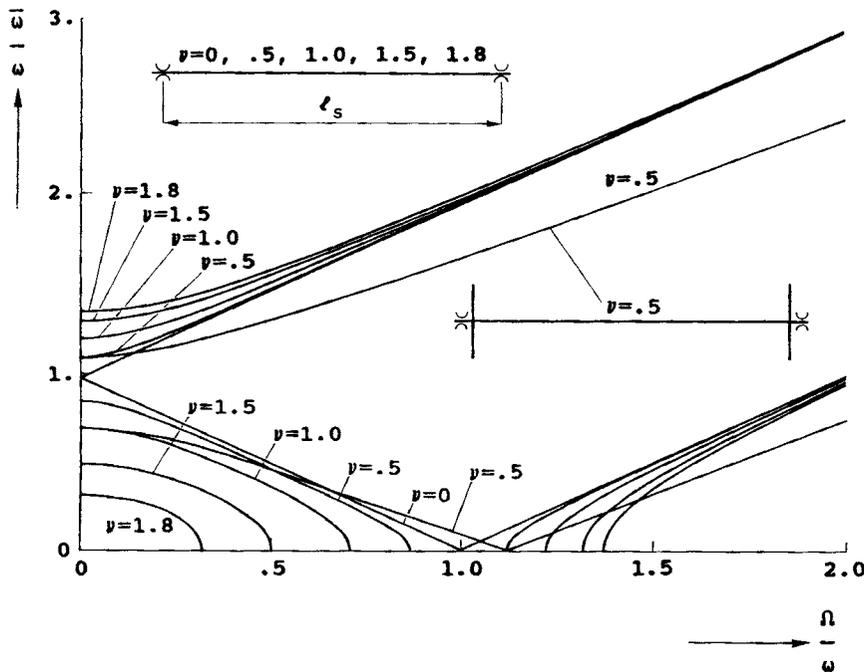


Figure 5. Natural frequencies of whirl $\bar{\omega}$ as function of the rotor speed Ω

unsymmetrical inertia of discs which are mounted on the shaft. This is demonstrated for the case of a simply supported shaft carrying unsymmetrical discs at both of the shaft ends. The relative rotation of the principal axes of the stiffness to those of rotor inertia is $\pi/2$, where $J_y^{\text{disc}} = 0$ and $J_z^{\text{disc}} = 0.012ml^3$. The dashed curve in Figure 5 shows the frequency characteristic of the first vibration mode ($i = 1$), for the shaft with $\nu = 0.5$, that carries the unsymmetrical discs. This example shows that the removal of the unstable vibrations has been realized by an appropriate combination of the inequalities in inertia and stiffness.

6. CONCLUSIONS

The finite element method and the geometric transfer function formulation have been applied to develop an algorithm for generating linearized dynamic equations of flexible mechanisms. The analytical approach leads to a system of linearized equations in which the matrix coefficients possess all physical and mathematical properties of the mechanism. Because of the approach with finite element notions, the method is applicable to a large class of spatial mechanisms including rotating components like flexible shafts.

The agreement for the numerical results shows that the general purpose method functions properly for the examples presented in this paper. The derivation of the linearized equations of motion for these examples is comparatively simple. It must, however, be pointed out that the computation scheme for generating the linearized equations is essentially developed for handling more complicated mechanisms. When more d.o.f. are taken into account the calculation of the coefficient matrices in equation (21) becomes prohibitively laborious. The only way to attack the general problem of linearization is to perform the corresponding manipulations with a computer.

APPENDIX I

Deformation functions D_i^k for the slider-truss and the spatial beam element

1. *(Slider) truss element.* The position of the slider-truss element is determined by the position vectors x^p and x^q of the end nodes p and q. A possible rotation of the element about the axis pq is not involved in the description of the element position. The number of degrees of freedom of the element as a rigid body is thus five, which give rise to a single deformation mode, associated with the elongation of the element. This elongation can be expressed as

$$e_1^k = D_1^k = \|x^q - x^p\| - l_0^k \tag{48}$$

where $\|x^q - x^p\|$ and l_0^k represent the actual length and the reference length of the element.

2. *Spatial beam element.* Figure 6 shows a spatial beam element in an x, y, z inertial co-ordinate system. The configuration of the element is determined by the position vectors x^p and x^q of the end nodes and the angular orientation of orthogonal triads $(\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z)$ rigidly attached to each end point. In the undeflected state the triads coincide with the axis pq and the principal axes of its cross section. The rotation part of the motion of the (flexible) beam is described by the rotation of the triads $(\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z)$ which are determined by rotation matrices \mathbf{R}^p and \mathbf{R}^q . If the beam is rigid then the rotation matrices are identical and in the initial undeflected state they are equal to the identity matrix. The components of the rotation matrices are expressed in terms of Euler parameters (λ_i^k) .⁹ With the vector $\mathbf{l}^k = x^q - x^p$, the deformation functions of the beam element can now be written as follows:¹⁴

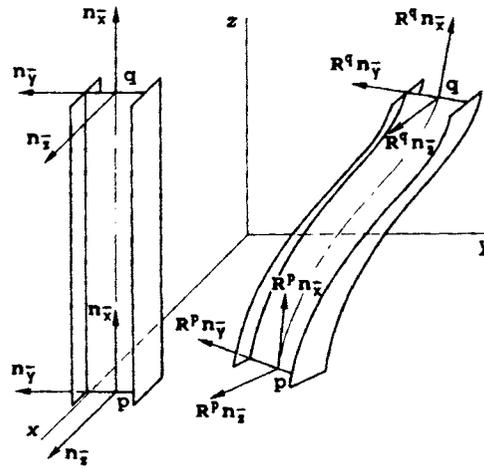


Figure 6. Beam element, initial and deformed state

elongation:

$$\epsilon_1^k = D_1^k = \|\mathbf{l}^k\| - l_0^k + \frac{1}{30l_0^k} [2(\epsilon_3^k)^2 + \epsilon_3^k \epsilon_4^k + 2(\epsilon_4^k)^2 + 2(\epsilon_5^k)^2 + \epsilon_5^k \epsilon_6^k + 2(\epsilon_6^k)^2] \quad (49a)$$

torsion:

$$\epsilon_2^k = D_2^k = [(\mathbf{R}^p \mathbf{n}_z, \mathbf{R}^q \mathbf{n}_y) - (\mathbf{R}^p \mathbf{n}_y, \mathbf{R}^q \mathbf{n}_z)] l_0^k / 2 \quad (49b)$$

bending:

$$\epsilon_3^k = D_3^k = -(\mathbf{R}^p \mathbf{n}_z, \mathbf{l}^k) \quad (49c)$$

$$\epsilon_4^k = D_4^k = (\mathbf{R}^q \mathbf{n}_z, \mathbf{l}^k) \quad (49d)$$

$$\epsilon_5^k = D_5^k = (\mathbf{R}^p \mathbf{n}_y, \mathbf{l}^k) \quad (49e)$$

$$\epsilon_6^k = D_6^k = -(\mathbf{R}^q \mathbf{n}_y, \mathbf{l}^k) \quad (49f)$$

Here, $\|\mathbf{l}^k\|$ and l_0^k represent the actual length and the reference length of the element; (\cdot, \cdot) stands for the inner product of two vectors. The terms in the expression for the elongation ϵ_1^k that are quadratic in the bending deformations represent the longitudinal deformation associated with the deflection of the beam element. The deformation mode co-ordinates in equations (49) possess the proper invariance with respect to rigid body motions of the beam element. Since the expressions for the bending deformations are defined with respect to orthogonal triads oriented according to the element axis and the principal axes of its cross section, they have a clear physical meaning. If the deformations (ϵ_i^k) remain sufficiently small ($\epsilon_i^k / l^k \ll 1$), then in the elastic range they are linearly related to known beam quantities as normal force σ_1^k , twisting moment σ_2^k and bending moments $\sigma_3^k, \sigma_4^k, \sigma_5^k, \sigma_6^k$ by the beam constitutive equations

$$\boldsymbol{\sigma}^k = \mathbf{S}^k \boldsymbol{\epsilon}^k \quad (50)$$

where \mathbf{S}^k is a symmetric matrix containing the elastic constants.

APPENDIX II

Determination of the third-order geometric transfer functions

The deformation functions of the individual elements can be taken together in a continuity map for the entire mechanism; we write symbolically

$$\mathbf{D} = \sum_k D^k: X \rightarrow E, \quad \text{or} \quad \mathbf{e} = \mathbf{D}(\mathbf{x}) \quad (51)$$

The continuity map in equation (51) constitutes the basic equations for the kinematic analysis and forms the basis for the determination of the geometric transfer functions. Substituting equations (4) and (5) into equation (51) yields the non-linear algebraic relation

$$\mathbf{F}^e = \mathbf{D} \circ \mathbf{F}^x, \quad \text{for all } (\mathbf{x}^m, \mathbf{e}^m) \quad (52)$$

Straightforward differentiation of equation (52) with respect to $(\mathbf{x}_i^m, \mathbf{e}_i^m)$ yields with the chain rule

$$D_i \mathbf{F}^e = \mathbf{D} \mathbf{D} \cdot D_i \mathbf{F}^x \quad (53)$$

$$D_{ij} \mathbf{F}^e = (D^2 \mathbf{D} \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x + \mathbf{D} \mathbf{D} \cdot D_{ij} \mathbf{F}^x \quad (54)$$

$$D_{ijk} \mathbf{F}^e = ((D^3 \mathbf{D} \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x) \cdot D_k \mathbf{F}^x + (D^2 \mathbf{D} \cdot D_{ik} \mathbf{F}^x) \cdot D_j \mathbf{F}^x \\ + (D^2 \mathbf{D} \cdot D_{jk} \mathbf{F}^x) \cdot D_i \mathbf{F}^x + (D^2 \mathbf{D} \cdot D_{ij} \mathbf{F}^x) \cdot D_k \mathbf{F}^x + \mathbf{D} \mathbf{D} \cdot D_{ijk} \mathbf{F}^x \quad (55)$$

where $D_i \mathbf{F}$ is the partial derivative of \mathbf{F} with respect to the i th degree of freedom. The differentiation operator \mathbf{D} working on $\mathbf{D}(\mathbf{x})$ represents partial differentiation with respect to the nodal co-ordinates (\mathbf{x}_i) . The derivative maps $\mathbf{D} \mathbf{D}$, $D^2 \mathbf{D}$ and $D^3 \mathbf{D}$ are composed from the corresponding derivative maps of the element deformation functions D^k .² Expressions for the first-order and second-order geometric transfer functions can be obtained from equations (53) and (54) respectively. For a detailed calculation the reader is referred to References 2 and 3. This need not be repeated here; only the derivation of the third-order geometric transfer functions appearing in the expressions for the geometrically non-linear stiffness matrix \mathbf{G}_0 is new and will be presented here. In accordance with the co-ordinate splitting in equation (3), the equations in (55) can be written in the form

$$\begin{bmatrix} D_{ijk} \mathbf{F}^{e^o} \\ D_{ijk} \mathbf{F}^{e^m} \\ D_{ijk} \mathbf{F}^{e^c} \end{bmatrix} = \begin{bmatrix} ((D^3 \mathbf{D}^o \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x) \cdot D_k \mathbf{F}^x \\ ((D^3 \mathbf{D}^m \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x) \cdot D_k \mathbf{F}^x \\ ((D^3 \mathbf{D}^c \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x) \cdot D_k \mathbf{F}^x \end{bmatrix} + \begin{bmatrix} (D^2 \mathbf{D}^o \cdot D_{ik} \mathbf{F}^x) \cdot D_j \mathbf{F}^x \\ (D^2 \mathbf{D}^m \cdot D_{ik} \mathbf{F}^x) \cdot D_j \mathbf{F}^x \\ (D^2 \mathbf{D}^c \cdot D_{ik} \mathbf{F}^x) \cdot D_j \mathbf{F}^x \end{bmatrix} \\ + \begin{bmatrix} (D^2 \mathbf{D}^o \cdot D_{jk} \mathbf{F}^x) \cdot D_i \mathbf{F}^x \\ (D^2 \mathbf{D}^m \cdot D_{jk} \mathbf{F}^x) \cdot D_i \mathbf{F}^x \\ (D^2 \mathbf{D}^c \cdot D_{jk} \mathbf{F}^x) \cdot D_i \mathbf{F}^x \end{bmatrix} + \begin{bmatrix} (D^2 \mathbf{D}^o \cdot D_{ij} \mathbf{F}^x) \cdot D_k \mathbf{F}^x \\ (D^2 \mathbf{D}^m \cdot D_{ij} \mathbf{F}^x) \cdot D_k \mathbf{F}^x \\ (D^2 \mathbf{D}^c \cdot D_{ij} \mathbf{F}^x) \cdot D_k \mathbf{F}^x \end{bmatrix} \\ + \begin{bmatrix} D^o \mathbf{D}^o & D^c \mathbf{D}^o & D^m \mathbf{D}^o \\ D^o \mathbf{D}^m & D^c \mathbf{D}^m & D^m \mathbf{D}^m \\ D^o \mathbf{D}^c & D^c \mathbf{D}^c & D^m \mathbf{D}^c \end{bmatrix} \begin{bmatrix} D_{ijk} \mathbf{F}^{x^o} \\ D_{ijk} \mathbf{F}^{x^c} \\ D_{ijk} \mathbf{F}^{x^m} \end{bmatrix} \quad (56)$$

where

$$D_{ijk} \mathbf{F}^{e^o} = D_{ijk} \mathbf{F}^{e^m} = D_{ijk} \mathbf{F}^{x^o} = D_{ijk} \mathbf{F}^{x^m} = 0 \quad (57)$$

The superscripts o, c and m combined with the operator D represent partial differentiation with respect to the corresponding nodal co-ordinates \mathbf{x}^o , \mathbf{x}^c and \mathbf{x}^m . If the mechanism is not in a singular configuration, then the unknown third-order geometric transfer functions $D^3\mathbf{F}^{xc}$ and $D^3\mathbf{F}^{ec}$ can be calculated by

$$\begin{aligned} D_{ijk}\mathbf{F}^{xc} = & - \left[\begin{array}{c} D^c D^o \\ D^c D^m \end{array} \right]^{-1} \left[\begin{array}{c} \left[\left((D^3 D^o \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x \right) \cdot D_k \mathbf{F}^x \right] \\ \left[\left((D^3 D^m \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x \right) \cdot D_k \mathbf{F}^x \right] \end{array} \right] \\ & + \left[\begin{array}{c} (D^2 D^o \cdot D_{ik} \mathbf{F}^x) \cdot D_j \mathbf{F}^x \\ (D^2 D^m \cdot D_{ik} \mathbf{F}^x) \cdot D_j \mathbf{F}^x \end{array} \right] + \left[\begin{array}{c} (D^2 D^o \cdot D_{jk} \mathbf{F}^x) \cdot D_i \mathbf{F}^x \\ (D^2 D^m \cdot D_{jk} \mathbf{F}^x) \cdot D_i \mathbf{F}^x \end{array} \right] \\ & + \left[\begin{array}{c} (D^2 D^o \cdot D_{ij} \mathbf{F}^x) \cdot D_k \mathbf{F}^x \\ (D^2 D^m \cdot D_{ij} \mathbf{F}^x) \cdot D_k \mathbf{F}^x \end{array} \right] \end{aligned} \quad (58)$$

and

$$\begin{aligned} D_{ijk}\mathbf{F}^{ec} = & ((D^3 D^c \cdot D_i \mathbf{F}^x) \cdot D_j \mathbf{F}^x) \cdot D_k \mathbf{F}^x + (D^2 D^c \cdot D_{ik} \mathbf{F}^x) \cdot D_j \mathbf{F}^x \\ & + (D^2 D^c \cdot D_{jk} \mathbf{F}^x) \cdot D_i \mathbf{F}^x + (D^2 D^c \cdot D_{ij} \mathbf{F}^x) \cdot D_k \mathbf{F}^x + D^c D^c \cdot D_{ijk} \mathbf{F}^{xc} \end{aligned} \quad (59)$$

Since the derivatives of $D^3\mathbf{D}$ and $D^2\mathbf{F}$ are commutative it follows from equations (58) and (59) that

$$D_{ijk}\mathbf{F}^x = D_{jki}\mathbf{F}^x = D_{kij}\mathbf{F}^x = D_{ikj}\mathbf{F}^x = D_{jik}\mathbf{F}^x = D_{kji}\mathbf{F}^x \quad (60)$$

and

$$D_{ijk}\mathbf{F}^e = D_{jki}\mathbf{F}^e = D_{kij}\mathbf{F}^e = D_{ikj}\mathbf{F}^e = D_{jik}\mathbf{F}^e = D_{kji}\mathbf{F}^e \quad (61)$$

In the algorithm that calculates the geometric transfer functions efficient use has been made for these symmetry properties. Nevertheless, the determination of the third-order geometric transfer functions is very time consuming. However, the quadratic velocity terms in equation (28) containing the third-order geometric transfer function $D^3\mathbf{F}$ can be calculated without explicitly determining $D^3\mathbf{F}$. From equation (58) we can deduce

$$\begin{aligned} D_{ijk}\mathbf{F}^{xc} \cdot (\dot{\mathbf{x}}_i^m, \dot{\mathbf{e}}_i^m)(\dot{\mathbf{x}}_j^m, \dot{\mathbf{e}}_j^m) = & - \left[\begin{array}{c} D^c D^o \\ D^c D^m \end{array} \right]^{-1} \left[\begin{array}{c} \left[\left((D^3 D^o \cdot \dot{\mathbf{x}}) \cdot \dot{\mathbf{x}} \right) \cdot D_k \mathbf{F}^x \right] \\ \left[\left((D^3 D^m \cdot \dot{\mathbf{x}}) \cdot \dot{\mathbf{x}} \right) \cdot D_k \mathbf{F}^x \right] \end{array} \right] \\ & + 2 \left[\begin{array}{c} (D^2 D^o \cdot (D_{ik} \mathbf{F}^x \cdot (\dot{\mathbf{x}}_i^m, \dot{\mathbf{e}}_i^m))) \cdot \dot{\mathbf{x}} \\ (D^2 D^m \cdot (D_{ik} \mathbf{F}^x \cdot (\dot{\mathbf{x}}_i^m, \dot{\mathbf{e}}_i^m))) \cdot \dot{\mathbf{x}} \end{array} \right] \\ & + \left[\begin{array}{c} (D^2 D^o \cdot (D^2 \mathbf{F}^x \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot D_k \mathbf{F}^x \\ (D^2 D^m \cdot (D^2 \mathbf{F}^x \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot D_k \mathbf{F}^x \end{array} \right] \end{aligned} \quad (62)$$

and with equation (59) we obtain

$$\begin{aligned} D_{ijk}\mathbf{F}^{ec} \cdot (\dot{\mathbf{x}}_i^m, \dot{\mathbf{e}}_i^m)(\dot{\mathbf{x}}_j^m, \dot{\mathbf{e}}_j^m) = & ((D^3 D^c \cdot \dot{\mathbf{x}}) \cdot \dot{\mathbf{x}}) \cdot D_k \mathbf{F}^x + (D^2 D^c \cdot ((D^2 \mathbf{F}^x \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m)) \cdot (\dot{\mathbf{x}}^m, \dot{\mathbf{e}}^m))) \cdot D_k \mathbf{F}^x \\ & + 2(D^2 D^c \cdot (D_{ik} \mathbf{F}^x \cdot (\dot{\mathbf{x}}_i^m, \dot{\mathbf{e}}_i^m))) \cdot \dot{\mathbf{x}} + D^c D^c \cdot (D_{ijk} \mathbf{F}^{xc} \cdot (\dot{\mathbf{x}}_i^m, \dot{\mathbf{e}}_i^m) \cdot (\dot{\mathbf{x}}_j^m, \dot{\mathbf{e}}_j^m)) \end{aligned} \quad (63)$$

With the aid of the first- and second-order geometric transfer functions $D\mathbf{F}^x$ and $D^2\mathbf{F}^x$ all of the terms in equations (62) and (63) can be calculated separately for the individual elements, yielding a computationally more efficient algorithm for evaluating the components of the dynamic stiffness matrix \mathbf{G}_0^N .

APPENDIX III

Inertia formulations for the spatial beam element

1. *Lumped formulation.* The inertia forces associated with the lumped-mass formulation of the spatial beam element can be characterized with the aid of the lumped-mass matrix \mathbf{M}_l^k and the quadratic velocity vector $(\mathbf{J}_l^k \cdot \dot{\lambda}^k) \cdot \dot{\lambda}^k$ as^{2,3}

$$\begin{bmatrix} (\mathbf{f}_{in}^x)_l^k \\ (\mathbf{f}_{in}^\lambda)_l^k \end{bmatrix} = \begin{bmatrix} (\mathbf{M}^x)_l^k & \mathbf{0} \\ \mathbf{0} & (\mathbf{M}^\lambda)_l^k \end{bmatrix} \begin{bmatrix} \ddot{x}^k \\ \ddot{\lambda}^k \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ (\mathbf{J}_l^k \cdot \dot{\lambda}^k) \cdot \dot{\lambda}^k \end{bmatrix} \quad (64)$$

The matrix $(\mathbf{M}^x)_l^k$ is a constant matrix whereas $(\mathbf{M}^\lambda)_l^k$ and \mathbf{J}_l^k depend on time, since they are functions of the Euler parameters λ_i^k . The vector $(\mathbf{f}_{in}^\lambda)_l^k$ represents the moment components associated with the time derivatives of the Euler parameters. The lumped formulation excludes the dynamic coupling between the translational and rotational motion since the lumped masses and rotational inertias are calculated by assuming that the element behaves like a rigid body.

2. *Consistent formulation.* The dynamic characteristics of the flexible beam element can be modelled more exactly by consistent-mass matrices. The inertia forces associated with the consistent-mass formulation of the spatial beam element can be characterized with the consistent-mass matrix \mathbf{M}_c^k and the quadratic velocity vectors $(\mathbf{J}_c^k \cdot \dot{\lambda}^k) \cdot \dot{\lambda}^k$ and $(\mathbf{L}_c^k \cdot \dot{\epsilon}^k) \cdot \dot{\lambda}^k$ as

$$\begin{bmatrix} (\mathbf{f}_{in}^x)_c^k \\ (\mathbf{f}_{in}^\lambda)_c^k \\ (\boldsymbol{\sigma}_{in})_c^k \end{bmatrix} = \begin{bmatrix} (\mathbf{M}^{xx})_c^k & (\mathbf{M}^{x\lambda})_c^k & (\mathbf{M}^{xe})_c^k \\ (\mathbf{M}^{\lambda x})_c^k & \mathbf{0} & (\mathbf{M}^{\lambda e})_c^k \\ (\mathbf{M}^{ex})_c^k & (\mathbf{M}^{e\lambda})_c^k & (\mathbf{M}^{ee})_c^k \end{bmatrix} \begin{bmatrix} \ddot{x}^k \\ \ddot{\lambda}^k \\ \ddot{\epsilon}^k \end{bmatrix} + \begin{bmatrix} ((\mathbf{J}^x)_c^k \cdot \dot{\lambda}^k) \cdot \dot{\lambda}^k \\ \mathbf{0} \\ ((\mathbf{J}^e)_c^k \cdot \dot{\lambda}^k) \cdot \dot{\lambda}^k \end{bmatrix} + \begin{bmatrix} ((\mathbf{L}^x)_c^k \cdot \dot{\epsilon}^k) \cdot \dot{\lambda}^k \\ \mathbf{0} \\ ((\mathbf{L}^e)_c^k \cdot \dot{\epsilon}^k) \cdot \dot{\lambda}^k \end{bmatrix} \quad (65)$$

The matrices $(\mathbf{M}^{xe})_c^k$ and $(\mathbf{M}^{ex})_c^k$ represent the principal dynamic coupling between the gross motion and the elastic deformation of the element. These matrices are functions of the Euler parameters (λ_i^k) . The matrices $(\mathbf{M}^{xx})_c^k$ and $(\mathbf{M}^{ee})_c^k$ are constant matrices associated with the Cartesian nodal co-ordinates and the flexible deformation mode co-ordinates of the element. The matrices $(\mathbf{M}^{x\lambda})_c^k$, $(\mathbf{M}^{\lambda x})_c^k$ and the components of $(\mathbf{J}^x)_c^k$ and $(\mathbf{J}^e)_c^k$ depend linearly on the flexible deformation mode co-ordinates (e_i^k) . This implies that the dynamics of the spatial beam element, undergoing only a deformation along the length of the element, is completely determined by the translational mass matrix $(\mathbf{M}^{xx})_c^k$. This matrix is the same mass matrix as occurs in linear finite element analysis representing the consistent mass matrix for a truss element.¹⁵ In References 2 and 3 detailed expressions are presented for the partitioned mass matrices and for the quadratic velocity vectors in equation (65).

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