

Replication of flexi-swaps

Ingmar Evers and Farshid Jamshidian describe a relatively new product known as a flexi-swap and discuss its application in securitisation. A flexi-swap gives a counterparty an option to amortise the interest rate swap at an accelerated pace. They show that it can be replicated semi-statically by a vanilla amortising swap plus a portfolio of Bermudan swaptions. The derivation employs a novel 'high-low interest rate path' argument, which results in a simple algebraic formula for the weights

A flexible swap, or 'flexi-swap', is a fixed versus floating swap that gives the fixed-side payer the option to amortise the swap in any way, so long as the notional is non-increasing and remains within specified lower and upper bands. Our purpose is to show that this swap can be replicated by a portfolio of an amortising swap and several receiver Bermudan swaptions.

Swaps effectively transfer risk from one party to the next. A borrower or lender may use swaps to convert floating payments to fixed, or vice versa, in order to hedge his particular interest rate risks. In the context of securitisations, the issuer may enter into a fixed-for-floating swap to mitigate the impact of increased borrower default due to higher debt service on the loans following interest rate rises. For the hedge to be effective, clearly the notional and maturity of the swap must be chosen to agree with those of the loans. This is trivial for loans with fixed notionals, or even for loans with (pre)specified amortisation schedules, requiring plain vanilla and amortising swaps respectively.

For loans with an uncertain notional typical of structured finance and securitisation, flexible swaps provide the suitable hedge. These mortgage- and asset-backed loans frequently amortise with prepayments and defaults on the assets side as a result of economic pressures or in order to maintain the credit quality of the structure. Whenever any loan repayments are triggered, a corresponding adjustment to the flexi-swap notional ensures that residual interest rate risks are mitigated. (See Pfister, 2002, for further details on the role of flexi-swaps in (European) securitisations.)

This article describes how the flexi-swap may be decomposed into receiver Bermudan swaptions and a single ordinary amortising swap for pricing purposes. A critical assumption in our approach is that the fixed-side payer chooses to decrease or maintain the notional of the swap in such a way that the value of the swap is maximised at every exercise date. In other words, the exercise strategy is driven by the underlying (interest) rate only, in contrast with the amortisation of the loan, which may be driven by collateral quality. Although additional factors may influence the exercise strategy, these factors do not enter into the flexi-swap contract and should therefore not influence its price. We call this an optimal exercise of the flexi-swap.

This notion of optimal exercise of a flexi-swap is more complex than that of an American-style or a Bermudan option. The latter can be exercised only once. By contrast, there is no such restriction on a flexible swap, as long as the notional stays in the band. For example, in year three, the fixed payer of a flexi-swap may decide to amortise the maximum allowed amount, thus bringing the notional down to the lower band. Then, for the next two years he may choose not to amortise any, thus keeping the notional constant, followed in year six, say, by another amortisation. As such, there can be several optimal exercises within the lifetime of the trade. Another interesting and at first surprising feature of a flexi-swap is that it may be optimal to amortise part of, but not all of, the maximum allowed amount, as discussed later.

The arguments given here do not actually prove that the optimal exercise policy of the proposed replicating portfolio coincides with the optimal amortisation policy of the flexi-swap in all scenarios. Although we suggest this is the case, and our arguments make this plausible, its rigorous mathematical formulation is beyond the scope of this article. Our ar-

guments only establish the replication for certain scenarios, those we call 'high-low' yield-curve paths. This does imply the uniqueness of the replicating portfolio. Namely, it implies if there is a replicating portfolio consisting of an amortising swap and Bermudan swaptions, then it is necessarily given by our formula.

Moreover, the proposed replicating portfolio emulates the principal feature of a flexi-swap: it enables the fixed-side payer to maintain the notional at each period within the specified band, at a level of his choice. This implies that the swaption portfolio provides at least as much optionality as the flexi-swap. Conversely, our earlier observation that optimal exercise of the flexi-swap may involve partial amortisation indicates that the flexi-swap contains as much optionality as the swaption portfolio.

The result presented here does not rely on any model for the underlying interest rates, but merely identifies an equivalence between the flexi-swap and other tradable instruments. Any interest rate model of choice capable of pricing Bermudan swaptions (see Brigo & Mercurio, 2001, for example, or references therein) can therefore be used, together with the replication formula, to price the flexi-swap. Accordingly, the replication does not apply to interest rate swaps only. The same arguments also hold for other categories, such as inflation swaps.

The replication of the flexi-swap is a perfect semi-static hedge in that it involves option exercise at discrete periods given by the possible swap amortisation dates. For similar replication methods, see, for example, Carr, Ellis & Gupta (1998) or Joshi (2001).

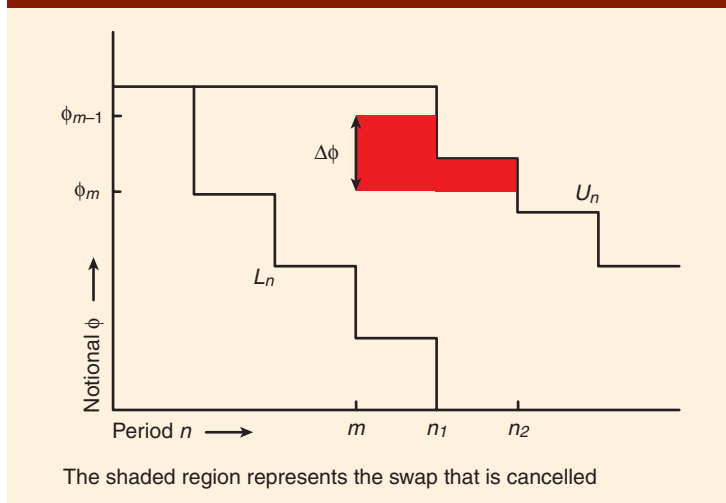
The following section illustrates the relevance of our analysis by discussing some applications, including a specific example of a recent flexi-swap transaction typical of those that the authors have encountered. We also point out that the optionality in a flexi-swap is very similar to the voluntary redemption option in sinking fund bonds. Next we will more precisely define the flexi-swap and provide the terminology with which the replication can be derived. The following section outlines our approach, which involves formulating equations for the notional amounts of the Bermudan swaptions and subsequently solving those equations. Finally, a numerical example of the replication is given.

Relevance

□ **Specific example.** The following transaction demonstrates the relevance of the flexi-swap to amortising loans. It concerns a €55 million credit facility for the financing of a number of large office buildings. The 10½-year loan and collateral are assigned to a special-purpose vehicle by a commercial real estate investor. The swap for this credit facility is one where the borrower pays fixed, and it specifies a notional schedule, as shown in the Appendix. Over the first three years, the notional is a straightforward decreasing amount, but over the remaining seven and a half years the notional is specified by a lower and an upper level, both also decreasing over time. The term sheet has a prepayment clause specifying that the client has the right to unwind the swap up to the lower level of the notional schedule subject to prepayment in the underlying credit facility – an event at the discretion of the borrower. This combination characterises the flexi-swap as considered in this article.

For the borrower (the real estate investor), these terms are attractive if

1. Decreasing the flexi-swap notional from ϕ_{m-1} to ϕ_m at period m



it is considered likely that one or several of the buildings may be sold before loan maturity, as is the case for this particular transaction. Any costs resulting from the premature termination of an ordinary swap, notably termination penalties, are avoided. In addition, specifying the upper and lower notional bounds is cheaper than a Bermudan swaption on the entire notional amount.

□ **Sinking fund bonds.** Structures similar to flexi-swaps have long been present in sinking-fund bonds. A vanilla sinking fund is a bond that obliges the issuer to redeem prescribed portions of the issue at given redemption dates. This is a direct counterpart of an amortising swap. More complex sinking fund bonds, however, give the issuer an option, called the voluntary option, to redeem at each redemption date more than (often double and sometimes triple) the mandatory redemption amount. When the voluntary option is present, there is often also a 'leftover option' clause that gives the issuer the right at any redemption date to substitute a prior voluntary redemption for the mandatory redemption. (See Jamshidian & Russell, 1990, for an analysis and pricing of these and other sinking-fund options.)

The similarity between flexi-swaps and complex sinking-fund bonds is evident. In both cases, there is a decreasing upper bound for the outstanding notional (or principal). Both structures give the fixed payer the option to amortise (or redeem) at an accelerated pace. Similar to the flexi-swap, the 'leftover option' enables the issuers to keep the outstanding principal constant (that is, not to redeem) at times when an accelerated redemption has been exercised previously. Both structures therefore allow the outstanding principal to remain constant, provided it is at or below the upper notional line.

The only difference between the two structures is in the treatment of the lower notional line. In a sinking fund, the amount that can be voluntarily redeemed is known in advance (twice or triple the mandatory amount). But, in a flexi-swap the maximum amortisable amount is the difference between the current outstanding notional and the lower notional bound. This difference is not known in advance. Rather, it depends on the extent to which the option to amortise has been previously exercised.

□ **Other applications.** In addition to amortising loans and securitisations thereof, flexi-swaps can also be applied to fix asset and project finance loans. Here, the contract enables the borrower to increase the notional of the swap, mirroring increased loans for possible additional investment. In this case, the 'accreting flexi-swap' enables the holder to increase the notional of the swap between pre-specified increasing notional schedules. This variation has an application in asset or project finance, in which loans may grow over time with increasing investments.

The analysis in this article assumes decreasing notionals bounded by two amortisation schedules. The corresponding replication is in terms of

an amortising swap and a series of Bermudan-style receiver swaptions. For the increasing bounds the replication involves an accreting swap and a series of Bermudan-style payer swaptions.

The results put forward here apply as long as the borrower is free to change the swap notional between the specified bounds at his own discretion. However, the contract may specify deal-specific conditions under which the swap can be decreased (or increased), such as mortgage prepayments, legal documentation to indicate newly acquired collateral for the loan or the completion of a project phase. In such cases, deal-specific analysis of the underlying loan is required to price the flexi-swap, reducing the swap's price compared with the price assuming optimal exercise.

However, although the exercise of the flexi-swap optionality is in practice closely linked to the underlying assets, the swap itself will frequently not include conditions specific to those underlying assets. The swap will probably not be exercised optimally, but because there are no asset-specific clauses in the term sheet the pricing must assume optimal exercise.

Definition of the flexi-swap

Let T_1, \dots, T_N be the swap payments dates. The accrual periods need not be regular, but for simplicity we assume each period to be a year, half a year, a quarter or a month. We label the periods $n = 0$ to $n = N$, so today corresponds to the start of period 0 and the swap matures at period N .

Let N_0 denote the initial notional.

Let L_n denote the lower band notional at period n , $1 \leq n \leq N-1$. (So, L_n corresponds to the higher amortisation rate.) We use the convention that L_n is the notional applicable on the interval between period n and period $n+1$. So, payment at period $n+1$ is proportional to L_n .

Let U_n denote the upper band notional at period n , $1 \leq n \leq N-1$. (So, U_n corresponds to the lower amortisation rate.) We use the convention that U_n is the notional applicable on the interval between period n and period $n+1$. So, payment at period $n+1$ is proportional to U_n .

It is assumed that both L_n and U_n are non-negative and decreasing, and that $L_n \leq U_n$. In practice L_n decreases faster than U_n , that is, $L_{n-1} - L_n > U_{n-1} - U_n$. The special case that $U_n = N_0$ for all n (that is, zero amortisation for upper notional) is also allowed.

A precise definition of the flexi-swap is as follows:

- The notional of the flexible swap at period n must lie (inclusively) between L_n and U_n .
- The notional of the flexible swap at period n must be less than or equal to the notional at the previous period $n-1$.
- The party paying fixed has the option at the start of each period n to choose the notional, subject to the two conditions above.

The fixed-rate payer (the option holder) will amortise as much as allowed if interest rates are very low, and will amortise as little as allowed if interest rates are very high.

□ **Replicating portfolio.** We show that the fixed payer can replicate the flexible swap by the following 'semi-static' trading strategy:

- An amortising swap, paying fixed, with amortisation schedule given by the upper notional sequence U_n .
- A portfolio of receiver Bermudan swaptions, with first exercise at period i , maturity at period j and with notional B_{ij} , where $1 \leq i < j \leq N$.

Our goal is to find enough equations that uniquely determine the swap notionals B_{ij} . Underlying this approach is the premise that the above instruments are indeed sufficient for a perfect hedge, in which case we need only determine their linear combination using an appropriate set of interest-rate paths.

Figure 1 makes the existence of the replication plausible. Intuitively, if we take the lower bound L_n to be represented by an amortising swap, we can expect the difference between the two bounds to consist of Bermudan swaptions. The figure indicates that the amortisation by any amount $\Delta\phi$ can be regarded as the exercise of a set of Bermudan swaptions, in this case one that matures on n_1 and one that matures on n_2 . In heuristic terms, any exercise is effectively the exercise of some Bermudan swaption, supporting our premise.

A similar argument states that all valid exercise paths for the flexi-swap

can be reproduced by the replicating portfolio, and vice versa. Arguably it follows that the replication is complete and that the optimal paths will also agree, but again a mathematically more rigorous proof is required to substantiate this conclusion.

Both the flexi-swap and the replicating portfolio admit fractional exercise, and indeed it will sometimes be optimal to exercise only part of the optionality. This is clear for the replicating portfolio as Bermudan swaptions of different maturities need not be optimally exercised at the same time. To see why partial amortisation may be optimal for the flexible swap, consider the following example. Suppose that interest rates have been high in the first two years, and no optional amortisation has taken place. So, the notional has been at the upper bound. Say in year three, rates become moderately low. The fixed payer knows that if he does not optionally amortise in year three, then he must do a mandatory amortisation in year four anyway, so as to keep the notional at the upper bound. It may well be optimal to amortise in year three this year four mandatory amortisation amount, while it may be not optimal to amortise in year three the mandatory portions that must be amortised in year five or later years.

High-low interest rate paths

Consider a flexi-swap with maturity at period N and a fixed-rate coupon K . We call a path of yield curves a high-low path, if at each period n either (i) interest rates are so low that all Bermudan receiver swaptions of coupon K and maturity $\leq N$ will be exercised, and the flexi-swap will be fully amortised, or (ii) interest rates are so high that no such receiver Bermudans will be exercised, and the flexible swap notional will be kept constant at the previous period level, or at the upper notional, whichever is less.

So, a high-low path oscillates at extremes. There are 2^{N-1} such high-low patterns, of which we consider $N(N-1)/2$. We assume that each such pattern is possible (that is, it has positive measure). This assumption is satisfied in practically all interest rate models.

By comparing the behaviour of a flexible swap and the replicating portfolio along such high-low paths, we will find enough equations that uniquely solve the notionals B_{ij} of the replicating portfolio.

The equations for Bermudan notionals

□ Period n constraint for a yield-curve path for which interest rates are low at period n but are high at all previous exercise dates.

Because of the given yield-curve path, we know that prior to period n no Bermudans receivers were exercised, and at period n all the available Bermudans that can be exercised are in fact exercised. These are all the Bermudans with start date at or before n and maturity greater than n . Their total notional is therefore $\sum_{i=1}^n \sum_{j=n+1}^N B_{ij}$.

So, looking at the replicating portfolio at period n , we have the notional U_n of the amortising swap, and we can exercise exactly $\sum_{i=1}^n \sum_{j=n+1}^N B_{ij}$ Bermudans. By exercising these Bermudans, the notional is effectively reduced to $U_n - \sum_{i=1}^n \sum_{j=n+1}^N B_{ij}$.

Since we wish this portfolio to replicate the flexi-swap, the notional should be reduced to that of the flexi-swap on this path. But, for such a path, the flexi-swap notional at period n will be L_n because the fixed payer amortises as much as possible at period n . Therefore, replication requires $U_n - \sum_{i=1}^n \sum_{j=n+1}^N B_{ij} = L_n$, or, equivalently:

$$\sum_{i=1}^n \sum_{j=n+1}^N B_{ij} = U_n - L_n, \quad 1 \leq n \leq N-1$$

□ Period n constraint for a yield-curve path such that for some $1 \leq m \leq n-1$, period m interest rates are low but in the subsequent periods $m+1, \dots, n$, interest rates are high.

Given this yield-curve path, we know that at period m all the available Bermudan receivers that could be exercised were in fact exercised. These are all the Bermudans with start date at or before m and maturity greater than m .

Now consider once more the situation at period n . Of the Bermudans that were exercised, those Bermudans with maturity $\leq n$ do not affect the

notional for period n . Only those with a maturity greater than n bring down the notional. The net notionals of these Bermudans is $\sum_{i=1}^m \sum_{j=n+1}^N B_{ij}$.

Again, since we want this portfolio to replicate the flexi-swap, the notional should be effectively reduced to that of a flexi-swap on this path. But, for such a path, the flexi-swap notional at period m will be L_m because the fixed payer amortises as much as possible at period m . From there on, the fixed payer keeps the notional constant, subject to not exceeding the upper boundary. So, on this path the notional of the flexi-swap at period n will be $\min(L_m, U_n)$. Therefore, replication requires that $U_n - \sum_{i=1}^m \sum_{j=n+1}^N B_{ij} = \min(L_m, U_n)$, or, equivalently:

$$\sum_{i=1}^m \sum_{j=n+1}^N B_{ij} = (U_n - L_m)^+, \quad 1 \leq m \leq n \leq N-1$$

The argument here is valid for $m < n$, but it also holds for $m = n$ by the previous formula, since $(U_n - L_n)^+ = U_n - L_n$ (because $U_n \geq L_n$ as assumed).

The above argument using high-low yield-curve paths led to $N(N-1)/2$ equations, precisely the same as the number of unknowns B_{ij} . This is despite the fact that there are 2^{N-1} possible high-low yield-curve scenarios. The reason is that many of the scenarios result in the same equation. For example, the high-low pattern (H, L, L, H, H) and (H, H, L, H, H) result in the same equation for the exercise strategy period $n = 5$ (subsequent to rates having changed from low to high at $m = 3$).

The solution

The above system of linear equations has a unique solution B_{ij} . Furthermore, the solution is non-negative, that is, $B_{ij} \geq 0$.

To solve the equation, set:

$$a_{mn} := (U_n - L_m)^+, \quad 1 \leq m \leq n \leq N-1$$

Note, $a_{nn} = U_n - L_n$. The system of equations we are trying to solve is:

$$\sum_{i=1}^m \sum_{j=n+1}^N B_{ij} = a_{mn}, \quad 1 \leq m \leq n \leq N-1 \quad (1)$$

To solve this system, set $n = N-1$ to get:

$$\sum_{i=1}^m B_{iN} = a_{m,N-1}, \quad 1 \leq m \leq N-1$$

From this we obtain:

$$B_{1N} = a_{1,N-1}, \quad B_{mN} = a_{m,N-1} - a_{m-1,N-1}, \quad 2 \leq m \leq N-1$$

Next, for $n \leq N-2$, subtract from equation (1) the same equation with n replaced by $n+1$. We get:

$$\sum_{i=1}^m B_{i,n+1} = a_{m,n} - a_{m,n+1}, \quad 1 \leq m \leq n \leq N-2$$

Solving this, we find that:

$$B_{1n} = a_{1,n-1} - a_{1n}, \quad 2 \leq n \leq N-1$$

$$B_{mn} = a_{m,n-1} - a_{mn} - a_{m-1,n-1} + a_{m-1,n}, \quad 2 \leq m < n \leq N-1$$

From these equations, one can show that in all circumstances $B_{ij} \geq 0$ for all i, j . In practice, most of the B_{ij} will be zero.

We can express the solution more conveniently if we adopt the following extended notation. Let us set $a_{0,k} = a_{k,N} = 0$ for all $0 \leq k \leq N$. Then the last equation also covers the previous equations. That is:

$$B_{mn} = a_{m,n-1} - a_{mn} - a_{m-1,n-1} + a_{m-1,n}, \quad 1 \leq m < n \leq N \quad (2)$$

□ **Simplified solution in special cases.** For the special case that $L_1 \leq U_{N-1}$, we have $U_n \geq L_m$ for all m and n . In this case, the solution above simplifies to $B_{ij} = 0$ for all i and j except the following:

$$B_{1N} = U_{N-1} - L_1$$

$$B_{nN} = L_{n-1} - L_n \quad \text{for } 1 \leq n \leq N-1$$

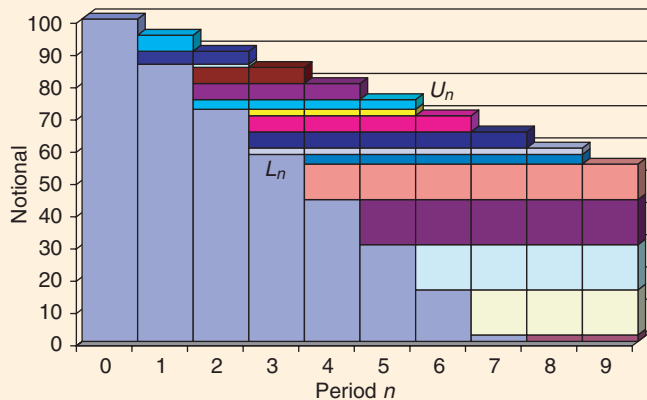
$$B_{1,n} = U_{n-1} - U_n \quad \text{for } 2 \leq n < N$$

A. Bermudan swaption notionalals B_{mn}

B_{mn}	2	3	4	5	6	7	8	9	10
1	5	4	-	-	-	-	-	-	-
2	-	1	5	5	3	-	-	-	-
3	-	-	-	-	2	5	5	2	-
4	-	-	-	-	-	-	-	3	11
5	-	-	-	-	-	-	-	-	14
6	-	-	-	-	-	-	-	-	14
7	-	-	-	-	-	-	-	-	14
8	-	-	-	-	-	-	-	-	2
9	-	-	-	-	-	-	-	-	-

Note: rows correspond to (Bermudan) start periods m and columns correspond to end periods n . In this example, there are 16 Bermudans with non-zero notional

2. Flexi-swap components – sample replication



The upper band U_n represents the notionals of the amortising swap over time. The horizontal bars separating the upper and lower bands represent those of the Bermudans

In the further special case that the lower amortisation rate is zero (that is, when the upper notionals are flat and all U_n are equal), then $B_{1,n} = 0$ also, for all $2 \leq n < N$.

Let us also consider the trivial special case that $U_n = L_n$ for all periods n , or in other words that the amortisation is predetermined. It is quickly confirmed that for all relevant $n \geq m + 1$ all terms on the right-hand side of equation (2) are zero due to both bounds being equal and decreasing. As expected, there are no Bermudan swaptions with non-zero notional.

Example result

We consider a swap with initial notional $N_0 = 100$ and with a maturity of $N = 10$ periods. The terms of this swap stipulate minimum and maximum amortisation rates of 5% and 14% of the initial notional per period. Consistent with our earlier definition, the fixed-side payer is allowed to specify any swap notional no less than $L_n = N_0(1 - 0.14n)^+$ and no greater than $U_n = N_0(1 - 0.05n)^+$ at every period $1 \leq n \leq 9$, with the restriction that those notionals may not increase over time.

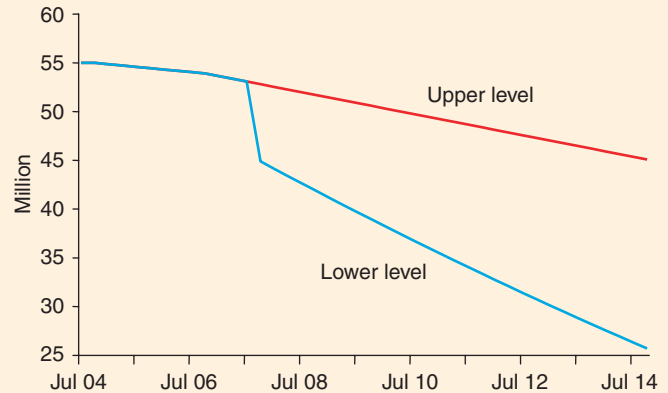
Using the above results we can now express this flexi-swap as an equivalent portfolio consisting of an ordinary amortising swap with notional amounts U_n and a set of Bermudan swaptions with notionals given by equation (2). These Bermudan notionals are summarised for this numerical example in table A. We find that there are 16 Bermudans with non-zero notional.

Note that for any swaption notional B_{mn} , the start (and first exercise date) is at period m , and the maturity (and final payment date) is at period n .

The same flexi-swap is also depicted in figure 2. It illustrates that the

Appendix

The figure below displays the notional schedule as specified in the term sheet of a recent flexi-swap transaction.



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'notional area' between the upper and lower bands U_n and L_n is effectively composed of the Bermudan swaptions stacked in order of start and end dates, providing an intuitive conceptual representation of the replication.

Conclusion

In conclusion, equation (2) is a concise and straightforward expression for the replicating Bermudan swaptions. The flexi-swap price is obtained by pricing these swaptions along with the specified amortising swap.

Clearly, the replication formula also provides a perfect hedge for the flexi-swap, which is of practical use particularly if the number of replicating swaptions is moderate. Depending on the upper and lower amortisation bands, though, the number of Bermudans may be too large for this hedge to be practically feasible. In that case, any hedging strategy applicable to portfolios of Bermudan swaptions themselves may be extended to the flexi-swap using our replication formula.

The flexi-swap gives us an example of a portfolio of Bermudan swaptions, which in aggregate differs qualitatively from a single Bermudan swaption. The portfolio of swaptions in this case results in a structure that exhibits more flexibility than ordinary options in terms of the number of times optimal exercise may occur and the notional amounts that will be exercised. The example of flexi-swap suggests that appropriate combinations of swaptions of different notionals and different start and end dates may lead to structures with surprising characteristics that can meet particular investment aims. ■

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