Let $G(s)=C(sI-A)^{-1}B$ be the transfer matrix of (Σ).

Let (T,F,G,K,H) be an element of G acting on (Σ). $G(s)$ is transformed in $G_1(s)$ such that

$$G_1(s)=H\left(I-C(sI-A-BF)^{-1}K\right)^{-1}G(s)\left(I-F(sI-A)^{-1}B\right)^{-1}G$$
$$=B_1(s)G(s)B_2(s) \quad [12].$$

$B_1(s)$ and $B_2(s)$ are bicausal matrices, i.e., proper with proper inverses.

Note that rational bicausal matrices have neither zeros nor poles at infinity. So they play the same role for the study of infinite structure as unimodular matrices in the finite case.

Consider an element of $\mathcal{G}$ which transforms (A,B,C) in its Morse's canonical form whose transfer matrix is $\Lambda(s)$. Then there exist rational bicausal matrices $B_1(s)$ and $B_2(s)$ such that

$$\Lambda(s)=B_1(s)G(s)B_2(s),$$

where $\Lambda(s)$ is a $(p\times m)$ rational matrix

$$\Lambda(s)=\begin{bmatrix} s^{-n_1} & & & & & \\ & s^{-n_2} & & & \bigcirc & \\ & & \ddots & & & \\ & & & s^{-n_{m'}} & & \\ & & & & 0 & \\ & \bigcirc & & & & \ddots \\ & & & & & 0 \end{bmatrix},$$

and $m'=\operatorname{rank} G(s)$, $\{n_1,\cdots,n_{m'}\}=\text{list } I_4$.

Since $G(s)=B_1^{-1}(s)\Lambda(s)B_2^{-1}(s)$ where $B_1^{-1}(s)$ and $B_2^{-1}(s)$ are bicausal matrices, the infinite structure of $G(s)$ is the same as the infinite structure of $\Lambda(s)$ [8]. From Definition 5 it follows that the n_i's are the infinite zero orders in the classical sense.

This statement extends the result of [20] to the noninvertible case.

V. Concluding Remark

Structural properties of irreducible systems are discussed in [21]. We show that for irreducible systems the state space may be decomposed into a direct sum of almost controllability subspaces, whose C-images decompose the output space in a special way. The dimensions of these subspaces are the infinite zeros orders.

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Decoupled Decomposition of the Riccati Equation

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Abstract—**The general matrix Riccati equation is decomposed into product of two factors where the first one is determined independent of the second factor. Condition for existence of solution of the Riccati equation is given via this decomposition and the existence of solution of the self-adjoint matrix Riccati equation arising in optimal control and Kalman filtering is directly established from that condition.**

I. Introduction

The Riccati equation has become increasingly important in many branches of applied mathematics. The self-adjoint Riccati equation arising in optimal control and estimation theory has received particular attention [1], [2]. In this correspondence, we consider the most general (not necessarily self-adjoint) matrix Riccati equation and study a decomposition of its solution as the product of two matrix factors. Decompositions of this nature were known in mathematics for a long time [3]. The special feature of our decomposition is that the first matrix factor can be determined independent of the second one. This decoupling helps one to solve the Riccati equation numerically in two simple steps. The general matrix Riccati equation studied here appears, among other fields, in the linear-quadratic differential games [4].

II. Decoupled Decomposition of Riccati Equations

We consider the general matrix Riccati equation

$$\dot{L}-F_{21}-F_{22}L+LF_{11}+LF_{12}L=0; \qquad L(t_0)=L_0, \tag{2.1}$$

where $L(t)\in\mathbb{R}^{(n-r)\times r}$, $1\leqslant r\leqslant n$ and F_{ij}, $i,j=1,2$ are arbitrary matrices and represent a partition of the $n\times n$ coefficient matrix

$$F=\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{2.2}$$

associated with (2.1). We assume that F is continuous in t for $t\geqslant t_0$.

Let $\Phi_1(t,s)$ and $\Phi_2(t,s)$ be the transition matrices corresponding to the dynamical systems $\dot{x}=F_{11}(t)x$ and $\dot{x}=F_{22}(t)x$, respectively. Then we have the following.

Theorem 1: Equation (2.1) has a unique solution in $[t_0,T]$ if the matrix

$$\pi(t)=\left(I+\int_{t_0}^{t}\Phi_1(t_0,s)F_{12}(s)\psi(s)\,ds\right) \tag{2.3}$$

has an inverse for all t in $[t_0,T]$, where $\psi(t)$ is the solution of the Volterra integral equation

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$$\psi(t)=K(t)\phi_1(t,t_0)+\int_{t_0}^{t}M(t,s)F_{12}(s)\psi(s)\,ds \tag{2.4}$$

with $K(t)$ satisfying the Lyapunov equation

$$\dot{K}(t)-F_{21}-F_{22}K(t)+K(t)F_{11}=0; \qquad K(t_0)=L_0 \tag{2.5}$$

and

$$M(t,s)=\int_s^t \Phi_2(t,\sigma)F_{21}(\sigma)\Phi_1(\sigma,s)\,d\sigma. \tag{2.6}$$

In fact, in the case when $\pi(t)$ has an inverse for all t in $[t_0,T]$, we have the decomposition

$$L(t)=\psi(t)\pi(t)^{-1}\Phi_1(t_0,t). \tag{2.7}$$

Proof: We first show that $L(t)$ given by (2.7) indeed satisfies (2.1). We note that

$$\frac{\partial M(t,s)}{\partial t}=F_{22}\int_s^t \Phi_2(t,\sigma)F_{21}\Phi_1(\sigma,s)\,d\sigma+F_{21}\Phi_1(t,s),$$

which gives us

$$\begin{aligned}\dot{\psi}(t)&=[F_{21}+F_{22}K(t)-K(t)F_{11}]\Phi_1(t,t_0)\\&\quad+K(t)F_{11}\Phi_1(t,t_0)+F_{22}\int_{t_0}^{t}M(t,s)F_{12}(s)\psi(s)\,ds\\&\quad+F_{21}\int_{t_0}^{t}\Phi_1(t,s)F_{12}(s)\psi(s)\,ds\\&=F_{22}\psi(t)+F_{21}\Phi_1(t,t_0)+F_{21}\int_{t_0}^{t}\Phi_1(t,s)F_{12}(s)\psi(s)\,ds.\end{aligned}$$

Now

$$\begin{aligned}\Phi_1(t,t_0)&+\int_{t_0}^{t}\Phi_1(t,s)F_{12}(s)\psi(s)\,ds\\&=\Phi_1(t,t_0)+\Phi_1(t,t_0)\int_{t_0}^{t}\Phi_1(t_0,s)F_{12}(s)\psi(s)\,ds\\&=\Phi_1(t,t_0)\pi(t),\end{aligned}$$

so that, from $L(t)$ given by (2.7),

$$\begin{aligned}\dot{L}(t)&=[F_{22}\psi(t)+F_{21}\Phi_1(t,t_0)\pi(t)]\pi(t)^{-1}\Phi_1(t,t_0)\\&\quad-\psi(t)\pi^{-1}(t)\Phi_1(t_0,t)F_{12}\psi(t)\pi^{-1}(t)\Phi_1(t,t_0)\\&\quad-\psi(t)\pi^{-1}(t)\Phi_1(t_0,t)F_{11}\\&=F_{21}+F_{22}L(t)-L(t)F_{11}-L(t)F_{12}L(t),\end{aligned}$$

which is (2.1). Clearly,

$$L(t_0)=\psi(t_0)=K(t_0)=L_0.$$

To prove uniqueness, let $L(t)$ be any solution of (2.1). Let

$$\chi(t)\triangleq L(t)\Phi_1(t,t_0)\pi(t).$$

Then

$$\begin{aligned}\dot{\chi}(t)&=[F_{21}+F_{22}L(t)-L(t)F_{11}-L(t)F_{12}L(t)]\Phi_1(t,t_0)\pi(t)\\&\quad+L(t)F_{11}\Phi_1(t,t_0)\pi(t)+L(t)\Phi_1(t,t_0)\Phi_1(t_0,t)F_{12}\psi(t)\\&=F_{21}\Phi_1(t,t_0)+F_{21}\int_{t_0}^{t}\Phi_1(t,s)F_{12}(s)\psi(s)\,ds\\&\quad+F_{22}\chi(t)+L(t)F_{12}[\psi(t)-\chi(t)]\end{aligned}$$

while

$$\dot{\psi}(t)=F_{21}\Phi_1(t,t_0)+F_{21}\int_{t_0}^{t}\Phi_1(t,s)F_{12}(s)\psi(s)\,ds+F_{22}\psi(t).$$

Subtracting, we get

$$\frac{d}{dt}[\chi(t)-\psi(t)]=F_{22}[\chi(t)-\psi(t)]-L(t)F_{12}[\chi(t)-\psi(t)]$$

with

$$\chi(t_0)-\psi(t_0)=0.$$

Therefore, $\chi(t)\equiv\psi(t)$ implying that any solution $L(t)$ of (2.1) must satisfy

$$L(t)\Phi_1(t,t_0)\pi(t)=\psi(t),$$

where $\psi(t)$ is the unique solution of (2.4). Therefore $\psi(t)\pi^{-1}(t)\Phi_1(t_0,t)$ is the unique solution of (2.1) if $\pi^{-1}(t)$ exists for all t in $[t_0,T]$. □

III. Riccati Equation Arising in Kalman Filtering

The self-adjoint Riccati equation arising in Kalman filtering has received particular attention in the literature. It has the specific structure $F_{12}'=F_{12}>0$, $F_{21}'=F_{21}\geq 0$, $F_{22}'=-F_{11}$ and a symmetric boundary condition $L_0=L_0'\geq 0$, where "prime" denotes transpose. It is well-known that in this case (2.1) has a unique solution in any finite interval $[t_0,T]$. We establish this result from Theorem 1.

Theorem 2: Equation (2.1) in the special case when $F_{12}'=F_{12}>0$, $F_{21}'=F_{21}\geq 0$, $F_{22}'=-F_{11}$ with the initial condition $L_0'=L_0\geq 0$ always admits a unique solution in any finite interval $[t_0,T]$.

Proof: All we have to prove is that the matrix

$$\pi(t)=I+\int_{t_0}^{t}\Phi_1(t_0,s)F_{12}(s)\psi(s)\,ds \tag{3.1}$$

has an inverse for all t in $[t_0,T]$ in the special case under consideration here. Let

$$\begin{aligned}\pi_1(t)&\triangleq\Phi_1(t,t_0)\pi(t)\\&=\Phi_1(t,t_0)+\int_{t_0}^{t}\Phi_1(t,s)F_{12}(s)\psi(s)\,ds.\end{aligned} \tag{3.2}$$

Then

$$\dot{\pi}_1(t)=F_{11}\pi_1(t)+F_{12}\psi(t), \qquad \pi_1(t_0)=I \tag{3.3}$$

and we know that

$$\dot{\psi}(t)=-F_{11}'\psi(t)+F_{21}\pi_1(t), \qquad \psi(t_0)=L_0\geq 0. \tag{3.4}$$

Suppose that $\pi(t)$ is not invertible at a point t^* in $(t_0,T]$; that is, $\det\pi(t^*)=0$. Then $\det\pi_1(t^*)=0$ and there exists a vector ξ such that

$$\pi_1(t^*)\xi=0, \tag{3.5}$$

implying that

$$\xi'\pi_1(t^*)'\psi(t^*)\xi=0. \tag{3.5$'$}$$

On the other hand, (3.3) and (3.4) give us

$$\frac{d}{dt}[\xi'\pi_1(t)'\psi(t)\xi]=\xi'\psi(t)'F_{12}\psi(t)\xi+\xi'\pi_1(t)'F_{21}\pi_1(t)\xi\geq 0 \tag{3.6}$$

while

$$\xi'\pi_1(t_0)'\psi(t_0)\xi=\xi'L_0\xi\geq 0.$$

Case 1: $\xi'L_0\xi>0$.
This implies that $\xi'\pi_1(t)'\psi(t)\xi>0$ for $t\geq t_0$ contradicting (3.5)$'$.
Case 2: $\xi'L_0\xi=0$.
We claim that in this case

$$\psi(t)\xi\equiv 0 \qquad \text{for } t\in(t_0,t^*). \tag{3.7}$$

Suppose that there is a point $t' \in (t_0, t^*)$ such that $\psi(t')\xi \neq 0$. Then from (3.6),

$$\frac{d}{dt}(\xi'\pi_1(t)'\psi(t)\xi)\bigg|_{t=t'} > 0 \qquad \text{because } F_{12} > 0,\ F_{21} \geq 0.$$

On the other hand, (3.6) implies that

$$\xi'\pi_1(t')'\psi(t')\xi \geq 0$$

so that from continuity it follows that

$$\xi'\pi_1(t')'\psi(t)\xi > 0 \qquad \text{for } t \in (t', t^*]$$

contradicting (3.5)'. Thus, (3.7) holds. Now

$$\pi_1(t^*)\xi = \Phi_1(t^*, t_0)\xi - \int_{t_0}^{t^*}\Phi_1(t^*, s)F_{12}(s)\psi(s)\xi\, ds,$$

which yields, from (3.5) and (3.7), that

$$0 = \Phi_1(t^*, t_0)\xi,$$

that is, $\xi = 0$.

Combining these two cases, we see that for every $t^* \in [t_0, T]$, there does not exist a vector ξ such that $\pi_1(t^*)\xi = 0$. Thus, $\pi_1(t)$ and therefore, $\pi(t)$ is invertible for all $t \in [t_0, T]$ and the theorem is established.

It follows from (2.7) and the discussion above that, in the case when $\pi(t)$ has an inverse in $[t_0, T]$, we have the decomposition

$$L(t) = \psi(t)\pi_1(t)^{-1} \tag{3.8}$$

where

$$\dot{\pi}_1(t) = F_{11}\pi_1(t) + F_{12}\psi(t), \qquad \pi_1(t_0) = I \tag{3.9}$$

$$\dot{\psi}(t) = -F_{11}'\psi(t) + F_{21}\pi_1(t), \qquad \psi(t_0) = L_0. \tag{3.10}$$

This is, of course, the well-known classical decomposition of Riccati equation (see [3], for example). The decomposition obtained in this paper basically decouples $\psi(t)$ from $\pi_1(t)$, enabling one to solve $L(t)$ numerically in two stages: first obtain $\psi(t)$ and then use this to obtain $\pi_1(t)$. There is another interesting feature of this decoupling, $\psi(t)$ satisfies a Volterra integral equation which always admits a unique solution and the obvious iteration

$$\psi^{(n+1)}(t) = K(t)\Phi_1(t, t_0) + \int_{t_0}^{t}M(t, s)F_{12}(s)\psi^{(n)}(s)\, ds \tag{3.11}$$

converges geometrically fast to that solution.

Finally, we remark that invertibility of $\pi(t)$ in the special case $F_{12}' = F_{12} > 0$, $F_{21}' = F_{21} \geq 0$, $F_{22}' = -F_{11}$ can be established indirectly by looking at the LQ-control problem (see [5] for details).

IV. Conclusion

We have obtained a decomposition of the solution of a general matrix Riccati equation as the product of two matrix factors where the first one can be determined independent of the second one. Furthermore, the first factor is the solution of a Volterra integral equation. Connection between matrix Riccati equation and Fredholm integral equation was observed already in [6]. Our result, however, is different and uses existence results of the solution of Volterra integral equations.

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A Solution of the Unit Circle Problem via Schwarz Canonical Form

BISWA NATH DATTA

Abstract—**The unit circle problem is the problem of finding the number of eigenvalues of a matrix A inside and outside the unit circle. It is associated with stability analysis of the system $x_{k+1} = Ax_k$. An algorithm is developed in this paper to solve the problem via Schwarz canonical form.**

I. Introduction

Let S be a real matrix of the form

$$S = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ -s_n & 0 & 1\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -s_2 & -s_1 \end{bmatrix} \tag{1}$$

where $s_i \neq 0$, $i = 1, 2, \cdots, n$.

In 1956 it was shown by Schwarz [6] that an arbitrary nonderogatory matrix A can be transformed by similarity to a matrix of the above form and that the matrix A is a stability matrix (that is, all the eigenvalues of A have negative real parts) iff $s_i > 0$, $i = 1, 2, \cdots, n$. The matrix S is known as the Schwarz canonical form of A.

Schwarz further showed that the numbers of positive and negative terms in a certain sequence formed out of the elements of S give complete information on the location of eigenvalues of A inside a given half-plane (see also [3]). The problem of counting the eigenvalues of a matrix inside a half-plane is known as the inertia problem. The inertia of a matrix A, In $(A) = (\pi(A), \nu(A), \delta(A))$ is an integer triple of the numbers of eigenvalues of A with positive, negative, and zero real parts, respectively. The unit circle problem is an analogous problem to the inertia problem. It is the problem of counting the number of eigenvalues of a matrix inside and outside the unit circle. This problem arises in stability analysis of discrete time control system

$$x_{k+1} = Ax_k. \tag{2}$$

The system (2) is asymptotically stable iff all the eigenvalues of A have moduli one.

In view of the Schwarz's result mentioned above, it is natural to investigate if the Schwarz canonical form can also be employed to solve the unit circle problem.

In [1] Anderson, Jury, and Mansur showed that given a polynomial $f(x)$, a lower Hessenberg matrix A with nonzero superdiagonal, similar to a companion matrix of $f(x)$, can be constructed with the property that

$$A^TPA - P = Q$$

where P and Q are diagonal matrices, and that the inertia of P supplies information on the numbers of zeros of $f(x)$ inside and outside the unit circle.

In this paper it is shown how the Schwarz matrix S can be used rather directly to solve the unit circle problem. Given S, a matrix H is constructed such that, whenever H is nonsingular, it is symmetric and the

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