verify that the desired stability behavior was obtained for $K = 2$, but that a rapid decrease in $u(t)$ resulted because of the decreasing time input. However, those convergence terms which insured that $\dot{V}$ was negative definite acted to force $\dot{e}$ to zero. The system error approached zero asymptotically, but very slowly, along the line $\dot{e} = 0$.

An effective means of circumventing this problem of slow convergence is to select all unity elements in the matrix R, i.e.,

$$V = \tfrac{1}{2}(e + \dot{e})^2 \tag{13}$$

which is positive definite except along the line $\dot{e} + e = 0$.[2] The derivative of V in (13) is

$$\dot{V} = (e + \dot{e})(\dot{e} + \ddot{e}) \tag{14}$$

which may be made negative definite by controlling $\ddot{e}$ such that $\ddot{e} < -\dot{e} - K(\dot{e} + e)$ in Region I of Fig. 2 and $\ddot{e} > -\dot{e} - K(\dot{e} + e)$ in Region II. Such a control forces the $\dot{e}$ versus e trajectory toward the line $\dot{e} + e = 0$ and then along that line toward the origin. The resulting control u satisfies

$$\begin{aligned}\dot{u} = &-476x_1 + 57x_3 + 1.85u + 0.03y_1 + 0.01y_2 - 0.25 \\ &+ 0.15t - 0.01\dot{e} - 0.01K(\dot{e} + e) + \{454|x_1| + 286|x_2| \\ &+ 44|x_3| + 1.25|u|\}\operatorname{sign}(\dot{e} + e).\end{aligned} \tag{15}$$

Simulation results given in Figs. 2 and 4 for $K = 1$ show that the desired convergence was achieved in this case.

Conclusions

Signal synthesis techniques have been derived for a larger class of systems than could be handled previously. Some of the more severe restrictions on the plant and model have been removed primarily by emphasizing relationships between plant and model output states. Improved results for the basic problem with unknown plant parameters have been obtained by using expanded Lyapunov functions to provide smoother synthesized plant inputs for reducing the output error asymptotically to zero. A useful extension would be to combine a parameter identification scheme with the results developed here. Such a procedure would permit the synthesis of a more nearly continuous plant input for incompletely specified parameters and would offer an important alternative to adding error derivative terms to the Lyapunov function.

The results of this paper may also be applied for an improved design of noninteracting multiple-input, multiple-output control systems by selecting a noninteracting model with the desired dynamic characteristics. This model may be used with a given nonlinear, time-varying plant with incompletely specified parameters to form a model reference adaptive control system.

References

[1] C.-C. Hang and P. C. Parks, "Comparative studies of model reference adaptive control systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 419–428, Oct. 1973.
[2] R. V. Monopoli, "The direct method of Liapunov applied to the design of controllers for a class of nonlinear and time-varying processes," NASA Tech. Rep. N63-23650, 1963.
[3] ——, "Design of controllers for nonlinear and time-varying plants," NASA Rep. CR-152, Jan. 1965.
[4] R. V. Monopoli and C. R. Gitlitz, "Control of linear time-varying plants with differential operators acting on the input," in *Proc. 5th Ann. Allerton Conf.*, Urbana, Ill., Oct. 1967, pp. 603–613.
[5] L. P. Grayson, "The status of synthesis using Liapunov's method," *Automatica*, vol. 3, pp. 91–121, Dec. 1965.
[6] P. N. Nikiforuk, M. M. Gupta, and H. H. Choe, "Control of unknown plants in reduced state space," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 489–496, Oct. 1969.
[7] P. N. Nikiforuk, "A deterministic type controller for a class of uncertain plants with partial differentiation," in *Proc. 3rd IFAC Symp. Identification and System Parameter Estimation*, The Hague, Netherlands, June 12–15, 1973.
[8] I. D. Landau and B. Courtiol, "Adaptive model following systems for flight control and simulation," *J. Aircr.*, vol. 9, pp. 668–674, Sept. 1972.

[2] Though unconventional, the Lyapunov function in (13) may be utilized for this case because $V > 0$ and $\dot{V} < 0$ are guaranteed by (15) for all points in the error phase plane for which $\dot{e} \neq -e$ and, moreover, the error e obviously approaches zero asymptotically if $\dot{e}$ does equal $-e$.

An Extension of the Stochastic Linear Regulator Problem

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***Abstract*—The stochastic linear regulator problem is extended to cover the case of non-Gaussian white noise. Using martingale theory, it is proved that the linear optimal regulator is also optimal for non-Gaussian white noise.**

I. Introduction

In this paper an extension is studied of the well-known stochastic linear regulator problem, in which it is desired to minimize

$$E\left\{\int_{t_0}^{t_1} [X_t'R_1(t)X_t + U_t'R_2(t)U_t]\, dt + X_{t_1}'P_1X_{t_1}\right\} \tag{1}$$

for the system described by

$$\begin{aligned}\dot{X}_t &= A(t)X_t + B(t)U_t + Z_t, \qquad t \geqslant t_0 \\ X_{t_0} &= X_0\end{aligned} \tag{2}$$

where Z_t, $t \geqslant t_0$, is Gaussian white noise. The problem will be considered where Z_t, $t \geqslant t_0$, is a generalization of white noise. This generalization will include the case where Z_t, $t \geqslant t_0$, is the derivative of a process with zero-mean, independent increments.

To solve this problem, use will be made of results in martingale theory and stochastic integrals that have recently become available. An introduction to martingale theory as applied in this paper is given by Wong [1]. The derivation that will be given is based upon results presented in the comprehensive paper by Doléans-Dade and Meyer [2].

II. Linear Stochastic Differential Equations and Martingales

Rather than working with differential equations such as (2), involving white noise, we shall consider stochastic differential equations of the form

$$\begin{aligned}dX_t &= A(t)X_t\, dt + B(t)U_t\, dt + dW_t, \qquad t \geqslant t_0, \\ X_{t_0} &= X_0.\end{aligned} \tag{3}$$

Here X_t, $t \geqslant t_0$, is an n-dimensional vector stochastic process. The process W_t, $t \geqslant t_0$, is a given local martingale with respect to the increasing family of sigma fields $\mathfrak{F}_t$, $t \geqslant t_0$, where $\mathfrak{F}_t$ is the sigma field generated by W_s and X_s, $t_0 \leqslant s \leqslant t$. In Section III an additional constraint will be imposed upon the process W. Furthermore, X_0 is a given second-order vector stochastic variable, independent of the process W, while U_t, $t \geqslant t_0$, is a given vector-valued second-order stochastic process. Finally, $A(t)$ and $B(t)$, $t \geqslant t_0$, are given, continuous matrix functions of appropriate dimensions. It will be assumed that $W_{t_0} = 0$, and that

$$EW_tW_t' = F(t) < \infty, \qquad \text{for each } t \geqslant t_0 \tag{4}$$

with the prime denoting the transpose.

For the definition of a local martingale the reader is referred to the literature (Wong [1], Doléans-Dade and Meyer [2]). In Section IV examples will be given of processes W that are included in the problem description.

The stochastic differential equation (3) is a representation of the stochastic integral equation

$$X_t = X_0 + \int_{t_0}^{t} A(s)X_s\, ds + \int_{t_0}^{t} B(s)U_s\, ds + W_t, \qquad t \geqslant t_0. \tag{5}$$

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The solution of this equation is given by

$$X_t = \Phi(t,t_0)X_0 + \int_{t_0}^{t} \Phi(t,s)B(s)U_s\, ds + \int_{t_0}^{t} \Phi(t,s)\, dW_s, \quad t \geqslant t_0. \tag{6}$$

Here Φ is the transition matrix corresponding to the homogeneous system $\dot{x}(t) = A(t)x(t)$, $t \geqslant t_0$. The third term on the right hand side of (6) is a stochastic integral [2] with respect to the martingale $\{W_t, \mathfrak{F}_t, t \geqslant t_0\}$.

Equation (6) shows that X_t, $t \geqslant t_0$, is the sum of the process

$$\Phi(t,t_0)X_0 + \int_{t_0}^{t} \Phi(t,s)B(s)U_s\, ds, \qquad t \geqslant t_0 \tag{7}$$

which is a process of bounded variation, and the process

$$\int_{t_0}^{t} \Phi(t,s)\, dW_s, \qquad t \geqslant t_0 \tag{8}$$

which is a local martingale with respect to $\mathfrak{F}_t$, $t \geqslant t_0$. This (and also (5)) shows that X is, by definition, a semi-martingale. For such processes the Doléans-Dade–Meyer "differentiation rule" [2], which is a generalization of the well-known Itô differentiation rule, applies. Let $\phi(x)$ be a twice differentiable function of the vector variable x, and let X_t, $t \geqslant t_0$, be a vector-valued semi-martingale. Then

$$\phi(X_t) = \phi(X_{t_0}) + \int_{t_0}^{t} \phi_x'(X_{s-})dX_s + \tfrac{1}{2}\mathrm{tr}\left[\int_{t_0}^{t} \phi_{xx}(X_{s-})d\Psi_s\right] + \sum_{t_0 \leqslant s \leqslant t} [\phi(X_s) - \phi(X_{s-}) - \phi_x'(X_{s-})(X_s - X_{s-})]. \tag{9}$$

Here ϕ_x is the gradient of ϕ with respect to x. Furthermore, tr denotes the trace of a matrix, while ψ_t, $t \geqslant t_0$, is a matrix-valued stochastic process, the (i,j)th element of which is $\langle X_i^c, X_j^c\rangle_t$. Here X_i^c is the i-th component of the continuous local martingale part of the process X_i. Also, if a scalar local martingale $\{M_t, \mathfrak{F}_t, t \geqslant t_0\}$ has continuous sample functions, $\langle M,M\rangle_t$, $t \geqslant t_0$, is the process that makes $\{M_t^2 - \langle M,M\rangle_t, \mathfrak{F}_t, t \geqslant t_0\}$ a local martingale. Finally, if $\{M_t, \mathfrak{F}_t, t \geqslant t_0\}$ and $\{N_t, \mathfrak{F}_t, t \geqslant t_0\}$ are two local martingales with continuous sample functions, $\langle M,N\rangle = \frac{1}{2}(\langle M+N, M+N\rangle - \langle M,M\rangle - \langle N,N\rangle)$.

A useful extension of the differentiation rule is obtained by applying it to the function $\psi(x,t)$ with respect to the semi-martingale $\{\mathrm{col}(X_t,t), \mathfrak{F}_t, t \geqslant t_0\}$, with $\{X_t, \mathfrak{F}_t, t \geqslant t_0\}$ a semi-martingale. Assuming that ψ has first and second partial derivatives with respect to the components of x and a first partial derivative with respect to t, we obtain

$$\psi(X_t,t) = \psi(X_{t_0},t_0) + \int_{t_0}^{t} \psi_t(X_{s-},s)\, ds + \int_{t_0}^{t} \psi_x'(X_{s-},s)\, dX_s + \tfrac{1}{2}\,\mathrm{tr}\left[\int_{t_0}^{t} \psi_{xx}(X_{s-},s)\, d\Psi_s\right] + \sum_{t_0 \leqslant s \leqslant t} [\psi(X_s,s) - \psi(X_{s-},s) - \psi_x'(X_{s-},s)(X_s - X_{s-})] \tag{10}$$

where ψ_t is the partial derivative of ψ with respect to t, ψ_x the gradient of ψ with respect to x, and ψ_{xx} the Jacobian of ψ with respect to x.

III. The Stochastic Linear Optimal Regulator Problem

Consider the system described by the stochastic differential equation (3), where an additional restriction on the process W will be imposed later. We shall study the problem of choosing U_t, $t \geqslant t_0$, such that

$$E\left\{\int_{t_0}^{t_1} [X_t'R_1(t)X_t + U_t'R_2(t)U_t]\, dt + X_{t_1}'P_1X_{t_1}\right\} \tag{11}$$

is minimized. Here R_1 and R_2 are continuous and symmetric matrix functions on $[t_0,t_1]$, with $R_1(t)$ nonnegative-definite and $R_2(t)$ positive-definite for each $t \in [t_0,t_1]$. P_1 is nonnegative-definite and symmetric. The process U_t, $t \geqslant t_0$, is restricted to be adapted to $\mathfrak{F}_t$, where $\mathfrak{F}_t$ is the sigma field generated by W_s and X_s, $t_0 \leqslant s \leqslant t$. This means that for each t, U_t is to be completely determined by $\mathfrak{F}_t$.

In order to solve this problem, a method will be followed that is a generalization of a derivation of Åström [3] for the case that W_t, $t \geqslant t_0$, is Brownian motion. Let the symmetric matrix function $P(t)$, $t_0 \leqslant t \leqslant t_1$, be the solution of the matrix Riccati equation

$$-\dot{P}(t) = R_1(t) - P(t)B(t)R_2^{-1}(t)B'(t)P(t) + A'(t)P(t) + P(t)A(t), \qquad t_0 \leqslant t \leqslant t_1,$$
$$P(t_1) = P_1. \tag{12}$$

It is well-known that this matrix differential equation has a unique solution under the conditions on A, B, R_1, and R_2 as stated (Kalman [4]). We now apply the differentiation rule (10) to the function $\psi(X_t,t) = X_t'P(t)X_t$ at $t = t_1$. This yields

$$X_{t_1}'P_1X_{t_1} = X_0'P(t_0)X_0 + \int_{t_0}^{t_1} X_{t-}'\dot{P}(t)X_{t-}\, dt + 2\int_{t_0}^{t_1} X_t'P(t)\, dX_t + \mathrm{tr}\left[\int_{t_0}^{t_1} P(t)\, d\Psi_t\right] + \sum_{t_0 \leqslant t \leqslant t_1} [X_t'P(t)X_t - X_{t-}'P(t)X_{t-} - 2X_{t-}P(t)(X_t - X_{t-})]. \tag{13}$$

Substitution of $\dot{P}$ from (12) and rearrangement of the last term yields, using also (3),

$$X_{t_1}'P_1X_{t_1} = X_0'P(t_0)X_0 - \int_{t_0}^{t_1} X_{t-}'[R_1(t) - P(t)B(t)R_2^{-1}(t)B'(t)P(t) + A'(t)P(t) + P(t)A(t)]X_{t-}\, dt + 2\int_{t_0}^{t_1} X_{t-}'P(t) \cdot [A(t)X_t\, dt + B(t)U_t\, dt + dW_t] + \mathrm{tr}\left[\int_{t_0}^{t_1} P(t)\, d\Psi_t\right] + \sum_{t_0 \leqslant t \leqslant t_1} (X_t - X_{t-})'P(t)(X_t - X_{t-}). \tag{14}$$

Now it follows from (5) that $d\Psi_t = d\Psi_t^W$, where the (i,j)th element of Ψ_t^W is $\langle W_i^c, W_j^c\rangle$, with W_i^c the continuous part of the ith component of W. Also, $X_t - X_{t-} = W_t - W_{t-}$. Using this, and rearranging, we obtain

$$\int_{t_0}^{t_1} [X_t'R_1(t)X_t + U_t'R_2(t)U_t]\, dt + X_{t_1}'P_1X_{t_1} = X_0'P(t_0)X_0 + \int_{t_0}^{t_1} [U_t + R_2^{-1}(t)B'(t)P(t)X_t]'R_2(t)[U_t + R_2^{-1}(t)B'(t)P(t)X_t]\, dt + 2\int_{t_0}^{t_1} X_{t-}'P(t)\, dW_t + \mathrm{tr}\left[\int_{t_0}^{t_1} P(t)\, d\Psi_t^W\right] + \mathrm{tr}\Big[\sum_{t_0 \leqslant s \leqslant t_1} P(t) \cdot (W_t - W_{t-})(W_t - W_{t-})'\Big]. \tag{15}$$

Taking the expectation of both sides of this equality it follows that

$$E\left\{\int_{t_0}^{t_1} [X_t'R_1(t)X_t + U_t'R_2(t)U_t]\, dt + X_{t_1}'P_1X_{t_1}\right\} = \mathrm{tr}[P(t_0)EX_0X_0'] + E\left\{\int_{t_0}^{t_1} [U_t + R_2^{-1}(t)B'(t)P(t)X_t]'R_2(t)[U_t + R_2^{-1}(t)B'(t)P(t)X_t]\, dt\right\} + \mathrm{tr}\left[\int_{t_0}^{t_1} P(t)Ed\Psi_t^W\right]$$

$$+ \operatorname{tr}\left[\sum_{t_0 \leqslant s \leqslant t_1} P(t)E(W_t - W_{t-})(W_t - W_{t-})'\right] \tag{16}$$

where use has been made of the fact that W is, by assumption, a local martingale. To further simplify this expression, consider

$$E\left\{\Psi_t^W + \sum_{t_0 \leqslant s \leqslant t} (W_s - W_{s-})(W_s - W_{s-})'\right\}, \qquad t \geqslant t_0 \tag{17}$$

the (i,j)th element of which is

$$E\{\langle W_i^c, W_j^c\rangle_t + \sum_{t_0 \leqslant s \leqslant t} (W_{i,s} - W_{i,s-})(W_{j,s} - W_{j,s-})\}. \tag{18}$$

In order to evaluate this quantity we apply the Doléans-Dade–Meyer rule to the function $\phi(W_{i,t}, W_{j,t}) = W_{i,t}W_{j,t}$ with respect to the local martingale $\{\operatorname{col}(W_{i,t}, W_{j,t}), \mathfrak{F}_t, t \geqslant t_0\}$. We obtain

$$\begin{aligned} W_{i,t}W_{j,t} = {} & \int_{t_0}^{t} (W_{j,s-}dW_{i,s} + W_{i,s-}dW_{j,s}) + \int_{t_0}^{t} d\langle W_i^c, W_j^c\rangle_s \\ & + \sum_{t_0 \leqslant s \leqslant t} [W_{i,s}W_{j,s} - W_{i,s-}W_{j,s-} \\ & - W_{j,s-}(W_{i,s} - W_{i,s-}) - W_{i,s-}(W_{j,s} - W_{j,s-})] \end{aligned} \tag{19}$$

or

$$\begin{aligned} W_{i,t}W_{j,t} = {} & \int_{t_0}^{t} (W_{j,s-}dW_{i,s} + W_{i,s-}dW_{j,s}) + \langle W_i^c, W_j^c\rangle_t \\ & + \sum_{t_0 \leqslant s \leqslant t} (W_{i,s} - W_{i,s-})(W_{j,s} - W_{j,s-}). \end{aligned} \tag{20}$$

Taking the expectation of both sides we obtain

$$F_{ij}(t) = E\{\langle W_i^c, W_j^c\rangle_t + \sum_{t_0 \leqslant s \leqslant t} (W_{i,s} - W_{i,s-})(W_{j,s} - W_{j,s-})\} \tag{21}$$

where $F_{ij}(t)$ is the (i,j)th element of the matrix function F defined in (4). With this result it follows that (16) may be rewritten in the form

$$\begin{aligned} & E\left\{\int_{t_0}^{t_1} [X_t'R_1(t)X_t + U_t'R_2(t)U_t]\, dt + X_{t_1}'P_1X_{t_1}\right\} \\ & = E\left\{\int_{t_0}^{t_1} [U_t + R_2^{-1}(t)B'(t)P(t)X_t]'R_2(t)[U_t \right. \\ & \quad \left. + R_2^{-1}(t)B'(t)P(t)X_t]\, dt\right\} + \operatorname{tr}\left[P(t_0)EX_0X_0' \right. \\ & \quad \left. + \int_{t_0}^{t_1} P(t)dF(t)\right]. \end{aligned} \tag{22}$$

We now introduce the additional assumption on the local martingale W that the behavior of the matrix function $F(t)$, $t_0 \leqslant t \leqslant t_1$, is not influenced by the choice of the process U_t, $t_0 \leqslant t \leqslant t_1$. This means that no matter how the process U is chosen (within the restrictions mentioned before), the matrix function F remains identically the same. It will be seen in Section IV what this additional restriction amounts to. Under this assumption it is easily seen that the criterion on the left hand side of (22) is minimized if the process U is chosen such that

$$U_t = -R_2^{-1}(t)B'(t)P(t)X_t, \qquad t_0 \leqslant t \leqslant t_1 \tag{23}$$

which is the well-known solution of the stochastic linear regulator problem. Moreover, (22) shows that for the optimal regulator the criterion takes the value

$$\operatorname{tr}\left[P(t_0)EX_0X_0' + \int_{t_0}^{t_1} P(t)dF(t)\right]. \tag{24}$$

IV. Applications

The best known application of the result obtained above is the case where the process W is a Gaussian process with zero-mean, independent increments such that

$$F(t) = \int_{t_0}^{t} V(s)\, ds \tag{25}$$

with $V(t)$, $t \geqslant t_0$, a nonnegative-definite matrix function. This is the familiar case where Z in (2) is Gaussian white noise. The minimal value of the criterion (24) may in this case be written in the form

$$\operatorname{tr}\left[P(t_0)EX_0X_0' + \int_{t_0}^{t_1} P(t)V(t)\, dt\right]. \tag{26}$$

The present result is applicable to the case where W is any process with zero-mean, independent increments. This class of processes includes besides Gaussian processes compensated Poisson processes (i.e., Poisson processes with the means subtracted), and generalized compensated Poisson processes (where the rate parameter of the Poisson process is a stochastic process and the jumps have stochastic magnitudes) (see e.g., Bretagnolle [5]). Stochastic control problems with Poisson white noise have been studied by Florentin [6], [7] and Robinson [8].

The class of processes considered for W is larger than the class of processes with zero-mean, independent increments, however. For example, any process represented as

$$W_t = \int_{t_0}^{t} \Lambda(X_s, W_s, s)\, dV_s, \qquad t \geqslant t_0 \tag{27}$$

with Λ a matrix function, and V_t, $t \geqslant t_0$, a process with zero-mean, independent increments, is a local martingale with respect to $\mathfrak{F}_t$, $t \geqslant t_0$. However, such processes generally do not satisfy the additional requirement that the matrix function F is not influenced by the process U. This condition is met by processes represented as

$$W_t = \int_{t_0}^{t} \Lambda(W_s, s)\, dV_s, \qquad t \geqslant t_0 \tag{28}$$

with V_t, $t \geqslant t_0$, a process with zero-mean, independent increments.

V. Conclusions

It has been demonstrated that the linear control law (23) is optimal for the stochastic linear regulator problem if the noise Z occurring in the system equation (1) is a much more general process than Gaussian white noise.

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References

[1] E. Wong, *Stochastic Processes in Information and Dynamical Systems*. New York: McGraw-Hill, 1971.

[2] C. Doléans-Dade and P. Meyer, "Intégrales stochastiques par rapport aux martingales locales," in *Seminaire de Probabilités IV* (Lecture Notes in Mathematics, vol. 124). Berlin: Springer, 1970.

[3] K. J. Åström, *Introduction to Stochastic Control Theory*. New York: Academic Press, 1970.

[4] R. E. Kalman, "Contributions to the theory of optimal control," *Bol. Soc. Mat. Mexicana*, vol. 5, pp. 102–119, 1960.

[5] J. L. Bretagnolle, "Processus a accroissements indépendants," in *Ecole d'Eté de Probabilités: Processus Stochastiques* (Lecture Notes in Mathematics, A. Badrikian and P. L. Hennequin, Ed., vol. 307). Berlin: Springer, 1973.

[6] J. J. Florentin, "Optimal control of continuous-time, Markov, stochastic systems," *J. Electr. Contr.*, vol. X, pp. 473–488, 1961.

[7] J. J. Florentin, "Optimal control of systems with generalized Poisson inputs," *J. Basic Eng.*, vol. 85, pp. 217–221, 1963.

[8] J. Robinson, "Time-optimal control in the presence of shot-noise," *Conf. Rec., 4th Asilomar Conf. Circuits and Systems*, Nov. 18–20, 1970, pp. 236–242.