Queueing Syst (2011) 67: 221–249
DOI 10.1007/s11134-011-9213-6

Busy period analysis of the level dependent PH/PH/1/K queue

Ahmad Al Hanbali

Received: 11 November 2009 / Revised: 30 November 2010 / Published online: 12 February 2011

Abstract In this paper, we study the transient behavior of a level dependent single server queuing system with a waiting room of finite size during the busy period. The focus is on the level dependent PH/PH/1/K queue. We derive in closed form the joint transform of the length of the busy period, the number of customers served during the busy period, and the number of losses during the busy period. We differentiate between two types of losses: the overflow losses that are due to a full queue and the losses due to an admission controller. For the M/PH/1/K, M/PH/1/K under a threshold policy, and PH/M/1/K queues, we determine simple expressions for their joint transforms.

Keywords PH/PH/1/K queue · Phase-type distributions · Level dependent queues · Busy period · Transient analysis · Absorbing Markov chains · Matrix analytical approach

Mathematics Subject Classification (2000) 60K25 · 60J05 · 68M20 · 90B22

1 Introduction

In practice, it is often the case that arrivals and their service times depend on the system state. For example, in roadway traffic networks it is well known that the vehicle service time deteriorates as a function of the occupancy on the roadway [7]. In human-based service systems, there is a strong correlation between the volume of work demanded from a human and her/his productivity. At the packet switch (router) in telecommunication systems, when the buffer size increases, a controller drops the

A. Al Hanbali (✉)
School of Management and Governance, Department Operational Methods for Production and Logistics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands
e-mail: a.alhanbali@utwente.nl

arriving packets with an increasing probability. Moreover, the transient performance measures of a system are important for understanding the system evolution. All these facts motivate us to study the transient measures of a state dependent queuing system.

The transient regime of queuing systems is much more difficult to analyze than the steady state regime. This explains the scarcity of transient research results in this field compared to the steady state regime. A good exception is the M/M/1 queue, which has been well studied in both transient and steady state regimes. This paper is devoted to the study of the more general case of the transient behavior of the level dependent PH/PH/1/K queue, i.e., the level dependent PH/PH/1 queue with finite waiting room of size $K-1$. In particular, we shall analyze the measures related to the busy period.

Takács in [15, Chap. 1] was among the first to derive the transient probabilities of the M/M/1/K queue, referred to as $P_{ij}(t)$. Basically, these are the probabilities that at time t the queue length is j given it was i at time zero. Building on these probabilities, Takács also determined the transient probabilities of the M/M/1 queue by taking the limit of $P_{ij}(t)$ for $K \to \infty$. For the M/G/1/K queue, Cohen [6, Chap. III.6] computed the Laplace transform of $P_{ij}(t)$ and the bivariate transform of the number of customers served and the number of losses due to overflow during the busy period. This is done using complex analysis. Specifically, the joint transform is presented as a fraction of two contour integrals that involve K and the Laplace-Stieltjes transform (LST) of the customers' service time. Rosenlund in [13] extended Cohen's result by deriving the joint transform of the busy period length, the number of customers served, and the number of losses during the busy period. In a similar way to [13], Rosenlund in [14] analyzed the G/M/1/K queue and gave the trivariate transform. The approach of Rosenlund is more probabilistic than Cohen's analysis. However, Rosenlund's final results for the trivariate transform for M/G/1/K and G/M/1/K queues are represented as a fraction of two contour integrals. For more recent works on the busy period analysis of the M/G/1/K queue we refer to [8, 16]. Recently, there has been an increased interest in the expected number of losses during the busy period in the M/G/1/K queue with equal arrival and service rate; see, e.g., [1, 12, 17]. In this case, the interesting phenomenon is that the expected number of losses during the busy period in the M/G/1/K queue equals one for all values of $K \geq 1$.

In this paper we shall assume that the distribution of the interarrival times and service times is phase-type. For this reason, the embedding of the queue length process at the instants of departures or arrivals becomes unnecessary in order to analyze its steady state distribution. We emphasize that this is a key difference between our approach and those used in [6, 13, 14]. For an algorithmic method of the LST of the busy period in the PH/PH/1 queue see, e.g., [10, 11]. Bertsimas et al. in [4] derived in closed form the LST of the busy period in the PH/PH/1 queue as a function of the roots of a specific function that involves the LST of the interarrival and service times.

In [2] we extended the results of Rosenlund in [13] for the M/M/1/K queue in several ways. First, we studied a level dependent M/M/1/K queue with admission control. Secondly, we considered the residual busy period that is initiated with $n \geq 1$ customers. Moreover, we derived the distribution of the maximum number of customers during the busy period and other related performance measures. In this paper we shall extend these results by considering the level dependent PH/PH/1/K queue. In a similar way to [2], this shall be done using the theory of absorbing Markov chains.

The key point is to model the event that the system becomes empty as absorbing. Contrary to the analysis in [2], the derivation of the joint transform does not use the explicit inverse of some Toeplitz matrices; here we shall proceed with a different approach that is based on the analyticity of probability generating functions.

The paper is organized as follows. In Sect. 1.1 we give a detailed description of the model and the assumptions made. Section 2 reports our results presented in a number of different theorems, propositions, and corollaries. More precisely, Theorem 1 gives our main result for the four-variate transform as a function of the inverse of a specific matrix. Proposition 1 presents a numerical recursion to invert this matrix. In Propositions 2, 3, and 4 we derive the closed-form expressions for the four-variate transform for the M/PH/1/K, the level dependent M/PH/1/K, and the PH/M/1/K queues.

1.1 Model

We consider a level dependent PH/PH/1/K queuing system, i.e., a level dependent PH/PH/1 queue with finite waiting room of size $K-1$ customers. The arrival process is a renewal process with phase-type interarrival times distribution and with Laplace-Stieltjes transform (LST) $\phi_i(w)$, $\mathrm{Re}(w) \geq 0$, in the case where the queue length is $i \in \{0, 1, \ldots, K\}$. The service times distribution is phase-type with LST $\xi_i(w)$, in the case where the queue length is $i \in \{0, 1, \ldots, K\}$. A phase-type distribution can be represented by an initial distribution vector α, a transient generator $\mathbf{T}$, and an absorption rate vector T^o, i.e., $\mathbf{T}^{-1}T^o = -e^T$, where e^T is a column vector with all entries equal to one. For more details we refer, for example, to [10, p. 44]. Then, it is well known that the LST of the interarrival times can be written as follows:

$$\phi_i(w) = f_i(w\mathbf{I} - \mathbf{F}_i)^{-1} F_i^o, \quad \mathrm{Re}(w) \geq 0, \tag{1}$$

where the initial probability distribution f_i is a row vector of dimension M_a, the transient generator $\mathbf{F}_i$ is an M_a-by-M_a matrix, and the absorption rate vector F_i^o is a column vector of dimension M_a. Similarly, the LST of the service times reads

$$\xi_i(w) = s_i(w\mathbf{I} - \mathbf{S}_i)^{-1} S_i^o, \quad \mathrm{Re}(w) \geq 0, \tag{2}$$

where s_i is a row vector of dimension M_s, $\mathbf{S}_i$ is an M_s-by-M_s matrix, and S_i^o is a column vector of dimension M_s.

We assume that an admission controller is installed at the entry of the queue that has the duty of dropping the arriving customers with probability p_i when the queue length is $i \in \{0, 1, \ldots, K\}$. In other words, the customers are admitted in the queue with probability $q_i = 1 - p_i$ when its queue length is i. The arrivals at the queue of size K are all lost. In the sequel, we shall refer to the latter type of losses as *overflow losses*. It should be clear that in this case $p_K = 1$ and $q_K = 0$.

We are interested in the queue behavior during the *busy period,* which is defined as the time interval that starts with an arrival that joins an empty queue and ends at the first time at which the queue becomes empty again. We note that an arrival at an empty queue is admitted in the system with probability q_0, $0 < q_0 \leq 1$. Similarly, we define the residual busy period as the busy period initiated with $n \geq 1$ customers. Note that for $n = 1$ the residual busy period and the busy period are equal. In the following, we shall assume that, unless otherwise stated, at the beginning of the residual busy period the distribution vector of the phases of the interarrival times and service times are distributed according to f_n and s_n.

Consider an arbitrary residual busy period. Let B_n denote its length. Let S_n denote the total number of served customers during B_n. Let L_n denote the total number of losses, i.e., arrivals that are not admitted to the queue either due to the admission control or to the full queue, during B_n. We shall differentiate between the two types of losses. Let L_n^c denote the total number of losses that are not admitted to the queue due to the admission control, during B_n. Let L_n^o denote the total number of the overflow losses that are not admitted to the queue because it is full, i.e., due to $p_K = 1$, during B_n. In this paper we determine the joint transform $\mathbb{E}[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}]$, $\mathrm{Re}(w) \geq 0$, $|z_1| \leq 1$, $|z_2| \leq 1$, and $|z_3| \leq 1$. We will use the theory of absorbing Markov chains. This is done by modeling the event that “the queue jumps to the empty state” as an absorbing event. Tracking the number of customers served and losses before the absorption occurs gives the desired result.

A word on the notation: throughout $x := y$ will designate that by definition x is equal to y, $1_{\{E\}}$ is the indicator function of any event E ($1_{\{E\}}$ is equal to one if E occurs and zero otherwise), x^T is the transpose vector of x, e_i is the unit row vector of appropriate dimension with all entries equal to zero except the i-th entry which is one, and $\mathbf{I}$ is the identity matrix of appropriate dimension. We use $\otimes$ as the Kronecker product operator defined as follows. Let $\mathbf{X}$ and $\mathbf{Y}$ be two matrices and $x(i, j)$ and $y(i, j)$ denote the (i, j)-entries of $\mathbf{X}$ and $\mathbf{Y}$ respectively. Then $\mathbf{X} \otimes \mathbf{Y}$ is a block matrix where the (i, j)-block is equal to $x(i, j)\mathbf{Y}$. Finally, let $\det(\mathbf{X})$ denote the determinant of the square matrix $\mathbf{X}$.

2 Results

Before reporting our main result, we shall first introduce a set of matrices, then we define our key absorbing Markov chain (AMC), and finally we order the AMC states in a proper way that yields a nice structure. The event that the queue becomes empty, i.e., the end of the busy period, is modeled as an absorbing event which justifies the need of the theory of AMCs.

Let us define the following K-by-K block matrices: the matrix $\mathbf{A}$ that is an upper bidiagonal block matrix with i-th upper diagonal element equal to $q_i(F_i^o f_i) \otimes \mathbf{I}$ and i-th diagonal element equal to $\mathbf{F}_i \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{S}_i$, the matrix $\mathbf{B}$ that is a lower diagonal matrix with i-th lower diagonal element equal to $\mathbf{I} \otimes (S_i^o s_i)$, and the matrix $\mathbf{C}$ that is a diagonal matrix with i-th diagonal element, $i = 1, \ldots, K - 1$, equal to $p_i(F_i^o f_i) \otimes \mathbf{I}$ and K-th element equal to $\mathbf{0}$, and the matrix $\mathbf{D}$ that is a zero block matrix with (K, K)-block element equal to $(F_K^o f_K) \otimes \mathbf{I}$. Note that F_i^o is a column vector and f_i is a row vector; thus $F_i^o f_i$ is a matrix. Similarly, $S_i^o s_i$ is a matrix. Moreover, note that $\mathbf{A} + \mathbf{B}$ represents the generator of a level dependent PH/PH/1/K queue restricted to strictly positive queue length; see, for example, [10, Chap. 3]. Let us define $\mathbf{Q}_K(w, z_1, z_2, z_3) = w\mathbf{I} - \mathbf{A} - z_1\mathbf{B} - z_2\mathbf{C} - z_3\mathbf{D}$. For ease of presentation, we shall refer to $\mathbf{Q}_K(w, z_1, z_2, z_3)$ as $\mathbf{Q}_K$. Appendix A gives a detailed description of the structure of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$.

Let $\mathcal{P}(t) := (Ph_s(t), Ph_a(t), N(t), S(t), L^c(t), L^o(t))$ denote the continuous-time Markov process with a discrete state space $\Omega := \{1, \dots, M_s\} \times \{1, \dots, M_a\} \times \{0, 1, \dots, K\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $Ph_s(t)$ represents the phase of the (if any) customer in service at time t, and $Ph_a(t)$ the phase of the interarrival time at time t, $N(t)$ represents the number of customers in the queue at time t, $S(t)$ the number of served customers from the queue until t, $L^c(t)$ the number of losses due to the admission control in the queue until t, $L^o(t)$ the number of overflow losses in the queue until t, and $\mathbb{N}$ the set of non-negative integers. States with $N(t) = 0$ are absorbing. We refer to this absorbing Markov process as AMC. The absorption of the AMC occurs when the queue becomes empty, i.e., $N(t) = 0$. We set the AMC initial state at time $t = 0$ to $\mathcal{P}(0) = (p_s, p_a, n, 0, 0, 0)$, $n \geq 1$, $p_s \in \{1, \dots, M_s\}$ with distribution vector equal to s_n and $p_a \in \{1, \dots, M_a\}$ with distribution vector equal to f_n. For this reason, the time until absorption of the AMC is equal to B_n, the residual busy period length. Moreover, it is clear that S_n (resp. L_n^o and L_n^c), the total number of departures (resp. losses) during the residual busy period, is equal to $S(B_n + \epsilon) = S_n$ (resp. $L^c(B_n + \epsilon) = L_n^c$ and $L^o(B_n + \epsilon) = L_n^o$), $\epsilon > 0$.

During a residual busy period, the processes $S(t)$, $L^c(t)$, and $L^o(t)$ are counting processes. To take advantage of this property, we order the transient states of the AMC, i.e., $(i, j, k, l, m, o) \in \Omega \setminus \{(\cdot, \cdot, 0, \cdot, \cdot, \cdot)\}$, increasingly first according to o, then m, l, k, j, and finally according to i. In the following we shall express the generator of the AMC as a function of the aforementioned matrices **A**, **B**, **C**, and **D** (see Appendix A for further details). The proposed ordering implies that the generator matrix of the transitions between the transient states of the AMC, denoted by **G**, is an infinite upper diagonal block matrix with diagonal blocks equal to $\mathbf{G}_0$ and upper diagonal blocks equal to $\mathbf{U}_0$, i.e.,

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_0 & \mathbf{U}_0 & \mathbf{0} & \cdots & \cdots \\ \mathbf{0} & \mathbf{G}_0 & \mathbf{U}_0 & \mathbf{0} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3}$$

We note that $\mathbf{G}_0$ denotes the generator matrix of the transitions which do not induce any modification in the number of overflow losses, i.e., $L_n^o(t)$. Moreover, $\mathbf{U}_0$ denotes the transition rate matrix of the transitions that represent an arrival at a full queue (an overflow), i.e., transitions between the transient states (i, j, K, l, m, o) and $(i, j', K, l, m, o + 1)$, where j' is the initial phase of the next interarrival time just after an overflow loss. For this reason, $\mathbf{U}_0$ is a block diagonal matrix with diagonal blocks equal to $\mathbf{U}_{00}$. The blocks $\mathbf{U}_{00}$ are in turn diagonal block matrices with entries equal to **D**. See Appendix A for a detailed description of the matrices **D**, $\mathbf{U}_{00}$ and $\mathbf{U}_0$. The block matrix $\mathbf{G}_0$ is also an infinite upper diagonal block matrix with diagonal blocks equal to $\mathbf{G}_1$, and upper diagonal blocks equal to $\mathbf{U}_1$. Therefore, $\mathbf{G}_0$ has the following canonical form:

$$\mathbf{G}_0 = \begin{pmatrix} \mathbf{G}_1 & \mathbf{U}_1 & \mathbf{0} & \cdots & \cdots \\ \mathbf{0} & \mathbf{G}_1 & \mathbf{U}_1 & \mathbf{0} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4}$$

where $\mathbf{U}_1$ denotes the transition rate matrix of the transitions that represent a dropped arriving customer by the admission controller, i.e., transitions between the transient states (i, j, k, l, m, o) and $(i, j', k, l, m+1, o)$. For this reason, $\mathbf{U}_1$ is a block matrix of diagonal entries equal to $\mathbf{C}$. See Appendix A for a full description of the matrices $\mathbf{U}_1$ and $\mathbf{C}$. The matrix $\mathbf{G}_1$ is the generator matrix of the transition between the transient states (i, j, k, l, m, o) and (i', j', k', l', m, o), i.e., the transitions that do not induce any modification in the number of overflow losses and of losses due to the admission controller. Observe that $\mathbf{G}_1$ has the following canonical form:

$$\mathbf{G}_1 = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \cdots \\ \mathbf{0} & \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5}$$

The upper diagonal blocks of $\mathbf{G}_1$ represent the transition between the transient states (i, j, k, l, m, o) and $(i', j, k-1, l+1, m, o)$, i.e., a transition that models a departure from the queue. For this reason, the upper diagonal blocks are equal to the aforementioned matrix $\mathbf{B}$. The diagonal blocks of $\mathbf{G}_1$ represent the transitions due to a modification in the interarrival phase, service phase, or an arrival that is admitted to the queue. For this reason, the diagonal blocks of $\mathbf{G}_1$ equal $\mathbf{A}$. Note that a full description of the matrices $\mathbf{A}$ and $\mathbf{B}$ is given in Appendix A.

In the following we model the event that the queue becomes empty, i.e., the end of the busy period, as an absorbing event. The joint transform is deduced by determining the last state visited before absorption.

We are now ready to formulate our main result.

Theorem 1 (Level dependent queue) *Assume that the residual busy period starts with n customers at time zero, and at time zero the phases of the interarrival time and the service time are distributed according to f_n and s_n. The joint transform of B_n, S_n, and L_n is then given by*

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = z_1 e_n \otimes f_n \otimes s_n \mathbf{Q}_K^{-1} (e_1 \otimes e)^T \otimes S_1^o.$$

Proof Let us define

$$\pi_{i,j,k,l,m,o}(t) := \mathbb{P}\big(\mathcal{P}(t) = (i, j, k, l, m, o) \mid \mathcal{P}(0) = (p_s, p_a, n, 0, 0, 0)\big).$$

The Laplace transform of $\pi_{i,j,k,l,m,o}(t)$ is given by

$$\tilde{\pi}_{i,j,k,l,m,o}(w) = \int_{t=0}^{\infty} e^{-wt} \pi_{i,j,k,l,m,o}(t)\, dt, \quad \operatorname{Re}(w) \geq 0.$$

Moreover, let us define the following row vectors:

$$\tilde{\Pi}_{j,k,l,m,o}(w) = \big(\tilde{\pi}_{1,j,k,l,m,o}(w), \ldots, \tilde{\pi}_{M_s,j,k,l,m,o}(w)\big),$$

$$\tilde{\Pi}_{k,l,m,o}(w) = \big(\tilde{\Pi}_{1,k,l,m,o}(w), \ldots, \tilde{\Pi}_{M_a,k,l,m,o}(w)\big),$$

$$\tilde{\Pi}_{l,m,o}(w) = \big(\tilde{\Pi}_{1,l,m,o}(w), \ldots, \tilde{\Pi}_{K,l,m,o}(w)\big).$$

The Kolmogorov backward equation of the absorbing state $(i, j, 0, l, m, o)$ reads

$$\frac{d}{dt}\pi_{i,j,0,l,m,o}(t) = \pi_{i,j,1,l-1,m,o}(t) S_1^o(i), \tag{6}$$

where $S_1^o(i)$ is the i-th entry of S_1^o. Since $(i, j, 0, l, m, o)$ is an absorbing state, it is easily seen that

$$\pi_{i,j,0,l,m,o}(t) = \mathbb{P}\big(B_n < t, Ph_s(B_n) = i, Ph_a(B_n) = j, S_n = l, L_n^c = m, \\ L_n^o = o \mid \mathcal{P}(0) = (p_s, p_a, n, 0, 0, 0)\big).$$

Hence, the Laplace transform of the left-hand side (l.h.s.) of (6) is equal to the joint transform $\mathbb{E}[e^{-wB_n} \cdot \mathbf{1}_{\{Ph_s(B_n)=i\}} \cdot \mathbf{1}_{\{Ph_a(B_n)=j\}} \cdot \mathbf{1}_{\{S_n=l\}} \cdot \mathbf{1}_{\{L_n^c=m\}} \cdot \mathbf{1}_{\{L_n^o=o\}}]$. Taking the Laplace transform on both sides in (6) and summing over all values of i and j gives

$$\begin{aligned}\mathbb{E}\big[e^{-wB_n} \cdot \mathbf{1}_{\{S_n=l\}} \cdot \mathbf{1}_{\{L_n^c=m\}} \cdot \mathbf{1}_{\{L_n^o=o\}}\big] &= \sum_{j=1}^{M_a} \tilde{\Pi}_{j,1,l-1,m,o}(w) S_1^o \\ &= \tilde{\Pi}_{1,l-1,m,o}(w) e^T \otimes S_1^o \\ &= \tilde{\Pi}_{l-1,m,o}(w)(e_1 \otimes e)^T \otimes S_1^o.\end{aligned}$$

Removing the condition on S_n, L_n^c, and L_n^o we deduce that

$$\begin{aligned}\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] &= \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{o=0}^{\infty} z_1^l z_2^m z_3^o \tilde{\Pi}_{l-1,m,o}(w)(e_1 \otimes e)^T \otimes S_1^o \\ &= z_1 \sum_{l=0}^{\infty} z_1^l \sum_{m=0}^{\infty} z_2^m \sum_{o=0}^{\infty} z_3^o \tilde{\Pi}_{l,m,o}(w)(e_1 \otimes e)^T \otimes S_1^o.\end{aligned} \tag{7}$$

We now derive the right-hand side (r.h.s.) of $\mathbb{E}[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}]$. Taking the Laplace transforms of the Kolmogorov backward equations of the AMC, we find that

$$\begin{aligned}\tilde{\Pi}_{l,m,o}(w)(w\mathbf{I} - \mathbf{A}) &= \mathbf{1}_{\{l,m,o=0\}} e_n \otimes f_n \otimes s_n + \mathbf{1}_{\{l\ge1\}} \tilde{\Pi}_{l-1,m,o}(w)\mathbf{B} \\ &\quad + \mathbf{1}_{\{m\ge1\}} \tilde{\Pi}_{l,m-1,o}(w)\mathbf{C} + \mathbf{1}_{\{o\ge1\}} \tilde{\Pi}_{l,m,o-1}(w)\mathbf{D},\end{aligned} \tag{8}$$

where $e_n \otimes f_n \otimes s_n$ represents the initial state vector of the AMC, and the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are given in Appendix A. Multiplying (8) by $z_1^l z_2^m z_3^o$ and summing the result first over all o, then m, and finally l yields that

$$\sum_{l=0}^{\infty} z_1^l \sum_{m=0}^{\infty} z_2^m \sum_{o=0}^{\infty} z_3^o \tilde{\Pi}_{l,m,o}(w)(w\mathbf{I} - \mathbf{A} - z_1\mathbf{B} - z_2\mathbf{C} - z_3\mathbf{D}) = e_n \otimes f_n \otimes s_n. \tag{9}$$

Note that $(w\mathbf{I} - \mathbf{A} - z_1\mathbf{B} - z_2\mathbf{C} - z_3\mathbf{D})$, $\mathrm{Re}(w) > 0$, is invertible since it has a dominant main diagonal. Inserting (9) into (7) completes the proof. □

Remark 1 Assume that the residual busy period starts with n customers at time zero, and at time zero the phases of the interarrival time and the service time are distributed according to some distribution vectors f_{n0} and s_{n0}. The joint transform of B_n, S_n, and L_n is then given by

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = z_1 e_n \otimes f_{n0} \otimes s_{n0} \mathbf{Q}_K^{-1} (e_1 \otimes e)^T \otimes S_1^o.$$

Proposition 1 *The joint transform B_1, S_1, L_1^c, and L_1^o is given by*

$$\mathbb{E}\big[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\big] = z_1 f_1 \otimes s_1 (\mathbf{X}_1)^{-1} e^T \otimes S_1^o,$$

where $\mathbf{X}_i$, $i = 1, \dots, K-1$, *and satisfies the following* (*backward*) *recursion*:

$$\mathbf{X}_i = w\mathbf{I} - \mathbf{F}_i \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{S}_i - z_2 p_i F_i^o f_i \otimes \mathbf{I} - z_1 q_i F_i^o f_i \otimes \mathbf{I}(\mathbf{X}_{i+1})^{-1} \mathbf{I} \otimes S_{i+1}^o s_{i+1},$$

with

$$\mathbf{X}_K = w\mathbf{I} - \mathbf{F}_K \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{S}_K - z_3 F_K^o f_K \otimes \mathbf{I}.$$

Proof According to Theorem 1, the joint transform of B_1, S_1, L_1^c, and L_1^o can be written as

$$\mathbb{E}\big[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\big] = z_1 f_1 \otimes s_1 \mathbf{Q}_K(1,1) e^T \otimes S_1^o,$$

where $\mathbf{Q}_K(1,1)$ is the $(1,1)$-block entry of $\mathbf{Q}_K^{-1}$. Let us partition the matrix $\mathbf{Q}_K$ as follows:

$$\mathbf{Q}_K = \left(\begin{array}{c|c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hline \mathbf{Q}_{21} & \mathbf{Q}_{K-1} \end{array}\right), \tag{10}$$

where $\mathbf{Q}_{11} := w\mathbf{I} - \mathbf{F}_1 \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{S}_1 - z_2 p_1 F_1^o f_1 \otimes \mathbf{I}$, $\mathbf{Q}_{12} := -e_1 \otimes q_1 F_1^o f_1 \otimes \mathbf{I}$, $\mathbf{Q}_{21} := -z_1 (e_1)^T \otimes \mathbf{I} \otimes S_2^o s_2$, and $\mathbf{Q}_{K-1}$ is obtained from the matrix $\mathbf{Q}_K$ by removing its first blocks row and first blocks column. Some simple linear algebra shows that the inverse of $\mathbf{Q}_K$ reads

$$\mathbf{Q}_K^{-1} = \left(\begin{array}{c|c} (\mathbf{Q}_{11}^*)^{-1} & -\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12}(\mathbf{Q}_{22}^*)^{-1} \\ \hline -\mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}(\mathbf{Q}_{11}^*)^{-1} & (\mathbf{Q}_{22}^*)^{-1} \end{array}\right), \tag{11}$$

where $\mathbf{Q}_{11}^* := \mathbf{Q}_{11} - \mathbf{Q}_{12}\mathbf{Q}_{K-1}^{-1}\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}^* := \mathbf{Q}_{K-1} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12}$. It is then readily seen that

$$\begin{aligned}
\mathbb{E}\big[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\big] &= z_1 f_1 \otimes s_1 \big(\mathbf{Q}_{11}^*\big)^{-1} e^T \otimes S_1^o \\
&= z_1 f_1 \otimes s_1 \big(\mathbf{Q}_{11} - \mathbf{Q}_{12}(\mathbf{Q}_{K-1})^{-1}\mathbf{Q}_{21}\big)^{-1} e^T \otimes S_1^o \\
&= z_1 f_1 \otimes s_1 \big(w\mathbf{I} - \mathbf{F}_1 \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{S}_1 - z_2 p_1 F_1^o f_1 \otimes \mathbf{I} \\
&\qquad - q_1 F_1^o f_1 \otimes \mathbf{I}\mathbf{Q}_{K-1}(1,1)\mathbf{I} \otimes S_2^o s_2\big)^{-1} e^T \otimes S_1^o,
\end{aligned} \tag{12}$$

where $\mathbf{Q}_{K-1}(1,1)$ is the (1, 1)-block entry of $\mathbf{Q}_{K-1}^{-1}$. $\mathbf{Q}_{K-1}$ is a tridiagonal block matrix. Repeating the manner of partitioning the matrix $\mathbf{Q}_K$ to $\mathbf{Q}_{K-1}$ one can show that

$$\mathbf{Q}_{K-1}(1,1) = w\mathbf{I} - \mathbf{F}_2 \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{S}_2 - z_2 p_2 F_2^o f_2 \otimes \mathbf{I} - q_2 F_2^o f_2 \otimes \mathbf{I}\mathbf{Q}_{K-2}(1,1)\mathbf{I} \otimes S_3^o s_3.$$

$\mathbf{Q}_{K-2}(1,1)$ is the (1, 1)-block entry of $\mathbf{Q}_{K-2}^{-1}$ and $\mathbf{Q}_{K-2}$ is obtained from the matrix $\mathbf{Q}_{K-1}$ by removing its first row and first column. For this reason, we deduce by induction that $\mathbb{E}[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}]$ satisfies the recursion defined in Proposition 1. □

2.1 M/PH/1/K queue

For the M/PH/1/K we have that $-\mathbf{F}_i = F_i^o f_i = \lambda$, $i = 1, \ldots, K$, $\mathbf{S}_i = \mathbf{S}$ and $S_i^o s_i = S^o s$, $i = 1, \ldots, K$. Let $\xi(w) = s(w\mathbf{I} - \mathbf{S})^{-1} S^o$ denote the LST of the service times. Moreover, we assume that $q_i = q$, $i = 1, \ldots, K-1$.

Lemma 1 *The function $x - z_1\xi(w + \lambda(1 - qx - pz_2))$ has $M_s + 1$ distinct non-null roots $r_1, \ldots, r_{M_s+1}$, such that $0 < |r_1| < |r_2| < \cdots < |r_{M_s+1}|$.*

Proof It is well known that $\xi(w)$, the LST of the service times which has a phase-type distribution of M_s phases, is a rational function. Therefore, the denominator of $\xi(w)$ is a polynomial in w of degree M_s and the numerator is a polynomial of degree $< M_s$. For this reason, the numerator of $x - z_1\xi(w + \lambda(1 - qx - pz_2))$ is a polynomial in x of degree $M_s + 1$. Therefore, the function $x - z_1\xi(w + \lambda(1 - qx - pz_2))$ has $M_s + 1$ roots. It is easily checked that zero is not a root of this function.

For clarity of presentation, we will assume that these roots are distinct. In Sect. 3 we shall relax this assumption by considering the case where $r_{i+l} = r_i + l\epsilon$, $\epsilon > 0$, $i \in \{1, \ldots, M_s + 1\}$ and $l = 0, \ldots, L-1$, and in our final result taking the limit $\epsilon \to 0$. This means that we have that r_i is a root of multiplicity L.

Let D_η denote the circle with center at the origin and with radius η, and $|\frac{pz_2 - z_3}{q}| < \eta < |r_1|$, where r_1 is the root with the smallest absolute value of

$$x - z_1\xi\bigl(w + \lambda(1 - qx - pz_2)\bigr) = 0. \tag{13}$$

□

We are now ready to present the main result of the M/PH/1/K queue.

Proposition 2 (M/PH/1/K queue) *The joint transform of B_n, S_n, L_n^o, and L_n^c for the M/PH/1/K queue is given by*

$$\mathbb{E}\bigl[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\bigr] = \frac{\frac{1}{2\pi\mathbf{i}} \int_{D_\eta} \frac{1}{x^{K-1-n}} \frac{1}{qx + pz_2 - z_3} \frac{dx}{x - z_1\xi(w + \lambda(1 - qx - pz_2))}}{\frac{1}{2\pi\mathbf{i}} \int_{D_\eta} \frac{1}{x^{K-1}} \frac{1}{qx + pz_2 - z_3} \frac{dx}{x - z_1\xi(w + \lambda(1 - qx - pz_2))}}.$$

Proof According to Theorem 1, the transform of B_n, S_n, L_n^c, and L_n^o for the M/PH/1/K queue can be reduced as follows (due to the Poisson arrivals we have that $f_n = 1$ and the vector e is of dimension one, i.e., $e = 1$ in Theorem 1):

$$\mathbb{E}\left[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\right] = z_1 e_n \otimes s\mathbf{Q}_K^{-1} e_1^T \otimes S^o, \tag{14}$$

where $\mathbf{Q}_K$ in this case is a K-by-K tridiagonal block matrix with upper diagonal blocks equal to $\mathbf{E}_0 = -q\lambda\mathbf{I}$, i-th diagonal blocks equal to $\mathbf{E}_1 = w\mathbf{I} + \lambda(1 - pz_2)\mathbf{I} - \mathbf{S}$, $i = 1, \ldots, K-1$, and K-th diagonal block equal to $\mathbf{E}_1^* = w\mathbf{I} + \lambda(1 - z_3)\mathbf{I} - \mathbf{S}$, and lower diagonal blocks equal to $\mathbf{E}_2 = -z_1 S^o s$. Therefore, $\mathbf{Q}_K$ has the following canonical form:

$$\mathbf{Q}_K = \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_0 & \mathbf{0} & \cdots & \cdots \\ \mathbf{E}_2 & \mathbf{E}_1 & \mathbf{E}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \mathbf{E}_1 & \mathbf{E}_0 \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{E}_2 & \mathbf{E}_1^* \end{pmatrix}. \tag{15}$$

Let $u = (u_1, \ldots, u_K) := e_n \otimes s\mathbf{Q}_K^{-1}$. Note that each entry of the row vector u is in its turn a row vector of dimension M_s and is a function of w, z_1, z_2, and z_3. Then (14) in terms of u can be rewritten as

$$\mathbb{E}\left[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\right] = z_1 u_1 S^o. \tag{16}$$

The definition of u gives $u\mathbf{Q}_K = e_n \otimes s$. Developing the latter equation yields

$$\mathbf{1}_{\{i\ge 2\}} u_{i-1}\mathbf{E}_0 + u_i\left[\mathbf{1}_{\{i\le K-1\}}\mathbf{E}_1 + \mathbf{1}_{\{i=K\}}\mathbf{E}_1^*\right] + \mathbf{1}_{\{i\le K-1\}} u_{i+1}\mathbf{E}_2 = \mathbf{1}_{\{i=n\}} s, \tag{17}$$

where $i = 1, \ldots, K$. Since u_1 is analytic, we deduce from (17) that u_i, $i = 2, \ldots, K$, are analytic. Multiplying (17) by x^i and summing over i we find that

$$\begin{aligned} \sum_{i=1}^{K} u_i x^i &= \left(u_1\mathbf{E}_2 + x^K u_K(x\mathbf{E}_0 + \mathbf{E}_1 - \mathbf{E}_1^*) + x^n s\right)\left(x\mathbf{E}_0 + \mathbf{E}_1 + \frac{1}{x}\mathbf{E}_2\right)^{-1} \\ &= \left(z_1 u_1 S^o s - x^n s + \lambda x^K(qx + pz_2 - z_3)u_K\right)\left(\mathbf{S} - \rho\mathbf{I} + \frac{z_1}{x}S^o s\right)^{-1}, \end{aligned} \tag{18}$$

where $\rho := w + \lambda(1 - qx - pz_2)$. Let $\mathbf{S}_* := \mathbf{S} - \rho\mathbf{I}$. Note that under the condition that $\mathrm{Re}(\rho) \ge 0$ the matrix $\mathbf{S}_*$ is nonsingular. Hence, the Sherman–Morrison formula, see, for example, [3, Fact 2.14.2, p. 67], yields

$$\left(\mathbf{S}_* + \frac{z_1}{x}S^o s\right)^{-1} = \mathbf{S}_*^{-1} - \frac{z_1}{x + z_1 s\mathbf{S}_*^{-1}S^o}\mathbf{S}_*^{-1}S^o s\mathbf{S}_*^{-1}. \tag{19}$$

The multiplication to the right of (18) by the column vector S^o and (19) gives

$$\sum_{i=1}^{K} u_i x^i S^o = \frac{x}{x + z_1 s \mathbf{S}_*^{-1} S^o} \big(z_1 u_1 S^o s - x^n s + \lambda x^K (qx + pz_2 - z_3) u_K\big) \mathbf{S}_*^{-1} S^o. \quad (20)$$

From (2) we know that $s\mathbf{S}_*^{-1} S^o = -\xi(\rho)$ and $\mathbf{S}_*^{-1} S^o = -(\xi^1(\rho), \dots, \xi^{Ms}(\rho))^T$, where $\xi^i(\rho) = e_i(\rho \mathbf{I} - \mathbf{S})^{-1} S^o$. Therefore, $\xi(\rho) = s(\xi^1(\rho), \dots, \xi^{Ms}(\rho))^T$ is a linear combination of $\xi^i(\rho)$, $i = 1, \dots, M_s$. Inserting $s\mathbf{S}_*^{-1} S^o$ and $\mathbf{S}_*^{-1} S^o$ into (20) yields

$$\sum_{i=1}^{K} u_i x^i S^o = \frac{-x}{x - z_1 \xi(\rho)} \Bigg[\big(z_1 u_1 S^o - x^n\big) \xi(\rho) + \lambda x^K (qx + pz_2 - z_3) \sum_{j=1}^{M_s} u_{Kj} \xi^j(\rho) \Bigg], \quad (21)$$

where $u_K = (u_{K1}, \dots, u_{KM_s})$. We recall that $u_i S^o$ is an analytic function. For this reason, the l.h.s. of (21) should be analytical for any finite x. This implies that the singular points, roots of $x - z_1 \xi(\rho)$, on the r.h.s. of (21) are removable.

Lemma 1 and the analyticity of $\sum_{i=1}^{K} u_i x^i S^o$ give

$$z_1 u_1 S^o \xi(\rho_i) + \lambda r_i^K (q r_i + pz_2 - z_3) \sum_{j=1}^{M_s} u_{Kj} \xi^j(\rho_i) = r_i^n \xi(\rho_i),$$
$$i = 1, \dots, M_s + 1, \quad (22)$$

where $\rho_i := w + \lambda(1 - q r_i - p z_2)$. The system of equations in (22) has $M_s + 1$ equations with $M_s + 1$ unknowns which are $z_1 u_1 S^o, u_{K1}, \dots, u_{KM_s}$. Using Cramer's rule, we find that

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = z_1 u_1 S^o = \frac{\det(\mathbf{M}^*)}{\det(\mathbf{M})}, \quad (23)$$

where $\det(\mathbf{M})$ is the determinant of the $(M_s + 1)$-by-$(M_s + 1)$ matrix $\mathbf{M}$ with i-th row equal to $(\xi(\rho_i)/[\lambda r_i^K \ (q r_i + p z_2 - z_3)], \xi^1(\rho_i), \dots, \xi^{M_s}(\rho_i))$, $i = 1, \dots, M_s + 1$. Therefore, $\mathbf{M}$ has the following canonical form:

$$\mathbf{M} = \begin{pmatrix} \frac{\xi(\rho_1)}{\lambda r_1^K (q r_1 + p z_2 - z_3)} & \xi^1(\rho_1) & \cdots & \xi^{M_s}(\rho_1) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\xi(\rho_i)}{\lambda r_i^K (q r_i + p z_2 - z_3)} & \xi^1(\rho_i) & \cdots & \xi^{M_s}(\rho_i) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\xi(\rho_{M_s+1})}{\lambda r_{M_s+1}^K (q r_{M_s+1} + p z_2 - z_3)} & \xi^1(\rho_{M_s+1}) & \cdots & \xi^{M_s}(\rho_{M_s+1}) \end{pmatrix}.$$

The matrix $\mathbf{M}^*$ is obtained from $\mathbf{M}$ by replacing its first column with

$$\left(\frac{\xi(\rho_1)}{\lambda r_1^{K-n}(qr_1+pz_2-z_3)},\ldots,\frac{\xi(\rho_{M_s+1})}{\lambda r_{M_s+1}^{K-n}(qr_{M_s+1}+pz_2-z_3)}\right)^T.$$

The Laplace expansion of the determinant along the first column of $\mathbf{M}$ and $\mathbf{M}^*$ gives

$$\begin{aligned}\mathbb{E}\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big]&=\frac{\sum_{i=1}^{M_s+1}\frac{\xi(\rho_i)(-1)^{i+1}}{\lambda r_i^{K-n}(qr_i+pz_2-z_3)}\det(\mathbf{M}^*(i,1))}{\sum_{i=1}^{M_s+1}\frac{\xi(\rho_i)(-1)^{i+1}}{\lambda r_i^{K}(qr_i+pz_2-z_3)}\det(\mathbf{M}(i,1))}\\&=\frac{\sum_{i=1}^{M_s+1}\frac{(-1)^i}{r_i^{K-1-n}(qr_i+pz_2-z_3)}\det(\mathbf{M}(i,1))}{\sum_{i=1}^{M_s+1}\frac{(-1)^i}{r_i^{K-1}(qr_i+pz_2-z_3)}\det(\mathbf{M}(i,1))},\end{aligned}\tag{24}$$

where $\mathbf{M}(i,1)$ (resp. $\mathbf{M}^*(i,1)$) is the M_s-by-M_s matrix that results by deleting the i-th row and the first column of $\mathbf{M}$ (resp. $\mathbf{M}^*$), and the second equality follows from $\xi(\rho_i)=r_i/z_1$ and $\mathbf{M}^*(i,1)=\mathbf{M}(i,1)$.

Let D_η denote the circle with center at the origin and with radius equal to η. Assume that $|\frac{pz_2-z_3}{q}|<\eta<|r_1|$ with $q\neq 0$. Let us define $f_i(x)\sim_i g_i(x)$ if $f_i(x)/g_i(x)=h(x)$ that is independent of i. Let $\mathrm{Res}_a f(z)$ denote the residue of the complex function $f(z)$ at point a. The sum of the residues of the following complex function:

$$\frac{1}{x^{K-1-n}}\frac{1}{qx+pz_2-z_3}\frac{1}{x-z_1\xi(w+\lambda(1-qx-pz_2))},$$

is equal to zero, including the residue at infinity, which is equal to zero ($q\neq 0$). Therefore, we deduce that

$$\begin{aligned}&\frac{\sum_{i=1}^{M_s+1}\frac{(-1)^i}{r_i^{K-1-n}(qr_i+pz_2-z_3)}\det(\mathbf{M}(i,1))}{\sum_{i=1}^{M_s+1}\frac{(-1)^i}{r_i^{K-1}(qr_i+pz_2-z_3)}\det(\mathbf{M}(i,1))}\\&\quad=\frac{\frac{1}{2\pi\mathbf{i}}\int_{D_\eta}\frac{1}{x^{K-1-n}}\frac{1}{qx+pz_2-z_3}\frac{dx}{x-z_1\xi(w+\lambda(1-qx-pz_2))}}{\frac{1}{2\pi\mathbf{i}}\int_{D_\eta}\frac{1}{x^{K-1}}\frac{1}{qx+pz_2-z_3}\frac{dx}{x-z_1\xi(w+\lambda(1-qx-pz_2))}},\end{aligned}\tag{25}$$

if and only if

$$(-1)^i\det\big(\mathbf{M}(i,1)\big)\sim_i \mathrm{Res}_{r_i}\frac{1}{x-z_1\xi(w+\lambda(1-qx-pz_2))}.\tag{26}$$

In the following we shall prove condition (26). Since the service times have a phase-type distribution, $\xi(w)$ is a rational function with denominator, $Q_\xi(w)$, of degree M_s and numerator of degree $< M_s$. Note that by Lemma 1 the roots of

$x - z_1\xi(w + \lambda(1 - qx - pz_2))$ are distinct. Therefore, we deduce that

$$\text{Res}_{r_i} \frac{1}{x - z_1\xi(w + \lambda(1 - qx - pz_2))} = \frac{Q_\xi(w + \lambda(1 - qr_i - pz_2))}{(-\lambda q)^{M_s} \prod_{j=1, j\neq i}^{M_s+1}(r_i - r_j)} = \frac{Q_\xi(\rho_i)}{\prod_{j=1, j\neq i}^{M_s+1}(\rho_i - \rho_j)}.$$

$\mathbf{M}(i, 1)$ is an M_s-by-M_s matrix with j-th row equal to $(\xi^1(\rho_j), \ldots, \xi^{M_s}(\rho_j))$ for $j = 1, \ldots, M_s + 1$ and $j \neq i$. We have (see Appendix B for the proof)

$$\det\bigl(\mathbf{M}(i, 1)\bigr) = C(-1)^{i-1} \frac{\prod_{j=1}^{M_s} \prod_{k=j+1}^{M_s+1}(\rho_k - \rho_j)}{\prod_{j=1}^{M_s+1} Q_\xi(\rho_j)} \times \frac{Q_\xi(\rho_i)}{\prod_{j=1, j\neq i}^{M_s+1}(\rho_j - \rho_i)}.$$

The latter two equations give (26) right away, which completes the proof. □

Remark 2 In the case where $|r_1| < |pz_2 - z_3|q^{-1}$, we choose the radius η such that $\eta < \min(|r_1|, |pz_2 - z_3|q^{-1})$. To capture this modification, it is necessary to correct the joint transform in Proposition 2 as follows:

$$\mathbb{E}\bigl[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\bigr] = \frac{\frac{1}{2\pi\mathbf{i}} \int_{D_\eta} \frac{1}{x^{K-1-n}} \frac{1}{qx+pz_2-z_3} \frac{dx}{x-z_1\xi(w+\lambda(1-qx-pz_2))} + \text{Res}_{z_0} f_1(z)}{\frac{1}{2\pi\mathbf{i}} \int_{D_\eta} \frac{1}{x^{K-1}} \frac{1}{qx+pz_2-z_3} \frac{dx}{x-z_1\xi(w+\lambda(1-qx-pz_2))} + \text{Res}_{z_0} f_2(z)},$$

where $z_0 = |pz_2 - z_3|q^{-1}$, and the functions $f_1(z)$ and $f_2(z)$ are the integrands of the contour integrations in the numerator and the denominator of $\mathbb{E}[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}]$.

Remark 3 For the M/G/1/K queue, note that Rosenlund [13] obtained the trivariate transform of B_1, S_1, and L_1. Recall that L_1 is the total number of losses during the busy period. Restricting Rosenlund's result to the M/PH/1/K queue, Proposition 2 extends his result in two ways. First, it gives the four-variate joint transform of B_n, S_n, L_n^c, and L_n^o, for the case when $n \geq 1$. Secondly, it allows the dropping of customers even when the queue is not full.

2.2 M/PH/1/K queue under threshold policy

Let $m \in \{1, \ldots, K\}$ denote the threshold of the M/PH/1/K queue length. According to the threshold policy, if the queue length at time t is i, the interarrival times and service times are then defined as follows. For $i \leq m - 1$, we have $-\mathbf{F}_i = F_i^o f = \lambda_0$, $\mathbf{S}_i = \mathbf{S}_0$, $s_i = s$, and $p_i = p_0$. For $m \leq i \leq K - 1$, we have $-\mathbf{F}_i = F_i^o f = \lambda_1$, $\mathbf{S}_i = \mathbf{S}_1$ and $s_i = s$, and $p_i = p_1$ and $p_K = 1$.

Let $\xi_i(w) = s(w\mathbf{I} - \mathbf{S}_i)^{-1} S_i^o = P_i(w)/Q_i(w)$ denote the LST of the service times when the queue length is below the threshold ($i = 0$) or above it ($i = 1$). Moreover, we let $\xi_i^l(w) = e_l(w\mathbf{I} - \mathbf{S}_i)^{-1} S_i^o = P_i^{*l}(w)/Q_i^l(w)$. Note that since $Q_0(w)$ is the common denominator of $\xi_0^l(w)$ we see that $\xi_0^l(w) = P_0^l(w)/Q_0(w)$ is a rational function where $P_0^l(w)$ is a polynomial of degree $< M_s$. Let $\mathbf{C}_0$ denote the matrix with (j,l)-entry equal to the coefficient of w^{j-1} of the polynomial $P_0^l(w)$. In the following, we shall assume that the matrix $\mathbf{C}_0$ is invertible. Note that the Erlang, hyperexponential, and Coxian distributions satisfy the latter assumption.

Lemma 2 *The function* $x - z_1\xi_l(w + \lambda(1 - q_l x - p_l z_2))$ *has* $M_s + 1$ *distinct non-null roots* $r_{1l}, \ldots, r_{(M_s+1)l}$, *such that* $0 < |r_{1l}| < \cdots < |r_{(M_s+1)l}|$, $l = 0, 1$.

Proof The proof results by analogy with the proof of Lemma 1. □

Before reporting our main result on the M/PH/1/K under Threshold Policy in Proposition 3, let us first introduce some notation.

Let D_{η_1} denote the circle with center at the origin and with radius η_1, $\frac{|p_1z_2-z_3|}{q_1} < \eta_1 < |r_{11}|$. According to Lemma 2, r_{11} is the root with the smallest absolute value. The contour integration $v(l)$, $l = 1, \ldots, M_s$, is given by

$$v(l) = z_1 \frac{\frac{1}{2\pi \mathbf{i}} \int_{D_{\eta_1}} \frac{1}{x^{K-m}} \frac{\xi_1^l(w+\lambda(1-q_1x-p_1z_2))}{q_1x+p_1z_2-z_3} \frac{dx}{x-z_1\xi_1(w+\lambda(1-q_1x-p_1z_2))}}{\frac{1}{2\pi \mathbf{i}} \int_{D_{\eta_1}} \frac{1}{x^{K-m}} \frac{1}{q_1x+p_1z_2-z_3} \frac{dx}{x-z_1\xi_1(w+\lambda(1-q_1x-p_1z_2))}}. \tag{27}$$

Let $\rho_{i0} = w + \lambda_0(1 - q_0 r_{i0} - p_0 z_2)$. Let us define $v_0(k,m)$ as follows:

$$v_0(k,m) = \frac{(-1)^{k-1}}{\prod_{l=1,l\neq m}^{M_s} \nu_l - \nu_m} \sum \nu_{m_1} \times \cdots \times \nu_{m_{M_s-k}}, \quad k, m = 1, \ldots, M_s, \tag{28}$$

where $1 \le m_1 < \cdots < m_{M_s-k} \le M_s$, $m_1, \ldots, m_{M_s-k} \neq k$, and $(\nu_1, \ldots, \nu_{M_s}) = (\rho_{10}, \ldots, \rho_{(i-1)0}, \rho_{(i+1)0}, \ldots, \rho_{(M_s+1)0})$. Note that for $k = M_s$, $\sum \nu_{m_1} \times \cdots \times \nu_{m_{M_s-k}}$ is equal to one by definition. Finally, let $\beta(i)$ denote the following sum:

$$\beta(i) = \sum_{l=1}^{M_s} v(l) \sum_{m=1,m\neq i}^{M_s+1} Q_0(\rho_{m0}) \sum_{k=1}^{M_s} c_0(l,k) v_0(k,m), \tag{29}$$

where $c_0(l,k)$ is the (l,k)-entry of $\mathbf{C}_0^{-1}$.

We are now ready to report our main result on the M/PH/1/K under Threshold Policy.

Proposition 3 (M/PH/1/K under threshold policy) *The joint transform of* B_1, S_1, L_1^c, *and* L_1^o *in the M/PH/1/K queue operating under the threshold policy is given by*

$$\mathbb{E}\left[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\right] = \frac{\sum_{i=1}^{M_s+1} \frac{z_1-\beta(i)}{r_{i0}^{m-2}} \frac{Q_0(\rho_{i0})}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_{j0}-\rho_{i0})}}{\sum_{i=1}^{M_s+1} \frac{z_1-\beta(i)}{r_{i0}^{m-1}} \frac{Q_0(\rho_{i0})}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_{j0}-\rho_{i0})}}, \tag{30}$$

where r_{i0} *and* r_{i1} *are given in Lemma 2,* $Q_0(w)$ *is the denominator of* $\xi_0(w)$, *and* $\beta(i)$ *is given in* (29).

Proof By analogy with Proposition 2, the joint transform B_1, S_1, L_1^c, and L_1^o for the M/PH/1/K queue can be written as follows:

$$\mathbb{E}\big[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\big] = z_1 e_1 \otimes s\mathbf{Q}_K^{-1} e_1^T \otimes S_0^o, \tag{31}$$

where in this case $\mathbf{Q}_K$ has the following structure:

$$\mathbf{Q}_K = \left(\frac{\mathbf{F}_{00}|\mathbf{F}_{01}}{\mathbf{F}_{10}|\mathbf{F}_{11}}\right).$$

The matrix $\mathbf{F}_{ll}$, $l = 0, 1$, is a block tridiagonal matrix with upper diagonal blocks equal to $\mathbf{E}_{0l} = -q_l\lambda_l\mathbf{I}$, diagonal blocks equal to $\mathbf{E}_{1l} = w\mathbf{I} + \lambda_l(1 - p_l z_2)\mathbf{I} - \mathbf{S}_l$ and lower diagonal blocks equal to $\mathbf{E}_{2l} = -z_1 S_l^o s$. Note that $\mathbf{F}_{00}$ is an $(m-1)$-by-$(m-1)$-block matrix and $\mathbf{F}_{11}$ is a $(K - m + 1)$-by-$(K - m + 1)$ block matrix. Moreover, the $(K - m + 1, K - m + 1)$-block entry of $\mathbf{F}_{11}$ is equal to $\mathbf{E}_{11}^* = w\mathbf{I} + \lambda_1(1 - z_3)\mathbf{I} - \mathbf{S}_1$. The matrix $\mathbf{F}_{01}$ is a block matrix with all its blocks equal to the zero matrix except the $(m - 1, 1)$-block, which is $\mathbf{E}_{00} = -q_0\lambda_0\mathbf{I}$. Finally, the matrix $\mathbf{F}_{10}$ is a block matrix with all blocks equal to the zero matrix except the $(1, m - 1)$-block, which is $\mathbf{E}_{21} = -z_1 S_1^o s$. Therefore, $\mathbf{F}_{00}$, $\mathbf{F}_{10}$, $\mathbf{F}_{10}$, and $\mathbf{F}_{11}$ have the following canonical form:

$$\mathbf{F}_{00} = \begin{pmatrix} \mathbf{E}_{10} & \mathbf{E}_{00} & \mathbf{0} & \cdots & \cdots \\ \mathbf{E}_{20} & \mathbf{E}_{10} & \mathbf{E}_{00} & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{E}_{00} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{E}_{20} & \mathbf{E}_{10} \end{pmatrix},$$

$$\mathbf{F}_{01} = \begin{pmatrix} \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{E}_{00} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \end{pmatrix},$$

$$\mathbf{F}_{10} = \begin{pmatrix} \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{E}_{21} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

$$\mathbf{F}_{11} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{01} & \mathbf{0} & \cdots & \cdots \\ \mathbf{E}_{21} & \mathbf{E}_{11} & \mathbf{E}_{01} & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \mathbf{E}_{11} & \mathbf{E}_{01} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{E}_{21} & \mathbf{E}_{11}^* \end{pmatrix}.$$

Equations (11) and (31) yield

$$\mathbb{E}\big[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\big] = z_1 e_1 \otimes s\big(\mathbf{F}_{00} - \mathbf{F}_{01}\mathbf{F}_{11}^{-1}\mathbf{F}_{10}\big)^{-1} e_1^T \otimes S_0^o$$
$$= z_1 e_1 \otimes s\big(\mathbf{F}_{00} - q_0\lambda_0 z_1 \mathbf{F}_{11}^{-1}(1,1) S_1^o s U^T U\big)^{-1} e_1^T \otimes S_0^o, \tag{32}$$

where U is a row vector of blocks with all entries equal to zero except the last, which is $\mathbf{I}$, and $\mathbf{F}_{11}^{-1}(1,1)$ is the $(1,1)$-block entry of $\mathbf{F}_{11}^{-1}$.

We shall now derive an expression for $z_1\mathbf{F}_{11}^{-1}(1,1)S_1^o$. Note that $z_1\mathbf{F}_{11}^{-1}(1,1)S_1^o$ is a column vector with size M_s. Let $v := z_1\mathbf{F}_{11}^{-1}(1,1)S_1^o$. First, observe that $\mathbf{F}_{11}$ has the same structure as the matrix $\mathbf{Q}_K$ in (15) with K replaced by $K-m+1$, λ by λ_1, $\mathbf{S}$ by $\mathbf{S}_1$, and $S^o s$ by $S_1^o s$. Second, note that the l-th entry of v can be written as follows:

$$v(l) = z_1 e_1 \otimes e_l (\mathbf{F}_{11})^{-1} e_1^T \otimes S_1^o, \quad l = 1, \dots, M_s. \tag{33}$$

Therefore, by analogy with the proof of Proposition 2, we find that $v(l)$ satisfies (27).

Note that $\mathbf{F}_{00} - q_0\lambda_0 v s U^T U$ has the same structure as the matrix $\mathbf{Q}_K$ in (15) with $K = m-1$, $\mathbf{E}_0 = \mathbf{E}_{00}$, $\mathbf{E}_1 = \mathbf{E}_{10}$, $\mathbf{E}_2 = \mathbf{E}_{20}$, and $\mathbf{E}_1^* = \mathbf{E}_{10} - q_0\lambda_0 v s$. Moreover, (32) has the same form as (14). By analogy with the proof of Proposition 2, we find that

$$\sum_{i=1}^{m-1} a_i x^i S_0^o = \frac{-x}{x - z_1\xi_0(\rho_0)}\Bigg[(z_1 a_1 S_0^o - x)\xi_0(\rho_0) + \lambda_0 q_0 x^{m-1} \sum_{j=1}^{M_s} a_{m-1j}\big(x\xi_0^j(\rho_0) - v(j)\xi_0(\rho_0)\big)\Bigg],$$

where $a = (a_1, \dots, a_{m-1}) := e_1 \otimes s(\mathbf{F}_{00} - q_0\lambda_0 v s U^T U)^{-1}$, $a_{m-1} = (a_{(m-1)1}, \dots, a_{(m-1)M_s})$, and $\rho_0 = w + \lambda_0(1 - q_0 x - p_0 z_2)$. Recall that r_{i0}, $i = 0, \dots, M_s + 1$, are the roots of $x - z_1\xi_0(w + \lambda_0(1 - q_0 x - p_0 z_2))$. The analyticity of $\sum_{i=1}^K a_i x^i S_0^o$ gives that

$$z_1 a_1 S_0^o \xi_0(\rho_{i0}) + \lambda_0 q_0 r_{i0}^{m-1} \sum_{j=1}^{M_s} a_{m-1j}\big(r_{i0}\xi_0^j(\rho_{i0}) - v(j)\xi_0(\rho_{i0})\big) = r_{i0}\xi_0(\rho_{i0}),$$

where $i = 1, \dots, M_s + 1$ and $\rho_{i0} = w + \lambda_0(1 - q_0 r_{i0} - p_0 z_2)$. Cramer's rule yields that

$$\mathbb{E}\big[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\big] = z_1 a_1 S_0^o = \frac{\sum_{i=1}^{M_s+1} \frac{\xi_0(\rho_{i0})(-1)^i}{r_{i0}^{m-1}} \det(\mathbf{N})}{\sum_{i=1}^{M_s+1} \frac{\xi_0(\rho_{i0})(-1)^i}{r_{i0}^{m}} \det(\mathbf{N})}$$
$$= \frac{\sum_{i=1}^{M_s+1} \frac{(-1)^i}{r_{i0}^{m-2}} \det(\mathbf{N})}{\sum_{i=1}^{M_s+1} \frac{(-1)^i}{r_{i0}^{m-1}} \det(\mathbf{N})}, \tag{34}$$

where $\mathbf{N}$ is an M_s-by-M_s matrix that has the following canonical form:

$$\mathbf{N} = \begin{pmatrix} \xi_0^1(\rho_{10}) - v(1)/z_1 & \cdots & \xi_0^{M_s}(\rho_{10}) - v(M_s)/z_1 \\ \vdots & \vdots & \vdots \\ \xi_0^1(\rho_{(i-1)0}) - v(1)/z_1 & \cdots & \xi_0^{M_s}(\rho_{(i-1)0}) - v(M_s)/z_1 \\ \xi_0^1(\rho_{(i+1)0}) - v(1)/z_1 & \cdots & \xi_0^{M_s}(\rho_{(i+1)0}) - v(M_s)/z_1 \\ \vdots & \vdots & \vdots \\ \xi_0^1(\rho_{(M_s+1)0}) - v(1)/z_1 & \cdots & \xi_0^{M_s}(\rho_{(M_s+1)0}) - v(M_s)/z_1 \end{pmatrix}.$$

Let $\mathbf{M}_0(i, 1)$ denote the following matrix:

$$\mathbf{M}_0(i, 1) = \begin{pmatrix} \xi_0^1(\rho_{10}) & \cdots & \xi_0^{M_s}(\rho_{10}) \\ \vdots & \vdots & \vdots \\ \xi_0^1(\rho_{(i-1)0}) & \cdots & \xi_0^{M_s}(\rho_{(i-1)0}) \\ \xi_0^1(\rho_{(i+1)0}) & \cdots & \xi_0^{M_s}(\rho_{(i+1)0}) \\ \vdots & \vdots & \vdots \\ \xi_0^1(\rho_{(M_s+1)0}) & \cdots & \xi_0^{M_s}(\rho_{(M_s+1)0}) \end{pmatrix}.$$

It is easily seen that $\mathbf{N}$ can be decomposed as follows:

$$\mathbf{N} = \mathbf{M}_0(i, 1) - \frac{1}{z_1} e^T v.$$

Since $\xi_0^l(w) = P_0^l(w)/Q_0(w)$, $l = 1, \ldots, M_s$, are rational functions with common denominator $Q_0(w)$ the decomposition of $\mathbf{M}_0(i, 1)$ gives

$$\mathbf{M}_0(i, 1) = \mathbf{D}(i)\mathbf{V}_0(i)\mathbf{C}_0,$$

where $\mathbf{D}(i)$ is an M_s-by-M_s diagonal matrix with j-th diagonal element, $j = 1, \ldots, M_s + 1$ and $j \neq i$, equal to $1/Q_0(\rho_{j0})$, $\mathbf{V}_0(i)$ is a Vandermonde matrix of the following canonical form:

$$\mathbf{V}_0(i) = \begin{pmatrix} 1 & \rho_{10} & \cdots & (\rho_{10})^{M_s} \\ \vdots & \vdots & \vdots & \\ 1 & \rho_{(i-1)0} & \cdots & (\rho_{(i-1)0})^{M_s} \\ 1 & \rho_{(i+1)0} & \cdots & (\rho_{(i+1)0})^{M_s} \\ \vdots & \vdots & & \vdots \\ 1 & \rho_{(M_s+1)0} & \cdots & (\rho_{(M_s+1)0})^{M_s} \end{pmatrix},$$

and $\mathbf{C}_0$ is a matrix with (j, l)-entry equal to the coefficient of w^{j-1} of the polynomial $P_0^l(w)$.

By Sylvester's determinant we have

$$\det(\mathbf{N}) = \frac{1}{z_1} \det\big(\mathbf{M}_0(i, 1)\big)\big(z_1 - v\mathbf{M}_0(i, 1)^{-1} e^T\big)$$

$$= \frac{1}{z_1} \det\big(\mathbf{M}_0(i,1)\big)\big(z_1 - v\mathbf{C}_0^{-1}\mathbf{V}_0(i)^{-1}\mathbf{D}(i)^{-1}e^T\big). \tag{35}$$

By analogy with Lemma 4 in Appendix B, we find that

$$\det\big(\mathbf{M}_0(i,1)\big) = \det(\mathbf{C}_0)(-1)^{i-1} \frac{\prod_{j=1}^{M_s}\prod_{k=j+1}^{M_s+1}(\rho_{k0}-\rho_{j0})}{\prod_{j=1}^{M_s+1} Q_0(\rho_{j0})} \frac{Q_0(\rho_{i0})}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_{j0}-\rho_{i0})}.$$

Let $\beta(i) := v\mathbf{C}_0^{-1}\mathbf{V}_0(i)^{-1}d$, where $d = \mathbf{D}(i)^{-1}e^T$. Therefore, d is a column vector of dimension M_s with j-th entry equal to $Q_0(\rho_{j0})$, $j = 1, \dots, M_s + 1$ and $j \neq i$. Let $v_0(k,l)$ denote the (k,l)-entry of $\mathbf{V}_0(i)^{-1}$. Note that the inverse of a Vandermonde matrix is known in closed form; see, e.g., [9]. We deduce from [9] the values of $v_0(k,l)$ that are given in (28). Let us denote $c_0(i,j)$ the (i,j)-entry of $\mathbf{C}_0^{-1}$; then it is easily seen that $\beta(i)$ is given by (29). Substituting $\beta(i)$ and $\det(\mathbf{M}_0(i,1))$ into (35) gives $\det(\mathbf{N})$. Inserting $\det(\mathbf{N})$ into (34) completes the proof. □

2.3 PH/M/1/K queue

For the level independent PH/M/1/K queue we have $-\mathbf{S}_i = S_i^o s_i = \mu$, $i = 1, \dots, K$, $\mathbf{F}_i = \mathbf{F}$ and $F_i^o f_i = F^o f$, $i = 1, \dots, K$. Let $\phi(w) = f(w\mathbf{I} - \mathbf{F})^{-1}F^o$ denote the LST of the interarrival times. Moreover, we assume that $q_i = q$, $i = 1, \dots, K-1$, and $q_K = 0$.

Lemma 3 *The function $x - (q + xpz_2)\phi(w + \mu(1 - z_1x))$ has $M_a + 1$ distinct non-null roots $o_1, \dots, o_{M_a+1}$, such that $0 < |o_1| < |o_2| < \cdots < |o_{M_a+1}|$.*

Proof The proof results by analogy with Lemma 1. □

Before reporting our result on the PH/M/1/K queue, let us introduce some notation.

Let D_δ denote the circle with center at the origin and with radius equal to δ with $\frac{q}{p|z_2|} < \delta < |o_1|$. o_1 is the root with the smallest absolute value defined in Lemma 3. Let $f(\delta)$, $g(\delta)$, $h(\delta)$, and $I(\delta)$ denote the following contour integrations:

$$f(\delta) = \frac{1}{2\pi\mathbf{i}} \int_{D_\delta} \frac{1}{x^{n-1}} \frac{1}{q + pz_2x} \frac{1}{w + \mu(1 - z_1x)} \times \frac{dx}{x - (q + pz_2x)\phi(w + \mu(1 - z_1x))}, \tag{36}$$

$$g(\delta) = \frac{1}{2\pi\mathbf{i}} \int_{D_\delta} \frac{1}{x^{n-1}} \frac{1}{q + pz_2x} \frac{dx}{x - (q + pz_2x)\phi(w + \mu(1 - z_1x))}, \tag{37}$$

$$h(\delta) = \frac{1}{2\pi\mathbf{i}} \int_{D_\delta} \frac{q + (pz_2 - z_3)x}{x^K(q + pz_2x)} \frac{dx}{x - (q + pxz_2)\phi(w + \mu(1 - z_1x))}, \tag{38}$$

$$I(\delta) = \frac{1}{2\pi \mathbf{i}} \int_{D_\delta} \frac{q + (pz_2 - z_3)x}{x^K(q + pz_2 x)}$$

$$\times \frac{1}{w + \mu(1 - z_1 x)} \frac{dx}{x - (q + pxz_2)\phi(w + \mu(1 - z_1 x))}. \tag{39}$$

Finally, let R be defined by

$$R = -\frac{(\mu z_1)^n}{(w+\mu)^{n-1}} \frac{1}{q\mu z_1 + p(w+\mu)z_2} \frac{1}{(w+\mu)(1-pz_2) - q\mu z_1}. \tag{40}$$

We are now ready to report our result on the PH/M/1/K queue.

Proposition 4 (PH/M/1/K queue) *The joint transform of B_n, S_n, L_n^o, and L_n^c for the PH/M/1/K queue with $p > 0$ $(p = 1 - q)$ and $n = 1, \dots, K$ is given by*

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \big((w+\mu)(1-pz_2) - q\mu z_1\big)\left(R + f(\delta) + \frac{g(\delta)I(\delta)}{h(\delta)}\right),$$

where $f(\delta)$, $g(\delta)$, $h(\delta)$, $I(\delta)$, and R are given in (36)–(40).

Proof Due to the exponential service times, we have that $s_n = 1$ and $S_1^o = \mu$. Then, according to Theorem 1, the joint transform B_n, S_n, L_n^c, and L_n^o in this case can be written as follows:

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \mu z_1 e_n \otimes f \mathbf{Q}_K^{-1} e_1^T \otimes e, \tag{41}$$

where $\mathbf{Q}_K$ in this case has the same structure as in (15) with $\mathbf{E}_0 = -qF^o f$, $\mathbf{E}_1 = (w+\mu)\mathbf{I} - \mathbf{F} - pz_2 F^o f$, $\mathbf{E}_1^* = (w+\mu)\mathbf{I} - \mathbf{F} - z_3 F^o f$, and $\mathbf{E}_2 = -z_1\mu\mathbf{I}$. Let $b = (b_1, \dots, b_K) := e_n \otimes f\mathbf{Q}_K^{-1}$. Note that each of the entries of the row vector b is in its turn a row vector of dimension M_a and is a function of w, z_1, z_2, and z_3. Equation (41) in terms of b can be rewritten as

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \mu z_1 b_1 e^T = \mu z_1 \sum_{j=1}^{M_a} b_{1j}. \tag{42}$$

By analogy with the derivation of (18), we find that

$$\sum_{i=1}^{K} b_i x^i = \big(\mu z_1 b_1 - x^n f + x^K (qx + pz_2 - z_3) b_K F^o f\big)$$

$$\times \big(\mathbf{F} - \theta\mathbf{I} + (qx + pz_2)F^o f\big)^{-1},$$

where $\theta := w + \mu(1 - z_1/x)$. Let $\mathbf{F}_* := \mathbf{F} - \theta\mathbf{I}$. Note that under the condition that $\mathrm{Re}(\theta) \geq 0$ the matrix $\mathbf{F}_*$ is nonsingular. Hence, the Sherman-Morrison formula, see, for example, [3, Fact 2.14.2, p. 67], yields

$$\big(\mathbf{F}_* + (qx+pz_2)F^o f\big)^{-1} = \mathbf{F}_*^{-1} - \frac{qx+pz_2}{1+(qx+pz_2)t\mathbf{F}_*^{-1}F^o}\mathbf{F}_*^{-1}F^o f\mathbf{F}_*^{-1}. \quad (43)$$

Multiplying to the right of $\sum_{i=1}^K b_i x^i$ with the column vector F^o and using (43) gives

$$\begin{aligned}\sum_{i=1}^K b_i x^i F^o &= \frac{1}{1+(qx+pz_2)f\mathbf{F}_*^{-1}F^o}\\ &\quad\times \big(\mu_1 z_1 b_1 - x^n f + x^K(qx+pz_2-z_3)b_K F^o f\big)\mathbf{F}_*^{-1}F^o. \quad (44)\end{aligned}$$

From (1) we have that $f\mathbf{F}_*^{-1}F^o = -\phi(\theta)$ and $\mathbf{F}_*^{-1}F^o = -(\phi^1(\theta),\ldots,\phi^{M_a}(\theta))^T$, where $\phi^i(\theta) = e_i(\theta\mathbf{I}-\mathbf{F})^{-1}F^o$. Therefore, $\phi(\theta) = f(\phi^1(\theta),\ldots,\phi^{M_a}(\theta))^T$ is a linear combination of $\phi^i(\theta)$, $i=1,\ldots,M_a$. Inserting $f\mathbf{F}_*^{-1}F^o$ and $\mathbf{F}_*^{-1}F^o$ into (44) yields

$$\sum_{i=1}^K b_i x^i F^o = -\frac{x^K(qx+pz_2-z_3)\phi(\theta)b_K F^o + \mu_1 z_1 \sum_{j=1}^{M_a} b_{1j}\phi^j(\theta) - x^n\phi(\theta)}{1-(qx+pz_2)\phi(\theta)}, \quad (45)$$

where $b_1 = (b_{11},\ldots,b_{1M_a})$. Note that $b_i F^o$ is a joint transform function. For this reason, the l.h.s. of (45) is analytical for any finite x, and the poles on the r.h.s. of (45) are removable. Note that the roots of $1-(qx+pz_2)\phi(w+\mu(1-z_1/x))$ are equal to the inverse of the roots of $x-(q+xpz_2)\phi(w+\mu(1-z_1x))$. Therefore, Lemma 3 and the analyticity of $\sum_{i=1}^K b_i x^i F^o$ give

$$\begin{aligned}&\frac{q+(pz_2-z_3)o_i}{o_i^{K+1}}\phi(\theta_i)b_K F^o + \mu_1 z_1\sum_{j=1}^{M_a} b_{1j}\phi^j(\theta_i) = \frac{1}{o_i^n}\phi(\theta_i),\\ &i=1,\ldots,M_a+1, \quad (46)\end{aligned}$$

where $\theta_i := w+\mu(1-z_1 o_i)$. The system of equations in (46) has M_a+1 equations with M_a+1 unknowns which are $b_K F^o, b_{11},\ldots,b_{1M_a}$. Using Cramer's rule we find that

$$\mathbb{E}\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big] = \mu z_1 b_1 e^T = \mu z_1\sum_{j=1}^{M_a} b_{1j} = -\frac{\det(\mathbf{H})}{\det(\mathbf{K})}, \quad (47)$$

where $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{pmatrix} \frac{q+(pz_2-z_3)o_1}{o_1^{K+1}}\phi(\theta_1) & \phi^1(\theta_1) & \cdots & \phi^{M_a}(\theta_1)\\ \vdots & \vdots & \vdots & \vdots\\ \frac{q+(pz_2-z_3)o_{M_a+1}}{o_{M_a+1}^{K+1}}\phi(\theta_{M_a+1}) & \phi^1(\theta_{M_a+1}) & \cdots & \phi^{M_a}(\theta_{M_a+1})\end{pmatrix},$$

and $\mathbf{H}$ is given by

$$\mathbf{H}=\begin{pmatrix} \frac{q+(pz_2-z_3)o_1}{o_1^{K+1}}\phi(\theta_1) & \phi^1(\theta_1) & \cdots & \phi^{M_a}(\theta_1) & \frac{1}{o_1^n}\phi(\theta_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{q+(pz_2-z_3)o_{M_a+1}}{o_{M_a+1}^{K+1}}\phi(\theta_{M_a+1}) & \phi^1(\theta_{M_a+1}) & \cdots & \phi^{M_a}(\theta_{M_a+1}) & \frac{1}{o_{M_a+1}^n}\phi(\theta_{M_a+1}) \\ 0 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

The Laplace expansion of the determinant along the first column of $\mathbf{K}$ and $\mathbf{H}$ gives

$$\begin{aligned}\mathbb{E}\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big] &= -\frac{\sum_{i=1}^{M_a+1}\frac{q+(pz_2-z_3)o_i}{o_i^{K+1}}\phi(\theta_i)(-1)^{i+1}\det(\mathbf{H}(i,1))}{\sum_{i=1}^{M_a+1}\frac{q+(pz_2-z_3)o_i}{o_i^{K+1}}\phi(\theta_i)(-1)^{i+1}\det(\mathbf{K}(i,1))} \\ &= -\frac{\sum_{i=1}^{M_a+1}\frac{1}{o_i^K}\frac{q+(pz_2-z_3)o_i}{q+pz_2o_i}(-1)^{i+1}\det(\mathbf{H}(i,1))}{\sum_{i=1}^{M_a+1}\frac{1}{o_i^K}\frac{q+(pz_2-z_3)o_i}{q+pz_2o_i}(-1)^{i+1}\det(\mathbf{K}(i,1))}, \end{aligned} \tag{48}$$

where the matrices $\mathbf{H}(i,1)$ and $\mathbf{K}(i,1)$ are obtained by deleting the i-th row and the first column of the matrices $\mathbf{H}$ and $\mathbf{K}$, and the second equality follows from $\phi(\theta_i)=o_i/(q+pz_2o_i)$.

Note that $\phi(w)$ is a rational function with denominator, $Q_\phi(w)$, of degree equal to M_a and numerator of degree $<M_a$. By analogy with the determinant of $\mathbf{M}(i,1)$ that is given in Lemma 4 in Appendix B, we find that

$$\begin{aligned}\det\big(\mathbf{K}(i,1)\big) &= C_k(-1)^{i-1}\frac{\prod_{j=1}^{M_a}\prod_{k=j+1}^{M_a+1}(\theta_k-\theta_j)}{\prod_{j=1}^{M_a+1}Q_\phi(\theta_j)}\frac{Q_\phi(\theta_i)}{\prod_{j=1,j\neq i}^{M_a+1}(\theta_j-\theta_i)} \\ &= C_k(-1)^{M_a+i-1}\frac{\prod_{j=1}^{M_a}\prod_{k=j+1}^{M_a+1}(\theta_k-\theta_j)}{\prod_{j=1}^{M_a+1}Q_\phi(\theta_j)} \\ &\quad\times \operatorname{Res}_{o_i}\frac{1}{x-(q+xpz_2)\phi(w+\mu(1-z_1x))}, \end{aligned} \tag{49}$$

where C_k is a constant that is a function of the polynomial parameters of the numerators of $\phi^i(w)$, $i=1,\dots,M_a$. Assume that $q/|pz_2|<\delta<|o_1|$. We find that

$$\begin{aligned}&\sum_{i=1}^{M_a+1}\frac{1}{o_i^K}\frac{q+(pz_2-z_3)o_i}{q+pz_2o_i}(-1)^{i+1}\det\big(\mathbf{K}(i,1)\big) \\ &\quad= C_k(-1)^{M_a}\frac{\prod_{j=1}^{M_a}\prod_{k=j+1}^{M_a+1}(\theta_k-\theta_j)}{\prod_{j=1}^{M_a+1}Q_\phi(\theta_j)}\big(-h(\delta)\big),\end{aligned}$$

where $h(\delta)$ is given in (38). Note that the minus sign that is next to $h(\delta)$ is due to the fact that the sum of all residues of the function

$$\frac{q+(pz_2-z_3)x}{x^K(q+pz_2x)}\frac{1}{x-(q+pxz_2)\phi(w+\mu(1-z_1x))},$$

including the residue at infinity, which is equal to zero ($K\geq 1$), is zero. We shall refer to the latter property of complex functions as the Inside-Outside property.

The expansion of the determinant of $\mathbf{H}(i,1)$ along the last column yields

$$\det\bigl(\mathbf{H}(i,1)\bigr)=\sum_{j=1,j\neq i}^{M_a+1}\frac{1}{o_j^{n-1}}\frac{(-1)^{M_a+j+1}}{q+pz_2o_j}\det(\mathbf{J}),\tag{50}$$

where $\mathbf{J}$ is obtained by deleting the j-th row and the last column of the matrix $\mathbf{H}(i,1)$. It is easily seen that $\mathbf{J}$ is an M_a-by-M_a matrix with the l-th row equal to $(\phi^1(\theta_l),\dots,\phi^{M_a}(\theta_l))$, $l=1,\dots,M_a+1$ and $l\neq i,j$, and the last row is equal to e. By analogy with the determinant of $\mathbf{M}(i,1)$ we find that

$$\begin{aligned}\det(\mathbf{J})&=\frac{C_J}{Q_\phi(0)}\prod_{l=1,l\neq i,j}^{M_a+1}\frac{\theta_l}{Q_\phi(\theta_l)}\prod_{l=1,l\neq i,j}^{M_a}\prod_{k=l+1,k\neq i,j}^{M_a+1}(\theta_k-\theta_l)\\&=\frac{C_J}{Q_\phi(0)}\prod_{l=1,l\neq i,j}^{M_a+1}\frac{\theta_l}{Q_\phi(\theta_l)}(-1)^{i+j-1}\frac{\prod_{l=1}^{M_a}\prod_{k=l+1}^{M_a+1}(\theta_k-\theta_l)}{\prod_{l=1,l\neq i}^{M_a+1}(\theta_l-\theta_i)\prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)}\\&=\frac{C_J(-1)^{i+j-1}}{Q_\phi(0)}\left(\prod_{l=1}^{M_a+1}\theta_l\right)\frac{\prod_{l=1}^{M_a}\prod_{k=l+1}^{M_a+1}(\theta_k-\theta_l)}{\prod_{l=1}^{M_a+1}Q_\phi(\theta_l)}\frac{Q_\phi(\theta_i)}{\theta_i\prod_{l=1,l\neq i}^{M_a+1}(\theta_l-\theta_i)}\\&\quad\times\frac{Q_\phi(\theta_j)}{\theta_j\prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)},\end{aligned}$$

where $Q_\phi(0)$ is due to the last row of $\mathbf{J}$ which is equal to $e=(1,\dots,1)=(P_\phi^1(0)/Q_\phi^1(0),\dots,P_\phi^{M_a}(0)/Q_\phi^{M_a}(0))$. It follows from the definitions of the matrices $\mathbf{J}$ and $\mathbf{K}$ that $C_J=C_k$. We note that

$$\begin{aligned}\prod_{l=1}^{M_a+1}\theta_l&=(\mu z_1)^{M_a+1}\prod_{l=1}^{M_a+1}\left(\frac{w+\mu}{\mu z_1}-o_l\right)\\&=(\mu z_1)^{M_a+1}\frac{\frac{w+\mu}{\mu z_1}Q_\phi(0)-(q+pz_2\frac{w+\mu}{\mu z_1})P_\phi(0)}{(-\mu z_1)^{M_a}}\\&=(-1)^{M_a}Q_\phi(0)\bigl[(w+\mu)(1-pz_2)-q\mu z_1\bigr],\end{aligned}$$

where the second equality follows from the fact that o_l, $l=1,\dots,M_a+1$, are the roots of $x-(q+xpz_2)\phi(w+\mu(1-z_1x))$ and $\phi(w)=P_\phi(w)/Q_\phi(w)$, and the last from $\phi(0)=1$. Inserting $\det(\mathbf{J})$ and $\prod_{l=1}^{M_a+1}\theta_l$ into (50) yields

$$\begin{aligned}
\det\big(\mathbf{H}(i,1)\big) &= \sum_{j=1,j\neq i}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{(-1)^{M_a+j+1}}{q+pz_2o_j}\det(\mathbf{J}) \\
&= C_J(-1)^i\big[(w+\mu)(1-pz_2)-q\mu z_1\big] \\
&\quad\times \frac{\prod_{l=1}^{M_a}\prod_{k=l+1}^{M_a+1}(\theta_k-\theta_l)}{\prod_{l=1}^{M_a+1}Q_\phi(\theta_l)} \frac{Q_\phi(\theta_i)}{\theta_i\prod_{l=1,l\neq i}^{M_a+1}(\theta_l-\theta_i)} \\
&\quad\times \sum_{j=1,j\neq i}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \frac{Q_\phi(\theta_j)}{\theta_j\prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)}.
\end{aligned} \tag{51}$$

Note that, for $p>0$ and $n=1,\ldots,K$, we have

$$\begin{aligned}
&\sum_{j=1,j\neq i}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \frac{Q_\phi(\theta_j)}{\theta_j\prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)} \\
&= (-1)^{M_a}\sum_{j=1}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \frac{(\theta_i-\theta_j)Q_\phi(\theta_j)}{\theta_j\prod_{l=1,l\neq j}^{M_a+1}(\theta_j-\theta_l)} \\
&= (-1)^{M_a}\left[\theta_i\sum_{j=1}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \frac{1}{\theta_j}\text{Res}_{o_j} \frac{1}{x-(q+pz_2x)\phi(w+\mu(1-z_1x))}\right. \\
&\quad\left.-\sum_{j=1}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j}\text{Res}_{o_j} \frac{1}{x-(q+pz_2x)\phi(w+\mu(1-z_1x))}\right] \\
&= (-1)^{M_a+1}\big(\theta_i\big(f(\delta)+R\big)+g(\delta)\big),
\end{aligned}$$

where the last equality follows for $p>0$ from the Inside–Outside property of the integrands of $f(\delta)$ and $g(\delta)$ that are given in (36) and (37),

$$\begin{aligned}
R &= \text{Res}_{\frac{w+\mu}{\mu z_1}} \frac{1}{x^{n-1}} \frac{1}{q+pz_2x} \frac{1}{w+\mu(1-z_1x)} \frac{1}{x-(q+pz_2x)\phi(w+\mu(1-z_1x))} \\
&= -\frac{(\mu z_1)^n}{(w+\mu)^{n-1}} \frac{1}{q\mu z_1+p(w+\mu)z_2} \frac{1}{(w+\mu)(1-pz_2)-q\mu z_1}.
\end{aligned} \tag{52}$$

Substituting (49) and (51) into (47) yields

$$\mathbb{E}\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big] = \big((w+\mu)(1-pz_2)-q\mu z_1\big)\left(R+f(\delta)+\frac{g(\delta)I(\delta)}{h(\delta)}\right),$$

where $I(\delta)$ is given in (39), which completes the proof. □

Remark 4 For the G/M/1/K queue, note that Rosenlund [14] obtained the four-variate transform of B_1, S_1, L_1, and the busy cycle defined as the time duration between two consecutive arrivals to an empty system. Restricting Rosenlund's result to the PH/M/1/K queue, Proposition 4 extends his result in two ways. First, it gives the four-variate joint transform of B_n, S_n, L_n^c, and L_n^o, for the case when $n \geq 1$. Secondly, it allows the dropping of customers even when the queue is not full. Note that in the particular case with $n = 1$ and $p = 1 - q = 0$, we have $f(\delta) = 0$, $g(\delta) = 1$, and $R = -1/(w + \mu(1 - z_1))$. Inserting these values into the joint transform in Proposition 4 yields

$$\mathbb{E}\big[e^{-wB_1} z_1^{S_1} z_2^{L_1^c} z_3^{L_1^o}\big] = \frac{\mu z_1 \sum_{i=1}^{M_a+1} \frac{1-z_3 o_i}{o_i^K} \frac{1-\phi(w+\mu(1-z_1 o_i))}{w+\mu(1-z_1 o_i)} \frac{Q_\phi(\theta_i)}{\prod_{l=1,l\neq i}^{M_a+1} \theta_l - \theta_i}}{\sum_{i=1}^{M_a+1} \frac{1}{o_i^K} \frac{Q_\phi(\theta_i)}{\prod_{l=1,l\neq i}^{M_a+1} \theta_l - \theta_i}}$$

$$= \frac{\int_{D_\delta} \frac{\mu z_1 (1-z_3 x)}{x^K} \frac{1-\phi(w+\mu(1-z_1 x))}{w+\mu(1-z_1 x)} \frac{dx}{x-\phi(w+\mu(1-z_1 x))}}{\int_{D_\delta} \frac{1-z_3 x}{x^K} \frac{dx}{x-\phi(w+\mu(1-z_1 x))}}.$$

We note that the last equation is in agreement with (11) in [14].

3 Discussion: non-distinct roots

Until now we have assumed that the roots in Lemmas 1, 2 and 3 are distinct. We shall now relax these assumptions and show that the results in Propositions 2, 3 and 4 still hold. In the following, we shall focus on extending the result in Proposition 2. This can be done similarly for Proposition 3 and 4.

Let us consider that $r_{i+l} = r_i + l\epsilon$, $\epsilon > 0$, $i \in \{1, \dots, M_s + 1\}$ and $l = 0, \dots, L - 1$, and take the limit in our final result for $\epsilon \to 0$. This means that r_i is a root of multiplicity L. In order to show that the results in Proposition 2 hold in this case, it is readily seen that one must prove that

$$\begin{aligned} &\mathrm{Res}_{r_i} \frac{1}{x^{K-1}} \frac{1}{qx + pz_2 - z_3} \frac{1}{x - z_1 \xi(w + \lambda(1 - qx - pz_2))} \\ &\quad = \lim_{\epsilon \to 0} \sum_{l=0}^{L-1} \frac{1}{r_{i+l}^{K-1}} \frac{1}{q r_{i+l} + pz_2 - z_3} \frac{Q_\xi(\rho_{i+l})}{\prod_{j=1, j\neq i+l}^{M_s+1} (\rho_{i+l} - \rho_j)}. \end{aligned} \tag{53}$$

First, note that when r_i is a root of multiplicity L, the complex residue reads

$$\begin{aligned} &\mathrm{Res}_{r_i} \frac{1}{x^{K-1}} \frac{1}{qx + pz_2 - z_3} \frac{1}{x - z_1 \xi(w + \lambda(1 - qx - pz_2))} \\ &= \frac{1}{(L-1)!} \frac{d^{L-1}}{dx^{L-1}} \left(\frac{1}{x^{K-1}(qx + pz_2 - z_3)} \frac{(x - r_i)^L}{x - z_1 \xi(w + \lambda(1 - qx - pz_2))} \right) \Bigg|_{x = r_i} \\ &= \frac{1}{(-\lambda q)^{L-1}(L-1)!} \frac{d^{L-1}}{dx^{L-1}} \end{aligned}$$

$$\times \left(\frac{1}{x^{K-1}(qx + pz_2 - z_3)} \frac{Q_\xi(\rho)}{\prod_{j=1, j\neq i,\ldots,i+L-1}^{M_s+1}(\rho - \rho_j)} \right)\Bigg|_{x=r_i}$$

$$= \frac{1}{(-\lambda q)^{L-1}(L-1)!} \lim_{\epsilon\to 0} \frac{1}{\epsilon^{L-1}} \sum_{l=1}^{L-1} \left(\frac{\binom{L-1}{l}(-1)^{L-1-l}}{(r_i + l\epsilon)^{K-1}(q(r_i + l\epsilon) + pz_2 - z_3)} \right.$$

$$\left. \times \frac{Q_\xi(\rho_i - \lambda q l \epsilon)}{\prod_{j=1, j\neq i,\ldots,i+L-1}^{M_s+1}(\rho_i - \lambda q l\epsilon - \rho_j)} \right), \tag{54}$$

where $\rho = w + \lambda(1 - qx - pz_2)$, $\rho_i = w + \lambda(1 - qr_i - pz_2)$, and the last equality follows from the following identity for the analytical function $f(x)$ around x_0:

$$\frac{d^n}{dx^n} f(x)\bigg|_{x_0} = \lim_{\epsilon\to 0} \frac{1}{\epsilon^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(x_0 + i\epsilon).$$

Note that the latter equation follows right away using the Taylor series of $f(x_0 + i\epsilon)$ around x_0 and the binomial series of $(x-1)^n$ and its derivatives.

The r.h.s. of (53) can be rewritten as

$$\lim_{\epsilon\to 0} \sum_{l=0}^{L-1} \frac{1}{r_{i+l}^{K-1}} \frac{1}{qr_{i+l} + pz_2 - z_3} \frac{Q_\xi(\rho_{i+l})}{\prod_{j=1, j\neq i+l}^{M_s+1}(\rho_{i+l} - \rho_j)}$$

$$= \lim_{\epsilon\to 0} \sum_{l=0}^{L-1} \frac{1}{(r_i + l\epsilon)^{K-1}} \frac{1}{q(r_i + l\epsilon) + pz_2 - z_3} \frac{Q_\xi(\rho_i - \lambda q l\epsilon)}{\prod_{j=1, j\neq i+l}^{M_s+1}(\rho_i - \lambda q l \epsilon - \rho_j)}, \tag{55}$$

where

$$\frac{Q_\xi(\rho_i + l\epsilon_0)}{\prod_{j=1, j\neq i+l}^{M_s+1}(\rho_i + l\epsilon_0 - \rho_j)} = \frac{(-1)^{L-1-l} Q_\xi(\rho_i + l\epsilon_0)}{\epsilon_0^{L-1} l!(L-1-l)! \prod_{j=1, l\neq 0,\ldots,L-1}^{M_s+1}(\rho_i + l\epsilon_0 - \rho_j)}$$

$$= \frac{\binom{L-1}{l}}{(L-1)!} \frac{(-1)^{L-1-l} Q_\xi(\rho_i + l\epsilon_0)}{\epsilon_0^{L-1} \prod_{j=1, l\neq 0,\ldots,L-1}^{M_s+1}(\rho_i + l\epsilon_0 - \rho_j)},$$

with $\epsilon_0 = -\lambda q \epsilon$. Inserting the last equation into (55) shows that the r.h.s. and l.h.s. of (53) are equal, which completes the proof.

Acknowledgements We first thank the anonymous reviewer for the valuable comments which helped to improve the presentation of the paper. The author would like to thank Prof. Onno Boxma for stimulating discussions during the paper's preparation when the author held a postdoctoral position at EURANDOM Institute, Eindhoven. The author thanks De Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO) for their financial support. In the Netherlands, the three universities of technology have formed the 3TU.Federation. This article is the result of joint research in the 3TU.Centre of Competence NIRICT (Netherlands Institute for Research on ICT).

Appendix A

In this appendix, we give the definition and the structure of some key matrices that we shall refer to frequently.

The matrix $\mathbf{A}$ is a K-by-K upper bidiagonal block matrix with i-th upper diagonal element equal to $q_i(F_i^o f_i)\otimes\mathbf{I}$ and i-th diagonal element equal to $\mathbf{F}_i\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{S}_i$, i.e.,

$$\mathbf{A}=\begin{pmatrix}\mathbf{F}_1\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{S}_1 & q_1(F_1^o f_1)\otimes\mathbf{I} & \mathbf{0} & \dots & \dots & \mathbf{0}\\ \mathbf{0} & \mathbf{F}_2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{S}_2 & q_2(F_2^o f_2)\otimes\mathbf{I} & \mathbf{0} & \dots & \mathbf{0}\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ & & & & & q_{K-1}(F_{K-1}^o f_{K-1})\otimes\mathbf{I}\\ \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{F}_K\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{S}_K\end{pmatrix}.$$

The matrix $\mathbf{B}$ is a K-by-K lower diagonal matrix with i-th lower diagonal element equal to $\mathbf{I}\otimes(S_i^o s_i)$. Therefore, $\mathbf{B}$ has the following canonical form:

$$\mathbf{B}=\begin{pmatrix}\mathbf{0} & \dots & \dots & \dots & \mathbf{0}\\ \mathbf{I}\otimes(S_2^o s_2) & \mathbf{0} & & \dots & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\otimes(S_3^o s_3) & \mathbf{0} & \dots & \vdots\\ \vdots & & \ddots & & \vdots\\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}\otimes(S_K^o s_K) & \mathbf{0}\end{pmatrix}.$$

The matrix $\mathbf{C}$ is a K-by-K diagonal matrix with i-th diagonal element, $i=1,\dots,K-1$, equal to $p_i(F_i^o f_i)\otimes\mathbf{I}$ and K-th element equal to $\mathbf{0}$, i.e.,

$$\mathbf{C}=\begin{pmatrix}p_1(F_1^o f_1)\otimes\mathbf{I} & \mathbf{0} & \dots & \dots & \mathbf{0}\\ \mathbf{0} & p_2(F_2^o f_2)\otimes\mathbf{I} & \mathbf{0} & \dots & \vdots\\ \vdots & & \ddots & & \vdots\\ \mathbf{0} & \dots & \mathbf{0} & p_{K-1}(F_{K-1}^o f_{K-1})\otimes\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \dots & & \mathbf{0} & \mathbf{0}\end{pmatrix}.$$

The matrix $\mathbf{D}$ is a K-by-K zero block matrix with (K,K)-block element equal to $(F_K^o f_K)\otimes\mathbf{I}$. Therefore, $\mathbf{D}$ has the following canonical form:

$$\mathbf{D}=\begin{pmatrix}\mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \vdots & & \vdots & \vdots\\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \cdots & \mathbf{0} & (F_K^o f_K)\otimes\mathbf{I}\end{pmatrix}.$$

The matrix $\mathbf{U}_1$ is an infinite size block diagonal matrix with diagonal blocks equal to $\mathbf{C}$, i.e.,

$$\mathbf{U}_1=\begin{pmatrix}\mathbf{C} & \mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{C} & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \ddots & \ddots & \ddots & \vdots\end{pmatrix}.$$

The matrix $\mathbf{U}_{00}$ is an infinite size block diagonal matrix with diagonal blocks equal to $\mathbf{D}$. Therefore, $\mathbf{U}_{00}$ has the following canonical form:

$$\mathbf{U}_{00} = \begin{pmatrix} \mathbf{D} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}.$$

The matrix $\mathbf{U}_0$ is an infinite size block diagonal matrix with diagonal blocks equal to $\mathbf{U}_{00}$, i.e.,

$$\mathbf{U}_0 = \begin{pmatrix} \mathbf{U}_{00} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{00} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}.$$

Appendix B

The matrix $\mathbf{M}(i,1)$ is an M_s-by-M_s matrix with j-th row equal to $(\xi^1(\rho_j),\ldots,\xi^{M_s}(\rho_j))$ for $j=1,\ldots,M_s+1$ and $j\neq i$. Therefore, $\mathbf{M}(i,1)$ has the following canonical form:

$$\mathbf{M}(i,1) = \begin{pmatrix} \xi^1(\rho_1) & \cdots & \xi^{M_s}(\rho_1) \\ \vdots & \vdots & \vdots \\ \xi^1(\rho_{i-1}) & \cdots & \xi^{M_s}(\rho_{i-1}) \\ \xi^1(\rho_{i+1}) & \cdots & \xi^{M_s}(\rho_{i+1}) \\ \vdots & \vdots & \vdots \\ \xi^1(\rho_{M_s+1}) & \cdots & \xi^{M_s}(\rho_{M_s+1}) \end{pmatrix}.$$

Recall that $\xi(\rho)=s(\rho\mathbf{I}-\mathbf{S})^{-1}S^o$ and $\xi^i(\rho)=e_i(\rho\mathbf{I}-\mathbf{S})^{-1}S^o$. Moreover, $\xi(\rho)$ is a linear combination of $\xi^1(\rho),\ldots,\xi^{Ms}(\rho)$, and it is a rational function with denominator, $Q_\xi(\rho)$, of degree equal to M_s and numerator of degree $<M_s$. In addition, $\xi^i(\rho)$, $i=1,\ldots,M_s$, are also rational functions with denominator of degree $\leq M_s$ and numerator of degree smaller than the denominator. Therefore, it is easily seen that $\xi^i(\rho)=P^i(\rho)/Q_\xi(\rho)$, $i=1,\ldots,M_s$, where $Q_\xi(\rho)$ is the denominator of $\xi(\rho)$. Let $\mathbf{C}$ denote the matrix with (j,i)-entry equal to the coefficient of ρ^{j-1} of the polynomial $P^i(\rho)$.

Lemma 4 *The determinant of* $\mathbf{M}(i,1)$ *is given by*

$$\det\bigl(\mathbf{M}(i,1)\bigr) = C(-1)^{i-1}\frac{\prod_{j=1}^{M_s}\prod_{k=j+1}^{M_s+1}(\rho_k-\rho_j)}{\prod_{j=1}^{M_s+1}Q_\xi(\rho_j)}\frac{Q_\xi(\rho_i)}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_j-\rho_i)}, \tag{56}$$

where $C=\det(\mathbf{C})$.

Proof We decompose $\mathbf{M}(i,1)$ as follows:

$$\mathbf{M}(i,1)=\mathbf{D}\cdot\mathbf{V}(i)\cdot\mathbf{C},$$

where $\mathbf{D}$ is the diagonal matrix with j-th diagonal entry equal to $1/Q_\xi(\rho_j)$, $j=1,\dots,M_s$ and $j\neq i$, $\mathbf{V}(i)$ is the Vandermonde matrix of the following form:

$$\mathbf{V}(i)=\begin{pmatrix}1 & \rho_1 & \dots & (\rho_1)^{M_s}\\ \vdots & & & \vdots\\ 1 & \rho_{i-1} & \dots & (\rho_{i-1})^{M_s}\\ 1 & \rho_{i+1} & \dots & (\rho_{i+1})^{M_s}\\ \vdots & & & \vdots\\ 1 & \rho_{M_s+1} & \dots & (\rho_{M_s+1})^{M_s}\end{pmatrix}.$$

Note that the determinant of $\mathbf{M}(i,1)$ reads

$$\det\bigl(\mathbf{M}(i,1)\bigr)=\det(\mathbf{D})\det\bigl(\mathbf{V}(i)\bigr)\det(\mathbf{C}).$$

It is well known that the determinant of the Vandermonde matrix is given by, see, for example, [5],

$$\det\bigl(\mathbf{V}(i)\bigr)=\prod_{j=1,j\neq i}^{M_s}\prod_{k=j+1,k\neq i}^{M_s+1}(\rho_k-\rho_j)=(-1)^{i-1}\frac{\prod_{j=1}^{M_s}\prod_{k=j+1}^{M_s+1}(\rho_k-\rho_j)}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_j-\rho_i)}.$$

Since $\mathbf{D}$ is the diagonal matrix with j-th diagonal entry equal to $1/Q_\xi(\rho_j)$, $j=1,\dots,M_s$ and $j\neq i$, it is readily seen that

$$\det(\mathbf{D})=\frac{1}{\prod_{j=1,j\neq i}^{M_s+1}Q_\xi(\rho_j)}.$$

Substituting the latter two equations into $\det(\mathbf{M}(i,1))$ immediately yields (56), which completes the proof. □

References

1. Abramov, V.: On a property of refusal systems. J. Appl. Probab. **34**, 800–805 (1997)
2. Al Hanbali, A., Boxma, O.: Busy period analysis of the state dependent M/M/1/K queue. Oper. Res. Lett. **38**(1), 1–6 (2010)
3. Bernstein, D.S.: Matrix Mathematics. Princeton University Press, Princeton (2005)
4. Bertsimas, D., Daisuke, N.: Transient and busy period analysis of the GI/G/1 queue: the method of stages. Queueing Syst. **10**(3), 153–184 (1992)
5. Buck, M., Coley, R., Robbins, D.: A generalized Vandermonde determinant. J. Algebr. Comb. **1**, 105–109 (1992)
6. Cohen, J.W.: The Single Server Queue. North-Holland, Amsterdam (1982)
7. Jain, R., Smith, J.: Modeling vehicular traffic flow using M/G/C/C state dependent queuing models. Transp. Sci. **31**(4), 324 (1997)
8. Lee, T.: M/G/1/K queue with vacation time and exhaustive service discipline. Oper. Res. **32**(4), 774–784 (1984)

9. Macon, N., Spitzbart, A.: Inverses of Vandermonde matrices. Am. Math. Mon. **65**(2), 95–100 (1958)
10. Neuts, M.: Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach. Johns Hopkins University Press, Baltimore (1981)
11. Ramaswami, V.: The N/G/1 queue and its detailed analysis. Adv. Appl. Probab. **12**(1), 222–261 (1980)
12. Righter, R.: A note on losses in M/G/1/N queues. J. Appl. Probab. **36**, 1240–1243 (1999)
13. Rosenlund, S.I.: Busy periods in time-dependent M/G/1 queue. Adv. Appl. Probab. **8**, 195–208 (1976)
14. Rosenlund, S.I.: The queue G/M/m/N: busy period and cycle, waiting time under reverse and random order service. Nav. Res. Logist. Q. **25**(1), 107–119 (1978)
15. Takács, L.: Introduction to the Theory of Queues. Oxford University Press, London (1962)
16. Takagi, H., Lamaire, R.: On the busy period of an M/G/1/K queue. Oper. Res. **42**(1), 192–193 (1994)
17. Wolff, R.: Losses per cycle in a single-server queue. J. Appl. Probab. **39**, 905–909 (2002)