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Pancyclicity of claw-free hamiltonian graphs

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Abstract

A graph G on n vertices is called *subpancyclic* if it contains cycles of every length k with $3 \leqslant k \leqslant c(G)$, where $c(G)$ denotes the length of a longest cycle in G; if moreover $c(G)=n$, then G is called *pancyclic*. By a result of Flandrin et al. a claw-free graph (on at least 35 vertices) with minimum degree at least $\frac{1}{3}(n-2)$ is pancyclic. This degree bound is best possible. We prove that for a claw-free graph to be subpancyclic we only need the degree condition $\delta > \sqrt{3n+1}-2$. Again, this degree bound is best possible. It follows directly that under the same condition a hamiltonian claw-free graph is pancyclic.

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1. Introduction

We use [6] for terminology and notation not defined here and consider finite simple graphs only.

Let $G=(V,E)$ be a graph with vertex set V and edge set E. We usually write n for $|V|$. By $N(v)$ we denote the set of neighbors of a vertex $v \in V$. The *neighborhood* of v is the subgraph of G induced by $N(v)$. The graph G is *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. If $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by the vertices of S.

A graph G is called *pancyclic* if it contains cycles of every length k with $3 \leqslant k \leqslant n$, and is called *subpancyclic* if it contains cycles of every length k with $3 \leqslant k \leqslant c(G)$, where $c(G)$ denotes the *circumference* of G, i.e. the length of a longest cycle in G. A graph is called *traceable* if it contains a Hamilton path.

In [5] Bondy proved the following result.

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Theorem 1. *Let G be a graph on at least three vertices. If* $\delta \geqslant \frac{n}{2}$, *then G is pancyclic or isomorphic to* $K_{\frac{n}{2},\frac{n}{2}}$.

This result is best possible. For example the graph $K_{\lceil\frac{n}{2}-1\rceil,\lfloor\frac{n}{2}+1\rfloor}$ with minimum degree $\lceil\frac{n}{2}-1\rceil$ is nonhamiltonian, and hence is not pancyclic.

It appeared that most of the strength of known conditions for graphs to be pancyclic is necessary to guarantee the existence of a Hamilton cycle. Mitchem and Schmeichel [11] suggested that the bound in Theorem 1 might be lowered if one makes the assumption that G is hamiltonian. In [1] Amar et al. proved the following result.

Theorem 2. *Let G be a hamiltonian graph on at least* 102 *vertices. If* $\delta \geqslant \frac{1}{5}(2n+1)$, *then G is pancyclic or bipartite, and the bound is best possible.*

Here we will prove a similar result for claw-free graphs. In [9] Flandrin et al. proved the following result.

Theorem 3. *Let G be a* 2*-connected claw-free graph on at least* 35 *vertices. If* $\delta \geqslant \frac{1}{3}(n-2)$, *then G is pancyclic.*

We will prove the following theorem.

Theorem 4. *Let G be a claw-free graph on at least* 5 *vertices. If* $\delta > \sqrt{3n+1}-2$, *then G is subpancyclic.*

Theorem 4 is best possible, as can be seen from the following family of graphs. For any integer $p \geqslant 2$, we define the graph G_p as follows. Let $H_1, \ldots, H_p$ be p disjoint copies of K_{3p-2}, and $u_i v_i$ an edge of H_i $(i=1,\ldots,p)$. Now G_p is obtained from $\bigcup_{i=1}^{p} H_i - u_i v_i$ by adding the edges $v_1u_2, v_2u_3, \ldots, v_{p-1}u_p$ and $v_p u_1$.

The graph G_p is both hamiltonian and claw-free. Furthermore, we have $\delta(G_p) = 3p-3$ and $|V(G_p)| = p(3p-2)$, implying that $\delta(G_p) = \sqrt{3n+1}-2$. It is obvious that G_p does not contain C_{3p-1} and hence G_p is not (sub)pancyclic.

As an immediate consequence of Theorem 4, we obtain the following result.

Corollary 5. *Let G be a hamiltonian claw-free graph on at least* 5 *vertices. If* $\delta > \sqrt{3n+1}-2$, *then G is pancyclic.*

Since Matthews and Sumner proved in [10] that every 2-connected non-hamiltonian claw-free graph contains a cycle of length at least $2\delta+4$, we also have the following corollary.

Corollary 6. *Let G be a* 2*-connected claw-free graph on at least* 5 *vertices. If* $\delta > \sqrt{3n+1}-2$, *then G contains cycles of every length l with* $3 \leqslant l \leqslant \min\{2\delta+4, n\}$.

Theorem 4 immediately follows from the following two theorems, which will be proved in the next two sections.

Theorem 7. *Let G be a claw-free graph. If* $\delta > \max\{2, \sqrt{2n+\frac{1}{4}}-\frac{3}{2}\}$, *then G contains* C_l *for every integer l with* $3 \leqslant l \leqslant \delta+1$.

Theorem 8. *Let G be a claw-free graph on at least* 5 *vertices. If* $\delta > \sqrt{3n+1}-2$ *and* $c(G) \geqslant \delta+3$, *then G contains* C_l *for every integer l with* $\delta+2 \leqslant l \leqslant c(G)$.

The condition $\delta > \sqrt{3n+1}-2$ is best possible in Theorem 8, as shown by the graphs G_p introduced after the statement of Theorem 4.

We do not believe that the condition $\delta > \max\{2, \sqrt{2n+\frac{1}{4}}-\frac{3}{2}\}$ is best possible in Theorem 7, but did not try to relax it, since it is already more than we need to prove Theorem 4. On the other hand, the upper bound $\delta+1$ is sharp, as shown by, again, the graphs G_p.

It is well known that line graphs are claw-free. Results on line graphs related to Theorem 4 appear in [4, 12]. For a trail T of a graph G, define $\imath(T)$ as the number of edges incident with at least one vertex of T. A general result on cycle lengths in line graphs was established by Broersma [8].

Theorem 9. *The line graph* $L(G)$ *of a graph G contains a cycle of length k if and only if* $\Delta(G) \geqslant k$ *or G contains a nontrivial closed trail T such that* $|E(T)| \leqslant k \leqslant \imath(T)$.

Before quoting a result from [12] we define, for a graph G with at least one edge,

$$\overline{\sigma}_2(G) = \min\{d(u)+d(v) \mid uv \in E(G)\}.$$

Theorem 10. *Let G be a graph on at least* 450 *vertices. If* $\overline{\sigma}_2(G) > \sqrt{2n+\frac{1}{4}}+\frac{1}{2}$, *then* $L(G)$ *is subpancyclic.*

Theorem 10 implies that Theorem 4 admits improvement for (large) line graphs.

Corollary 11. *Let G be a line graph on at least* 100 577 *vertices. If* $\delta > \sqrt{2n+\frac{1}{4}}-\frac{3}{2}$, *then G is subpancyclic.*

Proof. Assume $G = L(H)$. We may assume that G and H are connected. Since $|E(H)| = |V(G)| > \binom{449}{2}$, we have $|V(H)| \geqslant 450$. If H is a tree, then $c(G) = \Delta(H)$ and G is subpancyclic by Theorem 9. If H is not a tree, then

$$\begin{aligned}\overline{\sigma}_2(H) &= \delta(G)+2 > \sqrt{2|V(G)|+\tfrac{1}{4}}+\tfrac{1}{2} = \sqrt{2|E(H)|+\tfrac{1}{4}}+\tfrac{1}{2}\\ &\geqslant \sqrt{2|V(H)|+\tfrac{1}{4}}+\tfrac{1}{2},\end{aligned}$$

so G is subpancyclic by Theorem 10. □

Corollary 11 is close to best possible as shown by the graphs G'_p ($p \geqslant 3$) defined as follows. Let $H'_1, \ldots, H'_p$ be p disjoint copies of K_{2p-2}, and u'_i, v'_i two vertices of H'_i ($i = 1, \ldots, p$). Now G'_p is obtained from $\bigcup_{i=1}^{p} H'_i$ by adding the edges $v'_1 u'_2, v'_2 u'_3, \ldots, v'_{p-1} u'_p$ and $v'_p u'_1$.

The graph G'_p is a hamiltonian line graph. Furthermore, we have $\delta(G'_p) = 2p - 3$ and $|V(G'_p)| = p(2p-2)$, implying that $\delta(G'_p) = \sqrt{2n+1} - 2$. Obviously, G'_p does not contain C_{2p-1} and hence G'_p is not (sub)pancyclic.

Corollary 11 is related to Theorem 7, as will be exhibited in Section 2.

2. Proof of Theorem 7

We start with three lemmas.

Lemma 12 (Flandrin, Fournier and Germa [9]). *Let G be a claw-free graph. Then the neighborhood $N = G[N(x)]$ of any vertex $x \in V(G)$ has independence number $\alpha(N) \leqslant 2$. Let $\kappa = \kappa(N)$ be the connectivity of N. Then each of the following holds.*

1. *If $\kappa \geqslant 2$, then N is hamiltonian.*
2. *If $\kappa = 1$, then there exists a partition of $V(N)$ into the vertex sets of two complete subgraphs G_1 and G_2, the edges between G_1 and G_2 having a common end in one of these subgraphs.*
3. *If $\kappa = 0$, then N is the disjoint union of two complete subgraphs G_1 and G_2, or $N \cong K_1$.*

Lemma 13. *If G is a claw-free graph such that the neighborhood of every vertex is the disjoint union of two complete subgraphs, then G is a line graph.*

Proof. If G satisfies the hypothesis of the lemma, then G does not contain $K_4 - e$ as an induced subgraph. The result thus follows directly from Beineke's characterization of line graphs in terms of forbidden induced subgraphs in [2,3]. □

For n sufficiently large, the next lemma follows from Corollary 11. Yet we give an independent proof, in order to cover small values of n also.

Lemma 14. *Let G be a line graph. If $\delta > \max\{2, \sqrt{2n + \frac{1}{4}} - \frac{3}{2}\}$, then G contains C_l for every integer l with $3 \leqslant l \leqslant \delta + 1$.*

Proof. Assume $G = L(H)$. Then

$$(1) \quad \bar{\sigma}_2(H) = \delta(G) + 2 > \max\{4, \sqrt{2|E(H)| + \tfrac{1}{4}} + \tfrac{1}{2}\}.$$

If $\delta(H) = 1$, then by 1, $\Delta(H) \geqslant \delta(G) + 1$ and we are done by Theorem 9. Thus we may assume

$$\delta(H) \geqslant 2.$$

Let C be a shortest cycle of H. For an edge uv of C we have

$$l(C)\geqslant d(u)+d(v)-1\geqslant\overline{\sigma}_2(H)-1=\delta(G)+1.$$

Thus by Theorem 9, G contains C_l for every l with $|V(C)|\leqslant l\leqslant\delta(G)+1$. Also by Theorem 9, G contains C_l for every l with $3\leqslant l\leqslant\Delta(H)$. Thus we are done if $|V(C)|\leqslant\Delta(H)+1$. Henceforth assume

$$(2)\quad |V(C)|\geqslant\Delta(H)+2.$$

Suppose $\Delta(H)=3$. Then $\overline{\sigma}_2(H)\leqslant 6$. If $\overline{\sigma}_2(H)=6$, then H is 3-regular and by (1), $|E(H)|<15$, implying that $|V(H)|<10$; it is straightforward to check that H then has girth at most 4, contradicting (2). If $\overline{\sigma}_2(H)=5$, we obtain a contradiction in a similar way. It follows that $\Delta(H)\geqslant 4$ and, by (2), $|V(C)|\geqslant 6$.

Let u_0v_0 be an edge of H, and $u_1,\ldots,u_p$ the vertices in $N(u_0)\cup N(v_0)$. Since H has girth at least 6, the subgraph $H[\{u_0,v_0\}\cup N(u_0)\cup N(v_0)]$ is a tree. In particular,

$$p\geqslant d(u_0)+d(v_0)-2\geqslant\overline{\sigma}_2(H)-2.$$

Since $\delta(H)\geqslant 2, u_i$ has a neighbor v_i outside $\{u_0,v_0\}\cup N(u_0)\cup N(v_0)$ $(i=1,\ldots,p)$. From the fact that H has girth at least 6 we deduce that $v_i\neq v_j$ whenever $i\neq j$. In $\sum_{i=0}^{p}(d(u_i)+d(v_i))$, every edge of H is counted at most twice, so

$$2|E(H)|\geqslant\sum_{i=0}^{p}(d(u_i)+d(v_i))\geqslant(p+1)\overline{\sigma}_2(H)\geqslant(\overline{\sigma}_2(H)-1)\overline{\sigma}_2(H).$$

This contradiction with (1) completes the proof. □

Proof of Theorem 7. Let G be a claw-free graph with $\delta>\max\{2,\sqrt{2n+\frac{1}{4}}-\frac{3}{2}\}$. We are clearly done if G contains a vertex whose neighborhood is traceable. In the opposite case Lemma 13 applies by Lemma 12, and we are done by Lemma 14. □

3. Proof of Theorem 8

We start by giving some additional terminology and notation and some lemmas.

Let C be a cycle of a graph G and $u,v\in V(C)$. We denote by $\overrightarrow{C}$ the cycle C with a given orientation. By $u\overrightarrow{C}v$ we denote the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices in reverse order are denoted by $v\overleftarrow{C}u$. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. By u^{+i} and u^{-i} we denote the i-th successor and i-th predecessor of u on $\overrightarrow{C}$, respectively. Similar notation is used for paths.

A cycle C of a graph G is *reducible* if there exists a cycle C' such that $V(C')\subseteq V(C)$ and $|V(C')|=|V(C)|-1$. A cycle is *irreducible* if it is not reducible. A cycle C of a graph G is *k-extendable* if there exists a cycle C' such that $V(C)\subseteq V(C')$ and $|V(C')|=|V(C)|+k$.

A *chord* of a cycle C is an edge uv with $u,v\in V(C)$ and $uv\notin E(C)$.

Lemma 15. *Let G be a claw-free graph with $\delta>\sqrt{3n+1}-2$. If C is a cycle in G that has no chord, then $|V(C)|<\frac{4}{3}(\delta+2)$.*

Proof. Let G satisfy the hypothesis of the lemma and let C be a cycle without chords in G. Since C is a cycle, we have $n\geqslant 3$. By using $\delta>\sqrt{3n+1}-2$ we find that $\delta\geqslant 2$ and thus $|V(C)|<\frac{4}{3}(\delta+2)$ is true if $|V(C)|\leqslant 5$. So assume $|V(C)|>5$.

Since G is claw-free, C has no chords and $|V(C)|>5$, any vertex $v\in V(G)\backslash V(C)$ is adjacent to at most four vertices of C. Let V_i be the set of vertices in $V(G)\backslash V(C)$ that are adjacent to exactly i vertices of C $(i=1,2,3,4)$. Then

$$\sum_{v\in V(C)}(d(v)-2)=|V_1|+2|V_2|+3|V_3|+4|V_4|\leqslant 4(|V_1|+|V_2|+|V_3|+|V_4|).$$

Hence

$$\begin{aligned}n&\geqslant |V(C)|+|V_1|+|V_2|+|V_3|+|V_4|\geqslant |V(C)|+\frac{1}{4}\sum_{v\in V(C)}(d(v)-2)\\&\geqslant |V(C)|+\frac{1}{4}|V(C)|(\delta-2)=\frac{1}{4}|V(C)|(\delta+2).\end{aligned}$$

It follows that

$$\begin{aligned}|V(C)|&\leqslant\frac{4n}{\delta+2}<\frac{4n}{\sqrt{3n+1}}=\frac{4n\sqrt{3n+1}}{3n+1}<\frac{4n\sqrt{3n+1}}{3n}\\&=\frac{4}{3}\sqrt{3n+1}<\frac{4}{3}(\delta+2).\qquad\square\end{aligned}$$

Lemma 16. *Let G be a claw-free graph with $\delta>\sqrt{3n+1}-2$. If C is a cycle in G that has no chord and $|V(C)|\geqslant\delta+3$, then there exists a vertex $x\in V(G)\backslash V(C)$ that is adjacent to four vertices of C.*

Proof. Let G satisfy the hypothesis of the lemma and let C be a cycle without chords in G with $|V(C)|\geqslant\delta+3$. Assume that there is no vertex $x\in V(G)\backslash V(C)$ that is adjacent to four vertices of C. Let V_i be the set of vertices in $V(G)\backslash V(C)$ that are adjacent to exactly i vertices of C $(i=1,2,3)$. Since C has no chords, we have

$$\sum_{v\in V(C)}(d(v)-2)=|V_1|+2|V_2|+3|V_3|\leqslant 3(|V_1|+|V_2|+|V_3|).$$

Hence

$$\begin{aligned}n&\geqslant |V(C)|+|V_1|+|V_2|+|V_3|\geqslant |V(C)|+\frac{1}{3}\sum_{v\in V(C)}(d(v)-2)\\&\geqslant |V(C)|+\frac{1}{3}|V(C)|(\delta-2)=\frac{1}{3}|V(C)|(\delta+1)\\&\geqslant\frac{1}{3}(\delta+3)(\delta+1)>\frac{1}{3}(\sqrt{3n+1}+1)(\sqrt{3n+1}-1)=n,\end{aligned}$$

a contradiction. $\square$

The proofs of the following two lemmas are straightforward.

Lemma 17. *Let a and b be two vertices of a graph G, P an (a,b)-path in G, Q a path in G such that* $V(P)\cap V(Q)=\emptyset$, *and S a subset of V(Q) such that every vertex in S is adjacent to two consecutive vertices of P. Then there exists an (a,b)-path R in G such that* $V(P)\cup S\subseteq V(R)\subseteq V(P)\cup V(Q)$.

Lemma 18. *Let G be a claw-free graph and C be an irreducible cycle in G. If uv is a chord of C, then* $uv^+\in E(G)$ *or* $uv^-\in E(G)$, *and hence u has two consecutive neighbors on C.*

The proof of the following lemma is obtained by using arguments from [7].

Lemma 19. *Let* $\overrightarrow{C}$ *be an irreducible cycle in a claw-free graph G and xy a chord of C such that the cycle* $x\overrightarrow{C}yx$ *is chordless. Then* $x^+\overrightarrow{C}y^-$ *contains a vertex v such that* $|N(v)\cap V(C)|=2$.

Proof. Assume $|N(v)\cap V(C)|\geqslant 3$ for every vertex $v\in x^+\overrightarrow{C}y^-$. By Lemma 18 and the fact that $x\overrightarrow{C}yx$ is chordless, $xy^-\in E(G)$ and every vertex of $x^+\overrightarrow{C}y^-$ has two consecutive neighbors on $y^+\overrightarrow{C}x$. Hence by Lemma 17, there exists a (y^-,x)-path R with $V(R)=y^+\overrightarrow{C}x\cup x^+\overrightarrow{C}y^-=V(C)\backslash\{y\}$. But then $y^-\overrightarrow{R}xy^+$ contradicts the fact that C is irreducible. □

Lemma 20. *Let* $\overrightarrow{C}$ *be a cycle in a claw-free graph G with minimum degree* $\delta\geqslant 3$. *If there is a vertex* $v\in V(C)$ *such that* $|N(v)\cap V(C)|=2$ *and* $v^-v^+\notin E(G)$, *then C is k-extendable for every integer k with* $0\leqslant k\leqslant\delta-2$.

Proof. Let $\overrightarrow{C}$ be a cycle in a claw-free graph G with $\delta\geqslant 3$, let v be a vertex that has only two neighbors on C, the vertices v^- and v^+, and assume $v^-v^+\notin E(G)$. Let k be an integer with $0\leqslant k\leqslant\delta-2$. Set $N=G[N(v)]$. Then $|V(N)\backslash V(C)|\geqslant\delta-2$. We distinguish two cases.

Case 1: $\kappa(N)=0$ or $\kappa(N)=1$.

By Lemma 12, $V(N)$ can be partitioned into the vertex sets of two complete subgraphs G_1 and G_2. As $v^-v^+\notin E(G)$, we may assume without loss of generality that $v^-\in V(G_1)$ and $v^+\in V(G_2)$. Now we can extend the cycle C with k vertices by inserting sufficiently many vertices of G_1 between v^- and v, and/or sufficiently many vertices of G_2 between v and v^+.

Case 2: $\kappa(N)\geqslant 2$.

By Lemma 12, N is hamiltonian. Let $\overrightarrow{D}$ be a Hamilton cycle in N. For every vertex $x\in V(N)$ we denote by $(x)^{D^-}$ the predecessor of x on $\overrightarrow{D}$. Now we can extend the cycle C with k vertices by inserting sufficiently many vertices of $v^-\overrightarrow{D}(v^+)^{D^-}$ between v^- and v, and/or sufficiently many vertices of $(v^-)^{D^-}\overleftarrow{D}v^+$ between v and v^+. □

Proof of Theorem 8. Let G satisfy the hypothesis of the theorem and assume G does not contain cycles of every length l with $\delta+2\leqslant l\leqslant c(G)$. Set $m=\max\{i<c(G)\mid G$ does not contain $C_i\}$. Then $\delta+2\leqslant m\leqslant c(G)-1$ and G contains C_{m+1}. Let $\overrightarrow{C}$ be a cycle of length $m+1$. We distinguish two cases.

Case 1: C does not have a chord.

By Lemma 16, there exists a vertex $v\in V(G)\setminus V(C)$ that is adjacent to four vertices a,b,c,d of C. Since G is claw-free we may assume that $b=a^{+}$, $c=a^{+k}$ and $d=a^{+k+1}$ for some k with $2\leqslant k\leqslant\frac{1}{2}(m+1)$.

By Lemma 15, $m+1<\frac{4}{3}(\delta+2)$. Since $\delta\geqslant 3$, it follows that $k\leqslant\delta$. By Lemma 12, there exist (a,d)-paths of every length l with $2\leqslant l\leqslant\delta$ using only vertices of $v\cup N(v)$. Let P be such an (a,d)-path of length k. Then the cycle $a\overrightarrow{P}d\overrightarrow{C}a$ has length m, a contradiction.

Case 2: C has a chord.

Case 2.1: $\delta=3$.

Since $\delta=3$, we have $n\leqslant 7$. Thus, $|V(C)|=6$ or $|V(C)|=7$. If $|V(C)|=7$, C is a Hamilton cycle. By Lemma 19, $\delta=2$, a contradiction.

Now assume that $|V(C)|=6$. Let ab be a chord of C. Then $b=a^{+3}$, otherwise C is reducible. Using Lemma 18 we find that $ab^{+}\in E(G)$ or $ab^{-}\in E(G)$. This gives the contradiction that C is reducible.

Case 2.2: $\delta>3$.

Let ab be a chord of C such that $|a\overrightarrow{C}b|$ is minimal. By Lemma 19 there exists a vertex $v\in a^{+}\overrightarrow{C}b^{-}$ that is adjacent to only two vertices of C, v^{-} and v^{+}.

Consider the cycle $D=b\overrightarrow{C}ab$. By Lemma 18, $ab^{+}\in E(G)$. In particular, D has a chord. Let cd be a chord of D such that c precedes d on $b\overrightarrow{C}a$ and $|c\overrightarrow{C}d|$ is minimal. By Lemma 15, $|c\overrightarrow{C}d|<\frac{4}{3}(\delta+2)$.

Set $L=\{x\in c^{+}\overrightarrow{C}d^{-}\mid |N(x)\cap V(C)|=2\}$.

Claim 1. $|L|\geqslant\delta$.

Proof. Assume that $|L|\leqslant\delta-1$. Noting that $cd^{+}\in E(G)$ by Lemma 18, we apply Lemma 17 with $Q=c^{+}\overrightarrow{C}d^{-}$, $S=V(Q)\setminus L$ and either $P=d\overrightarrow{C}c$ or $P=d^{+}\overrightarrow{C}c$ to obtain a cycle F with $|V(C)|-\delta+1\leqslant|V(F)|\leqslant|V(C)|-1$. By Lemma 20, since $v\in V(F)$, F can be extended to cycles of any length k with $|V(F)|\leqslant k\leqslant|V(F)|+\delta-2$. Thus G contains a cycle of length $|V(C)|-1$, a contradiction. □

We assign indices to the elements of L, according to their order of occurrence on the path $c^{+}\overrightarrow{C}d^{-}$. Thus the ith element of L is denoted by x_i. Consider the vertices $x_1,x_4\in L$. Since $|L|\geqslant\delta\geqslant 4$ we have

$$|x_1\overrightarrow{C}x_4|\leqslant|c\overrightarrow{C}d|-2-(\delta-4)<\tfrac{4}{3}(\delta+2)-\delta+2\leqslant\delta+2.$$

Claim 2. $N(x_1)\cap N(x_4)=\emptyset$.

Proof. Assuming the contrary, let $p \in N(x_1) \cap N(x_4)$. By the definition of x_1 and x_4, $p \notin V(C)$. Also, $p \notin N(v)$, otherwise $G[\{p, v, x_1, x_4\}] \cong K_{1,3}$. Let C' be the cycle $x_4\overrightarrow{C}x_1 p x_4$. Then

$$|V(C)| - 1 \geqslant |V(C')| = |V(C)| - |x_1\overrightarrow{C}x_4| + 3 > |V(C)| - \delta + 1.$$

Since $v \in V(C')$, it follows from Lemma 20 that C' can be extended to a cycle of length $|V(C)| - 1$, a contradiction. □

Since $a\overrightarrow{C}ba$ is a cycle without chords and C is irreducible, by Lemma 15, we have

$$2 \leqslant |a^- \overrightarrow{C} b^-| < \tfrac{4}{3}(\delta + 2) - 2 = \tfrac{4}{3}\delta + \tfrac{2}{3}.$$

Then for the length of the cycle D, it follows that

$$\lceil |V(C)| - \tfrac{4}{3}\delta - \tfrac{1}{3} \rceil \leqslant |V(D)| \leqslant |V(C)| - 2.$$

Since $N(x_1) \cap N(x_4) = \emptyset$ we can apply Lemma 20 twice to obtain cycles of any length k with

$$|V(D)| \leqslant k \leqslant |V(D)| + 2\delta - 4.$$

Since $\delta \geqslant 4$, we have

$$|V(D)| + 2\delta - 4 \geqslant \lceil |V(C)| - \tfrac{4}{3}\delta - \tfrac{1}{3} \rceil + 2\delta - 4 = \lceil |V(C)| + \tfrac{2}{3}\delta - \tfrac{13}{3} \rceil \geqslant |V(C)| - 1$$

and hence G contains a cycle of length $|V(C)| - 1$, our final contradiction. □

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