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Existence of Multiple Normal Mode Trajectories on Convex Energy Surfaces of Even, Classical Hamiltonian Systems

E. W. C. VAN GROESEN

Mathematical Institute, Catholic University, Toernooiveld, 6525 ED Nijmegen, The Netherlands

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Hamiltonian systems of n degrees of freedom for which the Hamiltonian is a function that is even both in its joint n coordinate variables as well as in its joint n momentum variables are discussed. For such systems the number of distinct trajectories which correspond to particular periodic solutions (normal modes) with the same energy, is investigated. To that end a constrained dual action principle is introduced. Applying min–max methods to this variational problem, several results are obtained, among which the existence of at least n distinct trajectories if specific conditions are satisfied.

1. INTRODUCTION

With $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ consider Hamiltons equations

$$-J\dot{z} = H'(z), \tag{1.1}$$

where $z = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$, H' denotes the gradient of H and J is the symplectic matrix $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, with I the identity in $\mathbb{R}^n$. In recent times, global results have been obtained about the existence of (multiple) periodic trajectories of (1.1) on a prescribed regular energy surface. If Σ denotes such a regular level set of H, i.e., H is constant on Σ with non-vanishing gradient H', Rabinowitz [15] proved the existence of at least one periodic solution if Σ is the boundary of a star-shaped domain (see also Seifert [20], Weinstein [21], Rabinowitz [16] and Clarke [6]). With additional geometrical conditions on Σ, Ekeland and Lasry [9] proved the existence of at least n distinct Hamiltonian trajectories on Σ (see also Ambrosetti and Mancini [1] and Berestycki, Lasry, Mancini and Ruf [5]).

In this paper we shall consider energy surfaces that have certain symmetry properties. In that case it may be expected that there are particular periodic solutions which reflect the symmetry properties of Σ. We shall

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restrict ourselves to find these particular periodic solutions, to be called normal modes. Of course, this does not mean that these are the only possible periodic motions.

Since the nomenclature used in the literature differs from place to place (cf., e.g., Weinstein [21, 22], Rosenberg [17, 18]) we start to state the precise definition of the particular systems and solutions that will be considered.

DEFINITION. A function $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ is called an *even, classical Hamiltonian* if it satisfies the following condition:

$$H(q, p) = H(-q, p) = H(q, -p) = H(-q, -p) \qquad \forall (q, p) \in \mathbb{R}^n \times \mathbb{R}^n; \tag{1.2}$$

for such functions the system (1.1) is called an even, classical Hamiltonian system.

A *normal mode* (solution) of an even, classical Hamiltonian system is a periodic solution of (1.1) such that $q(0) = 0$ and $p(\tau) = 0$ for some $\tau > 0$.

The property that q and p vanish at certain instants of time is essential for the definition of a normal mode. That, in the above definition, q vanishes precisely for $t = 0$ is a convenient normalization of the initial time, which is possible since solutions of (1.1) are invariant for time translations.

The point $(q(\tau), 0) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a *restpoint* of the normal mode. As we shall see in section 2, a normal mode is completely determined by its behaviour between $t = 0$ and the time at which p vanishes for the first time (which is then a quarter of the minimal period). The projection of the trajectory of a normal mode on q-space is a symmetrical curve through the origin that connects two symmetrical restpoints, along which the periodic solution oscillates back and forth.

A particular class of Hamiltonians satisfying (1.2) is given by functions of the form

$$H(q, p) = \tfrac{1}{2} p \cdot Mp + V(q), \tag{1.3}$$

(where $\cdot$ denotes the innerproduct in $\mathbb{R}^n$), with M a positive definite $n \times n$-matrix, and with V an even function on $\mathbb{R}^n$.

Hamiltonian systems with a Hamiltonian given by (1.3) are, up to a canonical transformation, equivalent to Hamiltonian systems with a Hamiltonian of the form

$$\begin{cases} H(q, p) = \tfrac{1}{2} p \cdot p + V(q), \\ \text{with } V \in C^2(\mathbb{R}^n, \mathbb{R}),\ V(q) = V(-q) \qquad \forall q \in \mathbb{R}^n; \end{cases} \tag{1.4}$$

functions of this kind will be called even, *natural* Hamiltonians.

For the rest of this paper the following assumptions are supposed to hold.

A.1. The set Σ is the boundary of a compact, convex set $\Omega \subset \mathbb{R}^{2n}$, with 0 in the interior of Ω.

A.2. There exists an even, classical Hamiltonian H such that Σ is a regular levelset of H.

The results to be obtained may now be stated.

THEOREM 1. *There exists at least one normal mode trajectory on Σ.*

For the following it is convenient to define for $a>0$ an integer $[a]$ (somewhat different from the integer part of a) by $[a] := \min\{k \in \mathbb{N}: a \leqslant k\}$. Furthermore, for $\rho > 0$, let B_ρ denote the ball of radius ρ and the origin as center in $\mathbb{R}^{2n}$.

THEOREM 2. *With Σ and Ω as above, assume that there exist positive numbers r and R and $k \in \mathbb{N}$, $1 \leqslant k \leqslant n$, with*

$$R < \sqrt{2k+1}\, r \tag{1.5}$$

such that

$$B_r \subset \Omega \subset B_R. \tag{1.6}$$

Then there exist at least $[n/k]$ distinct normal mode trajectories on Σ.

THEOREM 3. *Suppose that Σ satisfies* A.1. *and, instead of* A.2., *the stronger condition*:

A.2*. *There exists an even, natural Hamiltonian H of the form* (1.4), *such that Σ is a regular level set of H, $\Sigma = H^{-1}(E)$ say.*

Suppose, moreover, that there exist a monotonically increasing function $U \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ with $U(0)=0$, a number $a>1$ and $k \in \mathbb{N}$, $1 \leqslant k \leqslant n$, with

$$a < 2k+1, \tag{1.7}$$

such that

$$U(|x|) \leqslant V(x) \leqslant U(a\,|x|) \qquad \forall x \in \{x \in \mathbb{R}^n\colon V(x) \leqslant E\}. \tag{1.8}$$

Then there exist at least $[n/k]$ distinct normal mode trajectories on Σ.

For natural Hamiltonians, Theorem 1 has already been proved by Pak and Rosenberg [14]; their proof uses the Jacobian functional (cf. also Seifert [20]) and is different from the proof to be presented below.

Condition (1.6) is akin to the condition required in [1] and [9] for the proof of multiple existence of periodic trajectories on sets Σ that merely satisfy condition A.1.: Ekeland and Lasry [9] prove the existence of n distinct periodic trajectories on Σ if Ω satisfies (1.6) with $R<\sqrt{2}r$, and Ambrosetti and Mancini [1] obtain the existence of $[n/k]$ distinct periodic trajectories on Σ if Ω satisfies (1.6) with $R<\sqrt{k+1}\ r$. Hence, for sets Σ for which condition A.2. is also satisfied, the results of Theorem 2 are slightly better.

In case of a natural Hamiltonian H, for $n=2$ the existence of at least two normal mode trajectories has been proved by Rosenberg [19] for a restricted class of potentials V; for $n\geqslant 2$ the existence of at least n distinct normal mode trajectories on Σ has been shown in [11] for the case that V is a homogeneous function, and this same result can easily be established for more general "similarity" potentials of the form

$$V(x)=f(j(x)),$$

where j is the gauge of any symmetric, smooth, starshaped domain D in $\mathbb{R}^n$ (with $0\in D$) and $f\in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is any monotonically increasing function with $f(0)=0$. With respect to Theorem 3, note that condition (1.8) is quite different from any condition of the form (1.6). In fact, (1.7) and (1.8) require the existence of two monotonically increasing functions r and R: $[0, E]\to\mathbb{R}_+$ with $R(e)<(2k+1)\,r(e)$ such that for any $e\in(0, E)$ the set $D_e := \{x\in\mathbb{R}^n\colon V(x)\leqslant e\}$ satisfies $B_{r(e)}\subset D_e\subset B_{R(e)}$, where now B_ρ denotes the ball of radius ρ in $\mathbb{R}^n$. We are not aware of any results of this nature in the literature.

The organisation of the paper is as follows.

In Section 2 it is shown that normal mode trajectories on Σ correspond to solutions of a boundary value problem $(H)_\Sigma$. In Section 3 this boundary value problem is replaced by a variational problem: the "constrained dual action principle." This variational problem is related to the formulation used by Ekeland and Lasry [9] and that used by Ambrosetti and Mancini [1], but differs from these formulations in that respect that our constrained formulation simplifies the application of min-max variational methods in the next sections. A constrained variational formulation of this kind has also been used in [12] to provide a simplified proof of the results of Ekeland and Lasry. In section 4 the proof of theorem 1 follows with elementary means. In section 5, Ljusternik-Schnirelmann theory is applied to the constrained variational formulation. If it is possible to find problem $(H_i)_{\Sigma_i}$, $i=1, 2$, of the same type as $(H)_\Sigma$ for which $\Omega_1\subset\Omega\subset\Omega_2$, it is shown in Section 6 that a lower bound for the number of distinct normal mode trajectories can be estimated in terms of properties of these problems $(H_i)_{\Sigma_i}$. If these problems allow one to obtain certain results explicitely, which turns out to be the case if the conditions of Theorem 2 or 3 are satisfied, the required lower bound for the number of distinct normal mode trajectories can be obtained.

2. The Boundary Value Problem for Normal Modes

In this section it is shown that normal modes of (1.1) are in an one-to-one correspondence with specific solutions of an eigenvalue problem for a two-point boundary value operator.

For $\lambda \neq 0$ consider the boundary value problem $(H)_\Sigma$:

$$(H)_\Sigma \quad \begin{aligned} &-J\dot{z} = \lambda H'(z) \qquad \text{for } t \in (0, 1),\ z = (z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n, \\ &z_1(0) = z_2(1) = 0, \\ &z(t) \in \Sigma \qquad \text{for } t \in [0, 1]. \end{aligned} \tag{2.1}$$

(Note that for $\lambda = 0$ this problem admits no solution as $0 \notin \Sigma$.)

In order to describe the relation between solutions of $(H)_\Sigma$ and normal modes of (1.1), we introduce the following mapping. For $l \in \mathbb{N}$ define a continuous piecewise linear function ζ_l on $\mathbb{R}$ as the odd periodic continuation of

$$\begin{aligned} \zeta_l(t) &= lt \qquad \text{for } t \in [0, 1/l] \\ &= 2 - lt \qquad \text{for } t \in [1/l, 2/l]. \end{aligned}$$

Then, for functions $z = (z_1, z_2)$: $[0, 1] \to \mathbb{R}^n \times \mathbb{R}^n$ the *reproducing map* $\mathscr{C}_l$ is defined by

$$\mathscr{C}_l z(t) := (\operatorname{sign}(\zeta_l(t)) \cdot z_1(|\zeta_l(t)|), \operatorname{sign}(\dot{\zeta}_l(t)) \cdot z_2(|\zeta_l(t)|)) \tag{2.2}$$

(where $\operatorname{sign}(\dot{\zeta}_l(t)) := 0$ in the points of discontinuity of $\dot{\zeta}_l$).

Lemma 2.1. (i) *Let $z = (z_1, z_2)$ be for some $\lambda \neq 0$ a solution of $(H)_\Sigma$. Define a function $\hat{z}$: $\mathbb{R} \to \mathbb{R}^{2n}$ by*

$$\hat{z}(t) := \mathscr{C}_1 z(t/\lambda). \tag{2.3}$$

Then $\hat{z}$ is a normal mode solution of (1.1) on Σ with period $T \leqslant 4|\lambda|$.

(ii) *If $\hat{z}$ is a normal mode solution of (1.1) on Σ with period T, then, for any $k \in \mathbb{N} \cup \{0\}$, the function*

$$z^{(k)}(t) := \hat{z}\left((2k+1)\frac{T}{4}\,t\right) \tag{2.4}$$

is a solution on $(H)_\Sigma$ with parameter $\lambda = (2k+1)(T/4)$.

Proof. (i) As $z_1(0) = z_2(1) = 0$, the function $\hat{z}$ defined by (2.3) is continuous. Because of the symmetry properties of H, it is readily verified that $\hat{z}$ satisfies $-J\dot{z} = H'(z)$ on each interval $(k, k+1)$, $k \in \mathbb{Z}$. But as $H'(\hat{z})$ is continuous, this shows that $\hat{z}$ is differentiable and satisfies this equation on all of $\mathbb{R}$.

(ii) If $\hat{z} = (\hat{z}_1, \hat{z}_2)$ is a normal mode of (1.1), let $\tau > 0$ be the first instant at which $\hat{z}_2$ vanishes. Then $z^{(0)}(t) := \hat{z}(t \cdot \tau)$ is a solution of $(H)_\Sigma$ with parameter τ. According to part (i), the transformation (2.3) defines a normal mode with period $T \leqslant 4\tau$, which coincides with the given normal mode $\hat{z}$ on $(0, \tau)$, and hence on all of $\mathbb{R}$. Consequently, $\tau = T/4$ because $T < 4\tau$ is not possible by the definition of τ, and $z_2^{(0)}(2k+1) = 0$ for all $k \in \mathbb{N}$. As, for $k \in \mathbb{N}$, the function $z^{(k)}$ defined by (2.4) is given by $z^{(k)} = z^{(0)}((2k+1)\,t)$, it is readily seen that $z^{(k)}$ is a solution of $(H)_\Sigma$ with $\lambda = (2k+1)\,\tau$. ∎

This lemma shows that a normal mode is completely determined by its behaviour between the time of crossing the origin in z_1-(= configuration-) space and the first time, a $\frac{1}{4}$-period later, of crossing the origin in z_2-(= momentum-) space. The projection of the trajectory into the configuration-, as well as into the momentum- space is a symmetric curve along which the solution oscillates back and forth. Therefore, as expressed by (2.4), one and the same normal mode trajectory gives rise to distinct solutions of $(H)_\Sigma$. Related to this observation is the next lemma, which is an easy consequence of the symmetry properties of the Hamiltonians under consideration.

LEMMA 2.2. *Let $z = (z_1, z_2)$ be a solution of $(H)_\Sigma$ for some $\lambda \neq 0$. Define $z_- := (z_1, -z_2)$ and, for any $l \in \mathbb{N}$,*

$$z^{(l)}(t) := \mathscr{C}_{2l+1}\, z(t).$$

Then the functions $-z$, z_- and $z^{(l)}$ are all solutions of $(H)_\Sigma$, with parameters λ, $-\lambda$ and $(2l+1)\cdot\lambda$, respectively, and all these solutions correspond to the same normal mode trajectory.

Let Q be the functional defined by

$$Q(z) := \int z \cdot (-J\dot{z}), \tag{2.5}$$

where $\cdot$ denotes the Euclidean $\mathbb{R}^{2n}$-innerproduct and where, here and in the following, $\int$ denotes integration with respect to the independent variable over $(0, 1)$.

For a solution z of $(H)_\Sigma$ we have

$$Q(z) = \lambda \int H'(z) \cdot z, \tag{2.6}$$

and thus, because $H'(z) \cdot z$ is sign definite on Σ, $Q(z) \neq 0$. For the solutions z_- and $z^{(l)}$ defined in the foregoing lemma, we have

$$Q(z_-) = -Q(z), \qquad Q(\mathscr{C}_{2l+1} z) = (2l+1) \cdot Q(z).$$

As we are interested to find distinct normal mode trajectories on Σ, we can restrict ourselves to look for solutions z of $(H)_\Sigma$ for which $Q(z)$ has a prescribed sign (positive say). Furthermore, because of the relation between the parameter λ and the period T of the normal mode, we call a solution z of $(H)_\Sigma$ a *solution with minimal period* if there do not exist a number $l \in \mathbb{N}$ and a function $\tilde{z}$ such that $z = \mathscr{C}_{2l+1} \tilde{z}$. Then we have

PROPOSITION 2.3. *The number of distinct normal mode trajectories of* (1.1) *on Σ equals the number of distinct pairs of solutions $\pm z$ of $(H)_\Sigma$ which have minimal period and $Q(z) > 0$.*

3. THE CONSTRAINED DUAL ACTION PRINCIPLE

In this section we shall replace problem $(H)_\Sigma$ by a problem in the Calculus of Variations in the large. This "constrained dual action principle" will be dealt with in the next sections to provide the required results.

First we transform problem $(H)_\Sigma$ to an equivalent problem with a homogeneous Hamiltonian. This procedure, standard nowadays (cf. Weinstein [21], Rabinowitz [15]), is based on the observation that trajectories of solutions of Hamiltons equations on Σ depend only on Σ and not on the particular choice of the Hamiltonian for which Σ is a regular level set.

Let $j: \mathbb{R}^{2n} \to \mathbb{R}$ be the *gauge* of the set $\Omega: j(z) := \inf\{\lambda > 0 : z \in \lambda\Omega\}$, and for $\beta > 1$ define

$$K(z) := j^\beta(z). \tag{3.1}$$

Then the function K satisfies

$K \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and $K(z) = 1$ iff $z \in \Sigma$;

K is positively homogeneous of degree β;

K is convex (and strictly convex if Ω is strictly convex);

if $z = (z_1, z_2)$, $K(z_1, z_2) = K(-z_1, z_2) = K(z_1, -z_2) = K(-z_1, -z_2)$.

As is easily verified, $\Sigma = K^{-1}(1)$ is a regular level set of K: the gradients of H and K satisfy

$$H'(z) = \phi(z)\, K'(z), \qquad \text{with } \phi(z) = \frac{1}{\beta} H'(z) \cdot z, \tag{3.2}$$

where ϕ is sign definite on Σ. Consequently, defining a regular time transformation

$$s(t) := \int_0^t \phi(z(\tau))\, d\tau \Big/ \int_0^1 \phi(z(\tau))\, d\tau,$$

the function $z(t)$ is a solution of $(H)_\Sigma$ for some $\lambda \in \mathbb{R}$ iff the function $w(s)$, related to z by

$$w(s(t)) = z(t), \qquad t \in [0, 1], \tag{3.3}$$

is for some $\sigma \in \mathbb{R}$ a solution of

$$(K)_\Sigma \quad \begin{aligned} &-J\dot{w} = \sigma K'(w), \qquad w = (w_1, w_2) \in \mathbb{R}^n \times \mathbb{R}^n, \\ &w_1(0) = w_2(1) = 0, \\ &K(w(s)) = 1, \end{aligned} \tag{3.4}$$

the relation between λ and σ being given by

$$\sigma = \lambda \int_0^1 \phi(z(\tau))\, d\tau. \tag{3.5}$$

Remark 3.1. Note that the functional Q is parameter independent. In particular, functions z and w related by (3.3) satisfy

$$Q(z) = Q(w). \tag{3.6}$$

As for $(H)_\Sigma$, we shall therefore look for solutions w of $(K)_\Sigma$ for which $Q(w) > 0$.

Using Eulers identity for the homogeneous function K, $K'(w) \cdot w = \beta K(w)$ for $w \in \mathbb{R}^{2n}$, the value of the parameter σ for a solution w of $(K)_\Sigma$ is easily seen to be given by

$$\sigma = 2/\beta \cdot Q(w). \tag{3.7}$$

If we define the β-homogeneous functional k by

$$k(w) := \int K(w),$$

it may be observed that the solutions w of $(K)_\Sigma$ (with $Q(w) > 0$) are precisely the solutions (with $Q(w) > 0$) of the following variational problem

$$\text{stat}\{Q(w) \colon k(w) = 1;\ w_1(0) = w_2(1) = 0\}, \tag{3.8}$$

where, here and in the following, $\operatorname{stat}\{F(u): u \in \mathscr{C}\}$ is shorthand for the problem of finding critical (stationary) points of the functional F on the set $\mathscr{C}$, and any critical point will be called a solution of this problem. Indeed, as $K'(w) \not\equiv 0$ for $w \in \{w: k(w)=1;\, w_1(0)=w_2(1)=0\}$, Lagranges multiplier rule for constrained functions applies and provides the equivalence between $(K)_\Sigma$ and (3.8). Problem (3.8) may be interpreted as the (homogenized) *constrained classical action principle* for normal modes on Σ.

Remark 3.2. If $\beta \neq 2$, it can be shown that the solutions of (3.8) are in an one-to-one correspondence with the solutions of the following (homogenized) classical action principle:

$$\operatorname{stat}\{-Q(v)+k(v): v_1(0)=v_2(1)=0\}. \tag{3.9}$$

Indeed, if w is a solution of $(K)_\Sigma$ with $Q(w)>0$, then $\sigma>0$ and $v := \sigma^{1/\beta-2} w$ is a solution of (3.9) (v satisfies $-J\dot{v}=K'(v)$), and if v is a solution of (3.9) then $w := k(v)^{-/\beta} v$ is a solution of $(K)_\Sigma$.

As the function K is convex, it is possible to replace (3.8) by an equivalent variational problem, obtained from (3.8) by dualization. In the literature, this procedure of dualization has been applied by several authors to variants of the unconstrained variational problem (3.9) (cf., e.g., [1, 3, 6, 8, 9]).

To that end, let $G: \mathbb{R}^{2n} \to \mathbb{R}$ be the conjugate function of K:

$$G(u) := \sup\{u \cdot z - K(z): z \in \mathbb{R}^{2n}\}. \tag{3.10}$$

Standard results from convex analysis show that

$G \in C^0(\mathbb{R}^{2n}, \mathbb{R})$ and G is convex;

G is positively homogeneous of degree α, where $1/\alpha + 1/\beta = 1$;

if $u=(u_1, u_2)$: $G(u_1, u_2)=G(-u_1, u_2)=G(u_1, -u_2)=G(-u_1, -u_2)$.

Denoting by $\partial G(u)$ the subdifferential of G at u, i.e., the set of points z for which the supremum in (3.10) is actually attained, the following relation holds: $u=K'(z)$ iff $z \in \partial G(u)$.

If K is strictly convex, $\partial G(u)$ is a singleton for each $u \in \mathbb{R}^{2n}$, which implies that G is differentiable on $\mathbb{R}^{2n}$. In that case

$$u=K'(z) \qquad \text{iff } z=G'(u). \tag{3.11}$$

For ease of presentation we shall assume in the following that Ω is strictly convex, i.e., that $G \in C^1(\mathbb{R}^{2n}, \mathbb{R})$.

Furthermore, we introduce the inverse of the mapping $-J(d/dt)$ on

$\{w: w_1(0) = w_2(1) = 0\}$. The inverse, to be denoted by L, can be written down explicitely:

$$\left.\begin{array}{l} u = -J\dot{w} \\ w_1(0) = w_2(1) = 0 \end{array}\right\} \quad \text{iff } w = Lu := \left(\int_0^t u_2, \int_t^1 u_1 \right). \tag{3.12}$$

Introducing the Banachspace $B := L_\alpha((0,1), \mathbb{R}^{2n})$ with the usual norm: $\|u\|_{L_\alpha} := (\int |u|^\alpha)^{1/\alpha}$, the mapping L can be extended to a self-adjoint, compact, mapping on B.

Finally, define a quadratic functional f and an α-homogeneous functional g as

$$f(x) := \frac{1}{2} \int x \cdot Lx, \qquad x \in B, \tag{3.13}$$

$$g(x) := \int G(x), \qquad x \in B. \tag{3.14}$$

Then f and g are both C^1-functionals, and the *constrained dual action principle* is defined to be the variational problem

$$(g)_{\mathcal{M}} \quad \operatorname{stat}\{g(x) : x \in \mathcal{M}\}, \tag{3.15}$$

where

$$\mathcal{M} := \{x \in B : f(x) = 1\}. \tag{3.16}$$

Note that the set $\mathcal{M}$ is a regular manifold in B as $f'(x) = Lx \neq 0$ for $x \in \mathcal{M}$. The multiplier rule applies and states that solutions of (3.15) satisfy for some $\mathbb{R}$:

$$(G)_{\mathcal{M}} \quad \begin{cases} \mu Lx = G'(x), \\ f(x) = 1. \end{cases} \tag{3.17}$$

Using Euler's identity for the function G, the value μ is seen to be

$$\mu = \frac{\alpha}{2} g(x). \tag{3.18}$$

PROPOSITION 3.3. *The set of solutions z of problem $(H)_\Sigma$ for which $Q(z) > 0$ is homeomorphic to the set of solutions of the variational problem $(g)_{\mathcal{M}}$.*

In view of the derivation of problem $(K)_\Sigma$ and Remark 3.1., this result is an immediate consequence of the following lemma.

LEMMA 3.4. (i) *Distinct solutions w of $(K)_\Sigma$ with $Q(w)>0$ are mapped onto distinct solutions x of $(G)_{\mathcal{M}}$ via the transformation*:

$$x = Q(w)^{-1/2}(-J\dot{w}). \tag{3.19}$$

(ii) *Distinct solutions x of $(G)_{\mathcal{M}}$ are mapped onto distinct solutions w of $(K)_\Sigma$ via the transformation*

$$w = \frac{\alpha}{2}\left(\frac{\beta}{\alpha}\right)^{1/\beta} \cdot g(x)^{1/\alpha}\, Lx. \tag{3.20}$$

(iii) *If x and w are corresponding solutions of $(G)_{\mathcal{M}}$ and $(K)_\Sigma$ respectively, the values $g(x)$ and $Q(w)$ are related by*:

$$g(x) = \left(\frac{2}{\alpha}\right)\cdot\left(\frac{2}{\beta}\right)^{\alpha-1} Q(w)^{\alpha/2}. \tag{3.21}$$

Proof. Let x be given by (3.19). Then $f(x)=\frac{1}{2}\int xLx=\frac{1}{2}Q(w)^{-1}\cdot \int w(-J\dot{w})=1$. As w satisfies $-J\dot{w}=(2/\beta)\,Q(w)\,K'(w)$ (cf. (3.7)), it follows that $K'(w)=(\beta/2)\,Q(w)^{-1/2}x$. Using (3.11), this may be written as $w=G'(\beta/2)\,Q(w)^{-1/2}x)$, from which it is seen that x satisfies $\mu Lx=G'(x)$ with $\mu=(2/\beta)^{\alpha-1}\,Q(w)^{\alpha/2}$. This proves (i) and, because of (3.18) also (iii).

If x is a solution of $(G)_{\mathcal{M}}$, let $v := \mu Lx = G'(x)$. Then $\mu x = -J\dot{v}$ and $v_1(0)=v_2(1)=0$. Using (3.11), it follows that v satisfies $-J\dot{v}=\mu K'(v)$, and from this that $K(v(t))$ is constant $(=k(v))$ for all $t\in[0,1]$. Now, take $b>0$ such that $w := bv$ satisfies $k(w)=1$, i.e., $b=k(v)^{-1/\beta}$. As $K'(bv)=b^{\beta-1}K'(v)$, the function $w=k(G'(x))^{-1/\beta}\,G'(x)$ is a solution of $(K)_\Sigma$ with $\sigma=\mu b^{2-\beta}$. From the extremality relation for the conjugate functions K and G it follows that $k(G'(x))+g(x)=\int xG'(x)=\alpha g(x)$ for all x. Together with $G'(x)=(\alpha/2)\,g(x)\,Lx$ (cf. (3.18)), the last expression for w can be rewritten to the form (3.20). ▮

Remark 3.5. Note that if we choose $\alpha=\beta=2$, the relation (3.21) is particularly simple: $g(x)=Q(w)$, and thus because of remark 3.1:

$$g(x) = Q(z) \tag{3.22}$$

if x and z are corresponding solutions of $(g)_{\mathcal{M}}$ and $(H)_\Sigma$, respectively. This is particularly interesting since the value of the functional g at a critical point is thus, via $Q(w)=\sigma$, directly related to the (quarter) period σ of the periodic solution of $(K)_\Sigma$. In this sense, the constrained dual action principle may be considered to be a *variational principle for the period* of the solutions of the homogenized problem $(K)_\Sigma$ when K is choosen to be homogeneous of degree 2. The constrained dual action principle may also be interpreted in the following way. If functions x and w are related as in

(3.19), the constraint $f(x)=1$ is satisfied and, for $\alpha=\beta=2$, $g(x)=g(-J\dot{w})/Q(w)$. As a consequence, solutions of

$$\text{stat}\left\{\frac{g(-J\dot{w})}{Q(w)}: w_1(0)=w_2(1)=0\right\} \tag{3.23}$$

are in an one-to-one correspondence with the solutions of the constrained classical action principle (3.8), and the critical values are, apart from sign, equal to the (quarter) period of the corresponding periodic solutions of $(K)_\Sigma$. In other words, (3.23) is an *unconstrained* variational formulation for the normal mode solutions

Remark 3.6. In the same way as in remark 3.2., if $\alpha \neq 2$ (i.e., $\beta \neq 2$), the solutions of $(g)_{\mathscr{M}}$ are in an one-to-one correspondence with the solutions of the following unconstrained dual action principle:

$$\text{stat}\{-f(x)+g(x): x \in B\}. \tag{3.24}$$

A variational principle of this kind, for arbitrary periodic solutions instead of for normal modes, has been used by Ambrosetti and Mancini [1] in their simplified proof of the result of Ekeland and Lasry [9]. In [12] it has been shown that for that case the constrained formulation is more convenient to deal with. The same applies for the normal modes: although (3.24) may be used to prove the results of this paper, the constrained formulation (3.15) has several technical advantages.

4. Existence of a Constrained Minimal Normal Mode

Here we shall prove Theorem 1 using the variational formulation (3.15). The following properties are elementary consequences of the fact that L is compact as a mapping from B into its dual $B^* = L_\beta((0,1), \mathbb{R}^{2n})$, and that g is convex and continuous on B:

(i) the functional $f: B \to \mathbb{R}$ is continuous with respect to weak convergence in B,

(ii) the functional $g: B \to \mathbb{R}$ is lower semi-continuous with respect to weak convergence in B.

According to Proposition 3.3, the next result provides us with a proof of Theorem 1.

Proposition 4.1. *The constrained minimization problem*:

$$\inf\{g(x): x \in \mathscr{M}\} \tag{4.1}$$

has at least one pair of solutions.

Proof. The set $\mathcal{M}$ is closed with respect to weak convergence in B because of (i) above. Because of property (ii) above, the existence of a pair of minimal elements of g on $\mathcal{M}$ follows from elementary arguments as soon as it is shown that g is coercive on $\mathcal{M}$ (i.e., $g(x_n) \to \infty$ for any sequence $\{x_n\} \subset \mathcal{M}$ for which $\|x_n\|_{L_\alpha} \to \infty$). But this is an immediate consequence of the fact that g is coercive on all of B: if $a \in \mathbb{R}$ is defined to be the minimum value of the function G on the unit sphere in $\mathbb{R}^{2n}$, then $a > 0$ and $G(x) \geqslant a\,|x|^\alpha$ for all $x \in \mathbb{R}^{2n}$. Hence

$$g(x) \geqslant a\,\|x\|^\alpha_{L_\alpha}, \tag{4.2}$$

which shows that g is coercive on B. ▌

Remark 4.2. Alternative proofs of theorem 1 can be obtained using the formulation (3.23). If α is taken to satisfy $\alpha < 2$, the existence of a solution of (3.23) can be proved with the aid of the mountain pass theorem of Ambrosetti and Rabinowitz [2]. If α satisfies $\alpha > 2$, the minimization problem

$$\inf\{-f(x) + g(x) \colon x \in B\} \tag{4.3}$$

has a solution because $-f+g$ is lower semi-continuous with respect to weak convergence, and coercive on B.

Remark 4.3. As $g'(x) = 0$ iff $x = 0$, the solutions of (4.1) are also the solutions of

$$\inf\{g(x) \colon f(x) \geqslant 1, x \in B\}. \tag{4.4}$$

Furthermore, if $g \in C^2(B, \mathbb{R})$, it is also possible to consider the following *inverse* extremum formulation of (4.1) (cf. [10]):

$$\sup\{f(x) \colon g(x) = 1; x \in B\}. \tag{4.5}$$

The solutions of (4.5) and (4.1) are the same except for some multiplicative factor.

5. Ljusternik–Schnirelmann Theory for Solutions with Minimal Period

In this section we shall describe the general idea how well-known min–max methods can be applied to problem (3.15). In the next section the proof of Theorems 2 and 3 will be completed.

To start with, observe that the functionals f and g are even on B. Hence, once the so-called Palais–Smale (P.S.) condition is verified, the Ljusternik–Schnirelmann theory for even functionals on symmetric sets in a Banach space may be applied, using as index theory (cf. Benci [4]) the Ljusternik–Schnirelmann category for sets in the quotient space obtained by identifying antipodal points or, somewhat simpler, the *genus* of symmetric sets in B (Krasnoselskii [13], Coffman [7]). Let us first verify the necessary compactness condition.

LEMMA 5.1. *The functional g restricted to the set $\mathcal{M}$, defined by* (3.16), *satisfies the P.S.-condition, i.e., any sequence $\{x_n\}$ which satisfies* (i) $x_n \in \mathcal{M}$, (ii) *$g(x_n)$ is uniformly bounded and* (iii) *the derivative of g along $\mathcal{M}$ at x_n tends to zero as $n \to \infty$, contains a subsequence that converges in B to some element of B.*

Proof. Let $\{x_n\}$ be any sequence satisfying (i), (ii) and (iii). From (ii) and (4.2) it follows that $\{x_n\}$ is uniformly bounded in $B = L_\alpha$. Hence there exists a subsequence, again to be denoted by $\{x_n\}$, that converges weakly in L_α to some $\hat{x} \in L_\alpha$. Since $f'(x) = Lx$, where L is a compact mapping from L_α into $L_\beta (= B^*)$, it holds that $Lx_n \to L\hat{x}$ in L_β. From this it easily follows that $\hat{x} \in \mathcal{M}$. In order to exploit (iii), note that the derivative of g along $\mathcal{M}$ at $x_n \in \mathcal{M}$ is given by $G'(x_n) - \mu_n L x_n$ for some $\mu_n \in \mathbb{R}$. Because of (iii) we have $\int G'(x_n)\, x_n - \mu_n \int Lx_n \cdot x_n \to 0$ as $n \to \infty$. Since the first integral in this expression is uniformly bounded, and $\int Lx_n \cdot x_n = 2f(x_n) \to 2f(\hat{x}) = 2$, the sequence $\{\mu_n\}$ is uniformly bounded. Taking a subsequence that converges in $\mathbb{R}$, to $\hat{\mu}$ say, the corresponding subsequence of $\{x_n\}$ satisfies $\mu_n L x_n \to \hat{\mu} L \hat{x}$ in L_β. Defining functions $\rho_n \in L_\beta$ by $\rho_n := G'(x_n) - \mu_n L x_n$, we have $G'(x_n) = \rho_n + \mu_n L x_n$, or according to (3.11), $x_n = K'(\rho_n + \mu_n L x_n)$. Since $\rho_n + \mu_n L x_n \to \hat{\mu} L \hat{x}$ in L_β, and since K' is a continuous mapping from L_β into L_α, it follows that x_n converges in L_α: $x_n \to K'(\hat{\mu} L \hat{x}) = \hat{x} \in \mathcal{M}$, which completes the proof. ∎

For closed, symmetric subsets Γ of $\mathcal{M}$ we denote the *genus* of Γ by $\gamma(\Gamma)$, i.e.,

> $\gamma(\Gamma) = k \in \mathbb{N} \cup \{0\}$ if k is the least number for which there exists an odd, continuous mapping $\phi\colon \Gamma \to \mathbb{R}^k \setminus \{0\}$, and $\gamma(\Gamma) = \infty$ if no such mapping exists.

For $[a, b] \subset \mathbb{R}$ let $g^{-1}([a, b])$ be the preimage of g in $\mathcal{M}$, i.e., $g^{-1}([a, b]) = \{x \in \mathcal{M}: a \leqslant g(x) \leqslant b\}$.

Then the results of the Ljusternik-Schnirelmann theory for the even functional g on the symmetric set $\mathcal{M}$ may be summarized in the following way:

Let m denote the minimum value of g on $\mathcal{M}$ (cf. Proposition 4.1)

$$m := \inf\{g(x): x \in \mathcal{M}\}. \tag{5.1}$$

Then:

the number of distinct pairs of critical points of g *on* $\mathscr{M}$ *with critical values in* $[m, b]$ *is not less than* $\gamma(g^{-1}[m, b])$. (5.2)

Since $\mathscr{M} = g^{-1}([m, \infty))$, and as $\gamma(\mathscr{M}) = \infty$, the existence of infinitely many pairs of distinct solutions of (3.15), and, consequently, of infinitely many pairs of distinct solutions of $(H)_\Sigma$, follows.

However, according to proposition 2.4 we are only interested in solutions of $(H)_\Sigma$ which have minimal period.

As the reproducing map $\mathscr{C}_k$ defined by (2.2) can be considered as a mapping from B into B, it can be shown that

$$\mathscr{C}_{2k+1} z = (2k+1)\, L\mathscr{C}_{2k+1} x \qquad \text{if } z = Lx, \text{ for } k \in \mathbb{N}. \tag{5.3}$$

Therefore, if we define

$$B_* := \{x \in B \mid \exists_{l \in \mathrm{N}} \exists_{y \in B} x = \mathscr{C}_{2l+1} y\},$$

and put $\tilde{B} := B \backslash B_*$, the set $\tilde{B}$ may be called the set of functions in B with minimal period. Consequently, Propositions 2.4 and 3.3 lead to

PROPOSITION 5.2. *The number of distinct normal mode trajectories on* Σ *equals the number of distinct pairs of solutions* $\pm x$ *of* $(g)_{\mathscr{M}}$ *that belong to* $\tilde{B}$.

In general it is difficult to decide whether a specific solution of (3.15) belongs to $\tilde{B}$ or to B_*. Nevertheless, using the same idea as in [12], we can argue as follows.

As is easily verified, the functionals f and g satisfy

$$f(\mathscr{C}_k x) = \frac{1}{k} f(x), \qquad g(\mathscr{C}_k x) = g(x) \qquad \text{for } x \in B,\ k \in \mathbb{N}. \tag{5.4}$$

Defining a number m_*:

$$m_* := \inf\{g(x) : \mathscr{M} \cap B_*\}, \tag{5.5}$$

the following relation between m and m_* holds.

LEMMA 5.3. $m_* = 3^{\alpha/2} m$.

Proof. If $x \in B_*$, with $x = \mathscr{C}_{2l+1} y$ for some $y \in B$, $l \in \mathbb{N}$, and $f(x) = 1$, then, writing $\hat{y} := (2l+1)^{-1/2} y$, $\hat{y}$ satisfies $f(\hat{y}) = 1$ and $g(\hat{y}) = (2l+1)^{-\alpha/2}$ $g(y) = (2l+1)^{-\alpha/2} g(x)$. Consequently, $m \leqslant 3^{-\alpha/2} m_*$. Moreover, if $\bar{x}$ is a minimal element of (4.1), i.e., $g(\bar{x}) = m$, then $\sqrt{3}\, \mathscr{C}_3 \bar{x} \in B_* \cap \mathscr{M}$ and $g(\sqrt{3}\, \mathscr{C}_3 \bar{x}) = 3^{\alpha/2} g(\bar{x})$. Hence $m^* = 3^{\alpha/2} m$. ∎

By definition of the number m_*, any solution x of (3.15) for which $g(x) < m_*$, has minimal period (belongs to $\tilde{B}$). This observation has two immediate consequences. The first one is a property of the solution of problem (4.1).

PROPOSITION 5.4. *The solutions of the minimization problem* (4.1) *have minimal period.*

The second consequence is a lower bound for the number of normal modes, obtained from (5.2) and Proposition 5.2.

To describe it, we shall use, here and in the following, the notation

$$\gamma(g^{-1}([a, b))) := \max\{\gamma(g^{-1}([a, c]): c < b\}.$$

PROPOSITION 5.5. *The number of distinct normal mode trajectories on Σ is not less than $\gamma(g^{-1}([m, m_*)))$.*

In the next section we shall estimate $\gamma(g^{-1}([m, m_*)))$ using the assumptions of Theorems 2 and 3.

6. Proof of Theorems 2 and 3

Let Ω_1 and Ω_2 be compact, convex domains in $\mathbb{R}^{2n}$ such that

$$\Omega_1 \subset \Omega \subset \Omega_2,$$

with 0 in the interior of Ω_1. Furthermore, for $i = 1, 2$, assume that $\Sigma_i := \partial\Omega_i$ where Σ_i is a regular level set of some function $H_i \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ that satisfies the symmetry properties (1.2). If j_i denotes the gauge of Ω_i, and $K_i := j_i^\beta$, then

$$K_1 \geqslant K \geqslant K_2 \qquad \text{on } \mathbb{R}^{2n},$$

and with G_i the conjugate of K_i, and $g_i = \int G_i$:

$$G_1 \leqslant G \leqslant G_2 \qquad \text{on } \mathbb{R}^{2n} \qquad \text{and} \qquad g_1 \leqslant g \leqslant g_2 \qquad \text{on } B.$$

In the same way as for the set Ω, we may consider the problem $(K_i)_{\Sigma_i}$ and the corresponding constrained dual action principles $(g_i)_{\mathscr{M}}$.

Defining $m_i := \inf\{g_i(x): x \in \mathscr{M}\}$, and $m_{i*} := 3^{\alpha/2} m_i$, we have $m_1 \leqslant m \leqslant m_2$, and $m_{1*} \leqslant m_* \leqslant m_{2*}$. Moreover, for any $\bar{m} < m_*$:

$$g^{-1}([m, m_*)) \supset g^{-1}([m, \bar{m}]) \supset g_2^{-1}([m, \bar{m}])).$$

Using the monotonicity of the genus (i.e., $\gamma(A_1) \leqslant \gamma(A_2)$ if $A_1 \subset A_2$), it follows that

$$\gamma(g^{-1}([m, m_*))) \geqslant \gamma(g_2^{-1}([m, m_{1*})). \tag{6.1}$$

This result, together with Proposition 5.5 yields a useful lower bound for the number of distinct normal modes as soon as we can find sets Ω_1 and Ω_2 for which $\gamma(g_2^{-1}([m, m_{1*})))$ can be calculated.

It is convenient to relate m_1 and $\gamma(g_2^{-1}([m, \bar{m}]))$ to properties of the original problems $(H_i)_{\Sigma_i}$.

LEMMA 6.1. *Let $\alpha = \beta = 2$ for simplicity. Then*:

(i) $m_1 = \inf\{Q(z)$: *z is a solution of $(H_1)_{\Sigma_1}$ with minimal period and* $Q(z) > 0\}$.

(ii) *For any $\bar{m}$*:

$$\gamma(g_2^{-1}([m, \bar{m}])) \geqslant \gamma(\{z\text{: } z \textit{ is a solution of } (H_2)_{\Sigma_2} \textit{ with minimal period and with } 0 < Q(z) \leqslant \bar{m}\}). \tag{6.2}$$

Proof. (i) If $x_1 \in \mathcal{M}$ is an element for which $g(x_1) = m_1$, x_1 has minimal period (Proposition 5.4). If z_1 denotes the corresponding solution of $(H_1)_{\Sigma_1}$, z_1 has minimal period, and $Q(z_1) = g(x_1)$ according to (3.22). Moreover, all solutions z of $(H_1)_{\Sigma_1}$ with minimal period correspond to solutions x of $(g_1)_{\mathcal{M}}$ which have minimal period, and as $Q(z) = g(x) \geqslant g(x_1)$, the result follows.

(ii) For any $\bar{m}$:

$$\begin{aligned} g_2^{-1}([m, \bar{m}]) &\supset \{x\text{: } x \text{ is a solution of } (g_2)_{\mathcal{M}} \text{ with } g_2(x) \leqslant \bar{m}\} \\ &\supset \{x\text{: } x \text{ is a solution of } (g_2)_{\mathcal{M}} \text{ with minimal period and } g_2(x) \leqslant \bar{m}\}. \end{aligned}$$

The last inclusion is a consequence of the fact that with any solution x of $(g)_{\mathcal{M}}$, $(2l+1)^{1/2}\,\mathscr{C}_{2l+1}x$ is also a solution of $(g)_{\mathcal{M}}$, and $g((2l+1)^{1/2}\,\mathscr{C}_{2l+1}x) = (2l+1)^{\alpha/2}\,g(x)$.

The last set is homeomorphic to the set of solutions z of $(H_2)_{\Sigma_2}$ which have minimal period and for which $0 < Q(z) = g_2(x) \leqslant \bar{m}$.

Since sets which are homeomorphic by an odd homeomorphism have the same genus, the result (6.2) follows. ∎

Proof of Theorem 2 for $k = 1$. Assume $B_{r_1} \subset \Omega \subset B_{r_2}$, where B_ρ is the ball of radius ρ in $\mathbb{R}^{2n}$. For H_i we can take $H_i(z) = (1/r_i^2)\,|z|^2$; then $\Sigma_i = H_i^{-1}(1)$ and $H_i(z) \equiv K_i(z)$ if $\beta = 2$.

Problem $(H_i)_{\Sigma_i}$ is particularly simple: the set Γ_i of solutions z of $(H_i)_{\Sigma_i}$ that have minimal period and $Q(z) > 0$ can be written down explicitly:

$$\Gamma_i = \left\{ x = (z_1, z_2) = \left(e \sin \frac{\pi}{2} t, e \cos \frac{\pi}{2} t \right) : e \in \mathbb{R}^n, |e| = r_i \right\}. \tag{6.3}$$

Moreover, for all $z \in \Gamma_i$: $Q(z) = \pi/4 \cdot r_i^2$. Consequently, $m_1 = \pi/4 \cdot r_1^2$ and, taking $\alpha = \beta = 2$, $m_{1*} = (3\pi/4)\, r_1^2$.

From (6.1) and (6.2) it follows that

$$\gamma(g^{-1}([m, m_{1*}))) \geqslant \gamma(\Gamma_2) \text{ provided } Q(\Gamma_2) = \frac{\pi}{4} \cdot r_2^2 < m_{1*}.$$

As the set Γ_2 has genus $\gamma(\Gamma_2) = n$, Proposition 5.5 yields the existence of at least n distinct normal mode trajectories provided that $r_2^2 < 3r_1^2$. This proves Theorem 2 for the case $k = 1$.

Proof of Theorem 3 *for* $k = 1$. Now suppose that $H(q, p) = \frac{1}{2}|p|^2 + V(q)$, where V is an even function on $\mathbb{R}^n$ that satisfies, with $a > 1$,

$$U(|x|) \leqslant V(x) \leqslant U(a\,|x|), \qquad x \in \mathbb{R}^n,$$

where $U \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ is monotonically increasing with $U(0) = 0$. Let $H_1(q, p) := \frac{1}{2}\,|p|^2 + U(a\,|q|)$ and $H_2(q, p) = \frac{1}{2}\,|p|^2 + U(|q|)$. If we define

$$\Omega_i := \{(q, p) \in \mathbb{R}^n \times \mathbb{R}^n : H_i(q, p) \leqslant E\},$$

then $\Omega_1 \subset \Omega \subset \Omega_2$, and $\Sigma_i = \partial\Omega_i = H_i^{-1}(E)$.

Again, problems $(H_i)_{\Sigma_i}$ are simple: the sets Γ_i of solutions z of $(H_i)_{\Sigma_i}$ that have minimal period and $Q(z) > 0$ are given by

$$\Gamma_1 = \left\{ z = \left(e \frac{1}{a} \hat{q}(t), e\hat{p}(t) \right) : e \in \mathbb{R}^n, |e| = 1 \right\},$$

and (6.4)

$$\Gamma_2 = \{ z = (e\hat{q}(t), e\hat{p}(t)) : e \in \mathbb{R}^n, |e| = 1 \},$$

where $(\hat{q}, \hat{p})$ is the (unique) solution of

$$\begin{cases} \dot{q} = \lambda p, & -\dot{p} = \lambda U'(q) \\ q(0) = p(1) = 0 \end{cases}$$

for which $\lambda > 0$, $q(t) > 0$ for $t \in (0, 1)$ and $\frac{1}{2}p^2 + U(q) = E$. If we put $\hat{m} := \int \hat{p} \cdot \dot{\hat{q}}$, then $Q(z) = \hat{m}$ for all $z \in \Gamma_2$ and $Q(z) = (1/a)\, \hat{m}$ for all $z \in \Gamma_1$.

Consequently, with $\alpha = \beta = 2$:

$$\gamma(g^{-1}([m, m_{1*})) \geqslant \gamma(\Gamma_2) \text{ provided } Q(\Gamma_2) = \hat{m} < m_{1*} = \frac{3}{a}\, \hat{m}.$$

As $\gamma(\Gamma_2)=n$, Proposition 5.5 yields the required result provided $a<3$. This proves Theorem 3 for the case $k=1$.

Proof of Theorems 2 *and* 3 *for* $k>1$. Let us briefly describe the modifications that are necessary to obtain the results if $k>1$. Define for $k\geqslant 1$

$$B_*^{(k)} := \{x\in B: \exists_{l\in \mathbb{N}, l\geqslant k} \exists_{y\in B}\, x=\mathscr{C}_{2l+1}\, y\},$$

and

$$\tilde{B}^{(k)} := B\backslash B_*^{(k)}.$$

Note that $B_*^{(1)}=B_*$ and $\tilde{B}^{(1)}=\tilde{B}$.

Noting that for each solution x of $(g)_{\mathrm{M}}$ that has minimal period, $(2l+1)^{1/2}\mathscr{C}_{2l+1}x$ is for each $l\in\mathbb{N}$ also a solution of $(g)_{\mathscr{M}}$, which, moreover, belongs to $\tilde{B}^{(k)}$ if $l<k$, we obtain the following generalization of Proposition 5.2.

PROPOSITION 6.2. *The number of distinct normal mode trajectories on* Σ *is not less than* $1/k$ *times the number of distinct pairs of solutions of* $(g)_{\mathscr{M}}$ *that belong to* $\tilde{B}^{(k)}$.

If we define $m_*^{(k)} := \inf\{g(x): x\in\mathscr{M}\cap B_*^{(k)}\}$, it is readily seen that $m_*^{(k)}=(2k+1)^{\alpha/2}m$.

Then, Proposition 5.5 may be replaced by

PROPOSITION 6.3. *The number of disttinct normal mode trajectories on* Σ *is not less than* $1/k\cdot\gamma(g^{-1}([m, m_*^{(k)})))$.

With $\gamma(g^{-1}([m, m_*^{(k)})))\geqslant\gamma(g_2^{-1}([m, m_{1*}^{(k)})))$, it follows from (6.2) that

$$\gamma(g_2^{-1}([m, m_{1*}^{(k)})))\geqslant\gamma(\Gamma_2) \text{ provided } Q(\Gamma_2)<m_{1*}^{(k)},$$

where Γ_2 is the set defined by (6.3), (6.4), in case of Theorems 2 and 3, respectively.

In case of theorem 2, $Q(\Gamma_2)<m_{1*}^{(k)}$ leads to $r_2^2<(2k+1)\,r_1^2$, and in case of Theorem 3 this condition reads $a<2k+1$. This completes the proof of the theorems.

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References

1. A. Ambrosetti and G. Mancini, On a theorem by Ekeland and Lasry concerning the number of periodic Hamiltonian trajectories, *J. Differential Equations* **43** (1982), 249–256.
2. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
3. J. P. Aubin and I. Ekeland, Second order evolution equations associated with convex Hamiltonians, *Canad. Math. Bull.* **23** (1980), 81–94.
4. V. Benci, A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations, *Comm. Pure Appl. Math.* **34** (1981), 393–432.
5. H. Berestycki, J. M. Lasry, G. Mancini, and B. Ruf, Existence of multiple periodic orbits on starshaped Hamiltonian surfaces, preprint.
6. F. H. Clarke, Periodic solutions to Hamiltonian inclusions; *J. Differential Equations* **40** (1981), 1–6.
7. C. V. Coffman, A minimum-maximum principle for a class of non-linear integral equations, *J. Analyse Math.* **22** (1969), 391–419.
8. I. Ekeland, Periodic solutions of Hamiltonian equations and a theorem of Rabinowitz, *J. Differential Equations* **34** (1979), 523–534.
9. I. Ekeland and J. M. Lasry, On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface, *Ann. of Math.* **112** (1980), 283–319.
10. E. W. C. van Groesen, Dual and Inverse formulations of constrained extremum problems; *J. Math. Modelling*, **1** (1980), 237–254.
11. E. W. C. van Groesen, On normal modes in classical Hamiltonian systems; *J. Nonlinear Mech.* **18** (1983), 55–70.
12. E. W. C. van Groesen, "A Simplified Proof for a Result of Ekeland and Lasry Concerning the Number of Periodic Hamiltonian Trajectories," Report 8229, Catholic University, Nijmegen, 1982.
13. M. A. Krasnoselskii, "Topological Methods in the Theory of Nonlinear Integral Equations," Pergamonn, Oxford, 1963.
14. C. H. Pak and R. M. Rosenberg, On the existence of normal mode vibrations in nonlinear systems, *Quart. Appl. Math.* **26** (1968), 403–416.
15. P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **31** (1978), 157–184.
16. P. H. Rabinowitz, Periodic solutions of a Hamiltonian system on a prescribed energy surface, *J. Differential Equations* **33** (1979), 336–352.
17. R. M. Rosenberg, Normal modes of non-linear dual-mode systems, *J. Appl. Mech.* **27** (1960), 263–268.
18. R. M. Rosenberg, On the existence of simple trajectories in admissible n-degree of freedom systems, *Adv. Appl. Mech.* **9** (1966), 155–242.
19. R. M. Rosenberg, On the existence of normal mode vibrations of nonlinear systems with two degrees of freedom, *Quart. Appl. Math.* **22** (1964), 217–234.
20. H. Seifert, Periodische Bewegungen mechanischer Systeme, *Math. Z.* **51** (1948), 197–216.
21. A. Weinstein, Periodic orbits for convex Hamiltonian systems, *Ann. of Math.* **108** (1978), 507–518.
22. A. Weinstein, Normal modes for nonlinear Hamiltonian systems, *Invent. Math.* **20** (1973), 47–57.