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THE PERMUTAHEDRON OF SERIES-PARALLEL POSETS

Annelie VON ARNIM
Institut für Operations Research, Universität Bonn, D-5300 Bonn 1, FRG

Ulrich FAIGLE
Faculty of Applied Mathematics, University of Twente, Enschede, Netherlands

Rainer SCHRADER
Institut für Operations Research, Universität Bonn, D-5300 Bonn 1, FRG

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We give a linear description of the permutahedron of series-parallel posets and characterize the facets.

1. Introduction

Schoute (1911) introduced the permutahedron on an n-element set $N=\{1,\dots,n\}$ as follows. With any permutation π of N we associate an *incidence vector* $x(\pi)=(\pi(1),\dots,\pi(n))\in\mathbb{R}^n$. The *permutahedron* is the polytope

$$\mathrm{Perm}(N)=\mathrm{conv}\{x(\pi)\colon \pi \text{ is a permutation of } N\}.$$

Independently, several authors (cf., e.g., Rado [4], Balas [1], Gaiha and Gupta [2], Young [6]) studied the permutahedron and derived a characterization of Perm(N) via the following linear inequalities

$$\begin{aligned} x(S) &\ge f(S), \quad S\subseteq N,\\ x(N) &= f(N), \end{aligned} \tag{1.1}$$

where $f(S)=\binom{|S|+1}{2}$.

Here we consider the more general problem on a partially ordered set $P=(N,\le)$ over an n-element ground set. The permutahedron of P now is restricted to the subset of permutations which are linear extensions of P, i.e.,

$$\mathrm{Perm}(P)=\mathrm{conv}\{x(\pi)\colon \pi \text{ is a linear extension of } P\}.$$

Given a linear function $c: N\to\mathbb{R}$, the linear programming problem

$$\max\{cx\colon x\in\mathrm{Perm}(P)\} \tag{1.2}$$

is equivalent to the following one-machine scheduling problem: For a set N of jobs with individual processing times c_i, $i\in N$, find a schedule on one machine consistent with precedence relations P among the jobs so that the average completion time is minimized. A schedule is a linear extension π of P and the completion time C_j of job j is its own processing time plus the sum over all processing times of jobs processed before it ($C_j = \sum_{i:\ \pi(i) \leq \pi(j)} c_i$). So in order to solve the scheduling problem, we have to minimize $\sum C_i$ or equivalently $\sum c_i(n+1-\pi(i))$ for all linear extensions.

We are interested in a linear description of Perm(P). The scheduling problem is NP-complete for arbitrary posets but polynomially solvable in the case of series-parallel posets (Sidney [5], see also Lawler [3]). So in general there is not much hope to obtain a full description of the permutahedron in terms of linear inequalities. In Section 3 we describe two classes of valid inequalities and show that they completely describe the permutahedron of series-parallel posets. Section 4 characterizes the facets among them.

2. Preliminaries

Let $P=(N, \leq)$ be a finite partially ordered set (poset) with $|P|=n$. An element x is *maximal* (*minimal*) if $x \leq y$ ($x \geq y$) implies $x=y$. We say that a subset $I \subseteq P$ is an *ideal* if $x \in I$ and $y \leq x$, then also $y \in I$. A *convex set* $C \subseteq P$ is a subset such that $x, y \in C$ and $x \leq z \leq y$ implies $z \in P$.

A *linear extension* L of P is an ordering $L = n_1 n_2 \ldots n_n$ of the ground set such that n_i occurs before n_j whenever $n_i \leq n_j$ or equivalent a permutation π of N s.t. $\pi(i) <' \pi(j)$ if $i \leq j$. Notice that $<'$ refers to the standard ordering on $\mathbb{N}$ and $\leq$ to the partial ordering on N.

A poset is *series-parallel* if it can be constructed inductively from single elements by repeated application of the following two operations to series-parallel posets $P_1=(N_1, \leq_1)$ and $P_2=(N_2, \leq_2)$.

(2.1) *Parallel composition*

$$P = P_1 \parallel P_2,$$

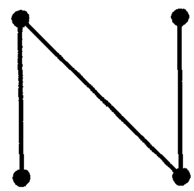
$$x \leq y \quad \text{if} \begin{cases} x \leq_1 y \text{ and } x, y \in N_1, \\ x \leq_2 y \text{ and } x, y \in N_2; \end{cases}$$

(2.2) *Series composition*

$$P = P_1 \oplus P_2,$$

$$x \leq y \quad \text{if} \begin{cases} x \leq_1 y \text{ and } x, y \in N_1, \\ x \leq_2 y \text{ and } x, y \in N_2, \\ x \in N_1,\ y \in N_2. \end{cases}$$

Equivalently, a poset is series-parallel if no four elements induce the following suborder:



If a poset can be written as the series composition of two nonempty posets, then it is *series-reducible* and *prime* otherwise.

3. The permutahedron of series-parallel posets

Let $P=(N, \leq)$ be an arbitrary poset, x an incidence vector of a linear extension of P and $I \subseteq P$ an ideal. A direct generalization of the restrictions (1.1) are the *ideal constraints*

$$\begin{aligned} x(I) &\geq f(I), \\ x(P) &= f(P). \end{aligned} \tag{3.1}$$

Secondly, consider a series-reducible convex set $C=A \oplus B$. Series-reducible convex sets induce valid inequalities in the following way. Let x be an incidence vector of a linear extension of P and k be the largest entry in x_i, $i \in A$. Then $x(A) \leq \sum_{i=0}^{|A|-1} k-i$ and $x(B) \geq \sum_{i=1}^{|B|} k+i$. If we combine these two, we obtain for all series-reducible convex sets $C=A \oplus B$ the *convex set constraints*

$$|A|\, x(B) - |B|\, x(A) \geq \tfrac{1}{2}\, |A|\,|B|(|A|+|B|). \tag{3.2}$$

Let $\mathrm{Perm}'(P)$ be the polytope defined by the ideal constraints (3.1) and the convex set constraints (3.2). Clearly, $\mathrm{Perm}(P) \subseteq \mathrm{Perm}'(P)$. We will show in this section that $\mathrm{Perm}(P) = \mathrm{Perm}'(P)$ if P is series-parallel. For this purpose we need two preparatory lemmas.

For a given vector $x \in \mathbb{R}^n$ we say that an ideal is *tight* if the corresponding ideal constraint holds with equality.

Lemma 3.1. *Let* $x \in \mathrm{Perm}'(P)$ *and* I *be a tight ideal with* $|I|=k$. *Then* $y=(x_i\colon i \in I) \in \mathrm{Perm}'(I)$ *and* $z=(x_i-k\colon i \in P \setminus I) \in \mathrm{Perm}'(P \setminus I)$.

Proof. It is easily verified that the vectors y and z both satisfy the convex set constraints. For every ideal $J \subseteq I$, $y(J)=x(J) \geq f(J)$. Now consider some ideal $J \subseteq P \setminus I$. Since $I \cup J$ is an ideal of P we get

$$z(J) = x(J) - |I|\,|J| \geq f(I \cup J) - x(I) - |I|\,|J| = f(J). \qquad \square$$

Lemma 3.2. *Let P be a series-parallel poset and x an extreme point of* $\mathrm{Perm}'(P)$. *Then there exists a tight proper subideal of P.*

Proof. Since P has a series-parallel decomposition we distinguish two cases.

(i) $P=P_1 \oplus P_2$. Then $P_1 \oplus P_2$ is a series-reducible convex set. Hence $|P_1|\, x(P_2) - |P_2|\, x(P_1) \geq \frac{1}{2}|P_1||P_2||P|$. Since $x(P)=x(P_1)+x(P_2)=f(P)$, we get $x(P_1)\leq f(P_1)$. This together with the ideal inequality for P_1 implies that P_1 is a tight ideal.

(ii) $P=P_1 \parallel P_2$. Suppose that no proper subideal of P is tight. Let $\varepsilon=\min\{x(I)-f(I)\colon I \text{ is a proper subideal of } P\}$. Choose a vector $c\in\mathbb{R}^n$ such that x is the unique optimal solution for $\min\{cx\colon x\in\mathrm{Perm}'(P)\}$. We may assume that $c(P_2)\,|P_1|\leq c(P_1)\,|P_2|$. Let $x'\in\mathbb{R}^n$ be given by

$$x_i' = \begin{cases} x_i + \dfrac{\varepsilon}{|P_2|}, & \text{if } i\in P_2,\\[2ex] x_i - \dfrac{\varepsilon}{|P_1|}, & \text{if } i\in P_1. \end{cases}$$

We claim that $x'\in\mathrm{Perm}'(P)$. The convex set constraints are obviously satisfied since any series-reducible convex set C is contained either in P_1 or P_2 and the convex set constraints are invariant if we add the same constant to every component in C.

For any proper ideal I we have

$$x'(I) = \sum_{I\cap P_1}\left(x_i - \frac{\varepsilon}{|P_1|}\right) + \sum_{I\cap P_2}\left(x_i + \frac{\varepsilon}{|P_2|}\right) \geq x(I)-\varepsilon \geq f(I).$$

Hence $x'\in\mathrm{Perm}'(P)$ and $cx'\leq cx+\varepsilon(c(P_2)/|P_2|-c(P_1)/|P_1|)\leq cx$, contradicting our assumption. □

We are now ready to give the linear characterization of the permutahedron of series-parallel posets.

Theorem 3.3. *Let P be a series-parallel poset. Then* $\mathrm{Perm}(P)=\mathrm{Perm}'(P)$.

Proof. We already know that $\mathrm{Perm}(P)\subseteq\mathrm{Perm}'(P)$. For the converse we perform induction on $|P|$. Let x be a vertex of $\mathrm{Perm}'(P)$ and $c\in\mathbb{R}^n$ such that x is the unique optimal solution to $\min\{cx\colon x\in\mathrm{Perm}'(P)\}$. By Lemma 3.2 there exists a tight proper subideal I of P. Then for the restricted vectors $y=(x_i\colon i\in I)$ and $z=(x_i-|I|\colon i\in P\setminus I)$ we know from Lemma 3.1 that $y\in\mathrm{Perm}'(I)$ and $z\in\mathrm{Perm}'(P\setminus I)$.

By induction, there exist incidence vectors of linear extensions $y'\in\mathrm{Perm}(I)$ and $z'\in\mathrm{Perm}(P\setminus I)$ so that $\sum_{i\in I} c_i y_i' \leq \sum_{i\in I} c_i y_i$ and $\sum_{i\in P\setminus I} c_i z_i' \leq \sum_{i\in P\setminus I} c_i z_i$. We can combine y' and z' to a vector $x'\in\mathrm{Perm}(P)$ by setting

$$x_i' = \begin{cases} y_i', & \text{if } i\in I,\\ z_i'+|I|, & \text{if } i\in P\setminus I. \end{cases}$$

Then $cx' \leq \sum_{i\in I} c_i y_i + \sum_{i\in P\setminus I} c_i(z_i + |I|) = cx$. Hence x is the incidence vector of a linear extension and $\mathrm{Perm}'(P) = \mathrm{Perm}(P)$. □

4. Dimension and facets

We say that the permutations $\pi_1, \ldots, \pi_k$ are linearly independent if their incidence vectors $(\pi_i(1), \ldots, \pi_i(n))$, $1 \leq i \leq k$, are linearly independent. For a permutation π of $P = \{p_1, \ldots, p_n\}$ and $p_{n+1} \notin P$ we define the *lifting* of π to $P \cup p_{n+1}$ by

$$\pi'(j) = \begin{cases} \pi(j), & \text{for } 1 \leq j \leq n, \\ p_{n+1}, & \text{for } j = n+1. \end{cases}$$

Lemma 4.1. *Let $\pi_1, \ldots, \pi_k \in \mathbb{R}^n$ be linearly independent permutations of P and π_0' a permutation of $P \cup p_{n+1}$ such that $\pi_0'(n+1) \neq p_{n+1}$. Then $\pi_0', \pi_1', \ldots, \pi_k'$ are linearly independent.*

Proof. Suppose not, then there exist $\lambda_1, \ldots, \lambda_k$ such that $\pi_0' = \sum_{i=1}^k \lambda_i \pi_i'$. For the last component of π_0' we get $\pi_0'(n+1) = p_{n+1} \sum_{i=1}^k \lambda_i$. On the other hand, summing over all components yields

$$\sum_{j=1}^{n+1} \pi_0'(j) = \sum_{j=1}^{n+1} \sum_{i=1}^{k} \lambda_i \pi_i'(j) = \sum_{i=1}^{k} \lambda_i \sum_{j=1}^{n+1} \pi_i'(j).$$

Since the component sum of the π_i' is constant for $i = 0, \ldots, k$, we get $\sum_{i=1}^k \lambda_i = 1$. We conclude $\pi_0'(n+1) = p_{n+1}$ which is a contradiction. □

Lemma 4.2. *Let $P = P_1 \| P_2$ be the parallel composition of P_1 and P_2; then* $\dim(\mathrm{Perm}(P)) = |P| - 1$ *and there exist $|P|$ linearly independent linear extensions of P.*

Proof. (Induction on $|P|$.) The case $|P| = 2$ is obvious. For $|P| = n+1$ we may assume that $|P_2| \geq 2$. Let p_{n+1} denote a maximal element of P_2 and consider the poset $P' := P_1 \| P_2 \setminus p_{n+1}$. By induction, we find n linearly independent extensions $\pi_1, \ldots, \pi_n$ for P'. Let $\pi_1(j)$ be the last element of P_1 in π_1, i.e., $\pi_1(j) \in P_1$ and $\pi_1(i) \notin P_1$ for $i > j$. Define now π_0' by

$$\pi_0'(i) = \begin{cases} \pi_1(i), & \text{for } i < j, \\ \pi_1(i+1), & \text{for } j \leq i \leq n, \\ \pi_1(j), & \text{for } i = n+1, \end{cases}$$

where $\pi_1(n+1) = p_{n+1}$. Then π_0' is a linear extension of P and, by Lemma 4.1, π_0' and the liftings $\pi_1', \ldots, \pi_n'$, which by construction are linear extensions of P, are linearly independent. □

We say that a series-parallel poset P has a *prime decomposition* if there are prime suborders $P_1, \ldots, P_k$, $k \geq 1$, such that $P = P_1 \oplus \cdots \oplus P_k$. If $k=1$, P is either a parallel composition or a singleton.

Theorem 4.3. *Let P be a series-parallel poset with prime decomposition $P_1 \oplus \cdots \oplus P_k$. Then*

$$\dim(\mathrm{Perm}(P)) = |P| - k$$

and there exist $\dim(\mathrm{Perm}(P)) + 1$ *linearly independent linear extensions of P.*

Proof. We prove the theorem by induction on the number k of components. The case $k=1$ has been proved in Lemma 4.2.

Assume now $k \geq 2$. Then the ideals $P_1 \cup \cdots \cup P_j$ for $j = 1, \ldots, k$ are tight for every linear extension of P. This implies $\dim(\mathrm{Perm}(P)) \leq |P| - k$. By induction hypothesis and by Lemma 4.2 there exist $r = \sum_{i=1}^{k-1} |P_i| - k + 2$ linearly independent extensions $\pi_1, \ldots, \pi_r$ for $P \setminus P_k$ and $s = |P_k|$ linearly independent linear extensions $\varrho_1, \ldots, \varrho_s$ for P_k. The result now follows from concatenating these linear extensions to $|P| - k + 1$ linearly independent extensions $\pi_1 \varrho_1', \ldots, \pi_r \varrho_1', \pi_1 \varrho_2', \ldots, \pi_1 \varrho_s'$ of P where $\varrho_i' = \varrho_i + r$. In order to see that they are linearly independent, subtract the first vector from each of the last $s-1$ vectors. □

We conclude by characterizing the constraints which define facets. The arguments used in the previous proof immediately give the following corollary.

Corollary 4.4. *Let P, P' be series-parallel posets. A convex set constraint defining a facet for* $\mathrm{Perm}(P)$ *defines a facet of both* $\mathrm{Perm}(P \oplus P')$ *and* $\mathrm{Perm}(P' \oplus P)$.

The set of constraints (3.1) and (3.2) used so far contain redundant inequalities. One easily checks that the convex set constraints for $C_1 = A_1 \oplus A_2$ and $C_2 = A_2 \oplus A_3$ imply the convex set constraints for $C_3 = A_1 \oplus (A_2 \cup A_3)$ and $C_4 = (A_1 \cup A_2) \oplus A_3$. Furthermore the ideal constraint for I and the convex set constraint for $C = I \oplus B$ imply the ideal constraint for $I' = I \cup B$.

We call a series-reducible convex set $C = A \oplus B$ *bipartite* if A and B are prime. Hence it suffices to require (3.1) for prime ideals and (3.2) for bipartite convex sets only.

Lemma 4.5. *Let $P = P_1 \| P_2$. Then every bipartite convex set C defines a facet of* $\mathrm{Perm}(P)$.

Proof. We proceed by induction on the cardinality of $|P \setminus C|$ and show that there are $|P| - 1$ linearly independent extensions for which C is tight. If $|P \setminus C| = 1$, we may assume that $P_1 = C$ and $P_2 = p_1$. Let $n = |P|$. By Theorem 4.3 there exist $n-2$ linearly independent extensions $x_1, \ldots, x_{n-2}$ of P_1. Given these, we define the $n-1$ linear extensions $x_1', \ldots, x_{n-1}'$ of P by

$$x_i'(j) = \begin{cases} x_i(j), & \text{for } j \leq |C|, \\ p_1, & \text{for } j = n, \end{cases}$$

for $i = 1, \dots, n-2$ and

$$x_{n-1}'(j) = \begin{cases} p_1, & \text{for } j = 1, \\ x_1(j-1), & \text{for } 1 < j \leq n. \end{cases}$$

The induction step is the same as in Lemma 4.2 and therefore omitted. □

Theorem 4.6. *Let P be a series-parallel poset with prime decomposition $P = P_1 \oplus \cdots \oplus P_k$.*

(i) *An ideal I defines a facet of* Perm(P) *if and only if $I \subset P_1$ and both I and $P_1 \setminus I$ are prime.*

(ii) *A convex set C defines a facet if and only if C is bipartite and $C \subset P_i$ for some i.*

Proof. (i) Observe that $\{x \in \text{Perm}(P)\colon x(I) = f(I)\} = \text{Perm}(P')$, where $P' := I \oplus (P_1 \setminus I) \oplus P_2 \oplus \cdots \oplus P_k$. Therefore an ideal I defines a facet iff the dimension of P' satisfies $\dim(\text{Perm}(P')) = \dim(\text{Perm}(P)) - 1 = |P| - (k+1)$. Hence (i) follows from Theorem 4.3.

(ii) The "if"-part follows from Corollary 4.4 and Lemma 4.5. Thus it remains to prove that if $C \not\subseteq P_i$ for all i, then C does not define a facet. Since $C = A \oplus B$ we have only to consider the case where $A \subseteq P_i$ and $B \subseteq P_{i+1}$. Now, $\{x \in \text{Perm}(P)\colon |A|\, x(B) - |B|\, x(A) = \frac{1}{2}\, |A|\,|B|\, (|A| + |B|)\} = \text{Perm}(P')$, where $P' := P_1 \oplus \cdots \oplus (P_i \setminus A) \oplus A \oplus B \oplus (P_{i+1} \setminus B) \oplus \cdots \oplus P_k$. Hence, if C defines a facet, then $\dim(\text{Perm}(P')) = \dim(\text{Perm}(P)) - 1$, contradicting Theorem 4.3. □

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