OR Spektrum (1993) 15:133-137

OR Spektrum

A decomposition of the class of semiconvex games*

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Received 21 April 1992 / Accepted in revised form 29 June 1993

Summary. The paper studies the class SC^N of cooperative games with player set N which have the semiconvexity property. SC^N is decomposed into an algebraic sum of convex cones of games for which generating sets are available. The union of these sets thus forms a generating set for SC^N. Special attention is paid to one of the considered cones in the decomposition of SC^N. In particular, the so called airport savings games $w_y, y \in \mathbb{R}^N$, defined by $w_y(S) = \sum_{j \in S} y_j - \max_{j \in S} y_j$ for $\emptyset \neq S \subseteq N$, are emphasized.

Zusammenfassung. In dieser Arbeit wird die Klasse SC^N von kooperativen Spielen mit Spielermenge N, welche die Semikonvexität-Eigenschaft besitzt, untersucht. SC^N wird in die algebraische Summe von konvexen Kegeln von Spielen zerlegt, für welche erzeugende Mengen verfügbar sind. Die Vereinigung dieser Mengen bildet dann eine erzeugende Menge für SC^N. Einem der betrachteten Kegel der Zerlegung wird besondere Aufmerksamkeit geschenkt. Es werden speziell die sog. „Airport savings" Spiele w_y, $y \in \mathbb{R}^N$, welche durch $w_y(S) = \sum_{j \in S} y_j - \max_{j \in S} y_j$ für $\emptyset \neq S \subset N$ definiert sind, betrachtet.

Key words: Semiconvexity, airport cost function, convexity, polyhedral cone, extreme direction

Schlüsselwörter: Semikonvexität, Flughafen-Kostenfunktion, Konvexität, polyhedrale Kegel, extremale Richtungen

1. Submodular cost and supermodular savings functions

As an application of game theoretic analysis to the cost allocation problem, Littlechild and Owen (1973) studied the problem of setting airport landing charges for different types of aircraft. Their game theoretic approach to the airport cost allocation problem is based on an appropriately defined set function, the so-called airport cost function $c_y: 2^N \to \mathbb{R}$. Here N is the set of planes which are to land at the airport and $y = (y_i)_{i \in N}$ denotes the costs for the planes $i \in N$ to construct a runway of appropriate length. Then the costs $c_y(S)$ of a subset S of planes is determined by that plane in S with the longest runway, i.e.,

$$c_y(S) = \max_{i \in S} y_i.$$

The airport cost function c_y satisfies the submodularity conditions

$$c_y(S) + c_y(T) \geq c_y(S \cup T) + c_y(S \cap T) \quad \text{for all } S, T \subseteq N.$$

As usual, an arbitrary cost function $c: 2^N \to \mathbb{R}$ induces a savings function $v: 2^N \to \mathbb{R}$ by means of $v(\emptyset) = 0$ and

$$v(S) = \sum_{j \in S} c(\{j\}) - c(S) \quad \text{for all } S \subseteq N, S \neq \emptyset.$$

Here, the expression $v(S)$ represents the cost savings that would result in the cost model if the participants in subset S cooperate instead of acting alone. Whenever the cost function c satisfies the submodularity conditions, then the induced savings function v satisfies the supermodularity conditions

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \text{for all } S, T \subseteq N. \tag{1}$$

With the cost vector $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ we associate the savings function $w_y: 2^N \to \mathbb{R}$ by means of $N = \{1, 2, .., n\}$, $w_y(\emptyset) = 0$ and

$$w_y(S) = \sum_{j \in S} y_j - \max_{j \in S} y_j \quad \text{for all } S \subseteq N, S \neq \emptyset. \tag{2}$$

It can easily be verified that the savings function w_y of (2) induced by an airport cost function and, thus, it satisfies the supermodularity conditions (1).

In the game theoretic context, the term convexity is preferred to the term supermodularity. This paper focuses

* This work is part of the Ph.D. thesis of the first author (Derks 1991)

on a related condition, the semiconvexity, which is a weaker condition than the convexity.

In Sect. 2 the notion of semiconvexity for a cooperative game is described. Section 3 is devoted to a decomposition of the class of semiconvex games into an algebraic sum of three convex cones of games. The decomposition is strongly based on the smallest convex cone of games containing the savings games of (2). Subsequently, a generating set for $\mathbf{SC}^N$ is presented. $\mathbf{SC}^N$ does not have extreme directions according to the fact that the $|N|$-dimensional linear subspace of additive games is contained in $\mathbf{SC}^N$. In Sect. 4 it is shown that the cone $\mathbf{SC}_0^N$ of 0-normalized semiconvex games is generated by 0-normalized games in the generating set of $\mathbf{SC}^N$ as presented in Sect. 3. Furthermore, we will show that these games are actually extreme directions of $\mathbf{SC}_0^N$.

2. The class of semiconvex games

First let us briefly go into the game theoretic setting. We consider a cooperative game in characteristic function form, or simply a *game*, with finite player set N to be a real-valued function v on the set 2^N of subsets of N with $v(\emptyset)=0$. A subset S of N is called a *coalition* and $v(S)$ is called its *value* in the game v. The value $v(S)$ is interpreted as the gain or savings of the coalition S in the case the members of S decide to cooperate. The class of all games with player set N is denoted by $\mathbf{G}^N$. Note that $\mathbf{G}^N$ is the $(2^{|N|}-1)$-dimensional Euclidean vector space indexed by the non-empty coalitions. Except for an example in Sect. 3 the player set N is supposed to consist of at least four players. Examples of games are the savings functions as defined in (2). We will refer to these games as the *(airport cost) savings games*.

An element $y=(y_i)_{i\in N}$ of $\mathbb{R}^N$ is called an *allocation*. For a coalition S let the allocation $e^S\in\mathbb{R}^N$ be given by $e_i^S=1$ for $i\in S$ and $e_i^S=0$ for $i\in N\setminus S$. With the allocation $y\in\mathbb{R}^N$ we associate the game $y:2^N\to\mathbb{R}$ by means of $y(\emptyset)=0$ and $y(S)=\sum_{j\in S}y_j$ for all $S\subseteq N$, $S\neq\emptyset$. Games which are associated with an allocation are called *additive*. The class of all additive games with player set N is denoted by $\mathbf{A}^N$. Notice that $\mathbf{A}^N$ is an $|N|$-dimensional linear sulspace of $\mathbf{G}^N$, which is generated by the additive games $u_{\{i\}}$ and $-u_{\{i\}}$ associated to the allocations $e^{\{i\}}$ and $-e^{\{i\}}$ respectively, with $i\in N$. We say the allocation $y\in\mathbb{R}^N$ majorizes the games $v\in\mathbf{G}^N$ (notation: $y\geq v$) if the additive game associated with the allocation y majorizes the game v, i.e. $y(S)\geq v(S)$ for all $S\subseteq N$.

Throughout the paper we pay special attention to the marginal contribution allocation of a game. Given an arbitrary game $v\in\mathbf{G}^N$ the corresponding *marginal contribution allocation* $b^v\in\mathbb{R}^N$ is defined by $b_i^v=v(N)-v(N\setminus\{i\})$ for all $i\in N$.

One of the main topics of research in cooperative game theory is how to allocate the value $v(N)$ in a game $v\in\mathbf{G}^N$ among the players. Since the introduction of the notion of a cooperative game, many solution concepts for these games have been proposed to solve the relevant allocation problem. Generally speaking, the concepts provide satisfactory and stable solutions only on a specific subclass of the game space. Perhaps the best-known solution concept is the so-called core: an allocation $y\in\mathbb{R}^N$ is said to be a *core allocation* of a game $v\in\mathbf{G}^N$ if y distributes the value $v(N)$ among the players in such a way that y majorizes v. In other words, the *core* $C(v)$ of a game $v\in\mathbf{G}^N$ consists of all allocations $y\in\mathbb{R}^N$ satisfying $y(N)=v(N)$ and $y\geq v$. Obviously, there are games without core allocations.

The solution part of cooperative game theory is mainly based on the traditional assumption that the grand coalition N will be formed. Note that the marginal contribution allocation b^v of a game v is derived from the marginal contributions of each player with respect to the formation of the grand coalition. We assert that a player i cannot demand a greater portion of the value $v(N)$ than the amount b_i^v since he would otherwise force the other players to divide an amount less than $v(N)-b_i^v=v(N\setminus\{i\})$ among themselves. Thus, in distributing $v(N)$ among the players only those efficient allocations should be considered which are majorized by the marginal contribution allocation b^v.

Now if a non-empty coalition $S\neq N$ exists with $\sum_{j\in S}b_j^v<v(S)$ then the players in S will not be satisfied by an (efficient) allocation which is majorized by b^v. It seems, therefore, reasonable to examine the formation of the grand coalition only in a game v that fulfils

$$\sum_{j\in S} b_j^v \geq v(S) \quad \text{for each non-empty coalition } S. \tag{3}$$

Games with this property are called *semibalanced* and are first discussed in Tijs and Lipperts (1982).

Semibalancedness is a necessary condition for the non-emptiness of the core since for each game v and core allocation $y\in C(v)$ we have

$$y_i=y(N)-y(N\setminus\{i\})\leq v(N)-v(N\setminus\{i\})=b_i^v \quad \text{for all } i\in N \tag{4}$$

and, therefore, $v(S)\leq y(S)\leq b^v(S)$ for each coalition $S\subseteq N$, implying the semibalancedness of v.

It may happen that the semibalancedness inequalities $b^v(S)\geq v(S)$, $S\subseteq N$, even hold whenever an arbitrary marginal contribution b_i^v in the sum $b^v(S)$ is replaced by the individual value $v(\{i\})$ of the player i involved. Remark that we interpret the marginal contribution of any player as a large payoff to the player and the individual value as a small payoff. Due to these reasonings, the class of *semiconvex* games is defined by

$$\mathbf{SC}^N=\{v\in\mathbf{G}^N: b^v\geq v \text{ and } b^v(S\setminus\{i\})+v(\{i\})\geq v(S) \text{ for all } i\in N, \text{ all } S\subseteq N \text{ with } i\in S\}.$$

Semiconvex games are introduced in Driessen and Tijs (1985) as an adjunct to the study of the τ-value concept. There it is established that $\mathbf{SC}^N$ is a $(2^{|N|}-1)$-dimensional cone in $\mathbf{G}^N$ which includes the class of convex games (i.e. games satisfying the supermodularity conditions (1)). More generally, a game v with the property that it fulfils the supermodularity conditions

$$v(N)+v(S\setminus\{i\})\geq v(N\setminus\{i\})+v(S) \quad \text{for all } i\in S\in N, \tag{5}$$

is semiconvex. To see this notice that (5) is equivalent to

$$b_i^v \geq v(S) - v(S \setminus \{i\}) \quad \text{for all } i \in S \in N, \tag{6}$$

and this expresses the condition that the marginal contribution of each player to a coalition is largest for the grand coalition. The semiconvexity condition $b^v(S\setminus\{i\}) + v(\{i\}) \geq v(S)$ for an arbitrary non-empty coalition $S = \{i, i_2, \ldots, i_{|S|}\}$ now follows from the $|S|$ inequalities $b_i^v \geq v(\{i\})$, $b_{i_2}^v \geq v(\{i, i_2\}) - v(\{i\}), \ldots, b_{i_{|S|}}^v \geq v(S) - v(S \setminus \{i_{|S|}\})$.

3. A decomposition of the class $\mathbf{SC}^N$

Our main goal is to decompose the class of semiconvex games into an algebraic sum of convex cones of games for which generating sets are available. The union of these generating sets forms a generating set for $\mathbf{SC}^N$.

The savings game w_y of (2) associated with an arbitrary allocation $y \in \mathbb{R}^N$ is semiconvex. The class of these savings games is not a convex cone as is illustrated by the following example. Consider the player set $N = \{1,2,3\}$ (the example can easily be extended to arbitrary large N) and the savings games w_x and w_y associated with the allocations $x = (0,1,2)$ and $y = (2,1,0)$. In fact, the games w_x, w_y and the sum game $w = w_x + w_y$ are as follows.

coalition S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
value $w_x(S)$	0	0	0	0	0	0	1	1
value $w_y(S)$	0	0	0	0	1	0	0	1
value $w(S)$	0	0	0	0	1	0	1	2

Suppose w equals w_z for a certain allocation $z \in \mathbb{R}^3$. Then for a two-person coalition S we have $w(S) = w_z(S) = \sum_{j \in S} z_j - \max_{j \in S} z_j = \min_{j \in S} z_j$. From this and $w(\{1,3\}) = 0$, we first derive that $z_1 = 0$ or $z_3 = 0$ and subsequently, $w(\{1,2\}) \leq 0$ or $w(\{2,3\}) \leq 0$; a contradiction. We assert that the sum game w is not a savings game.

Now the idea is to embed the class of these savings games in a convex cone generated by a finite number of appropriately defined savings games. For the sake of notation, the savings game associated with the allocation e^S is shortly denoted by $w_S: 2^N \to \mathbb{R}$. Thus, w_S is given by

$$w_S(T) = \begin{cases} |S \cap T| - 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We note that for coalitions S with $|S| \geq 2$ the marginal contribution allocation b^{w_S} of the savings game w_S equals e^S and in case $|S| \leq 1$ then the savings game w_S equals the zero game and, thus, $b^{w_S} = 0$.

The next theorem states that each savings game associated with a non-negative allocation is a non-negative linear combination of the savings games w_S, $S \subseteq N$, i.e. the class of non-negative savings games is embedded in the convex cone $\mathbf{W}^N$ generated by the savings games w_S, $S \subseteq N$.

Theorem 3.1. *Let w_y be the savings game of (2), with $y \in \mathbb{R}_+^N$. Then $w_y \in \mathbf{W}^N = \mathrm{Cone}(\{w_S: S \subseteq N\})$.*

Proof. Without loss of generality, the players may be ordered so that $N = \{1,2,\ldots,n\}$ and $0 = y_0 \leq y_1 \leq y_2 \leq \ldots \leq y_n$. For any $j \in N$, define the set $S_j = \{j, j+1, \ldots, n\}$ of players with index j or a larger index. It is evident that the vector equality $y = \sum_{j=1}^n (y_j - y_{j-1}) e^{S_j}$ holds. Now we assert that

$$w_y = \sum_{j=1}^{n} (y_j - y_{j-1}) w_{S_j}. \tag{8}$$

To prove (8) let $T \subseteq N$, $T \neq \emptyset$. Further, let $m \in T$ be the player in T with the largest index. Then we get $S_j \cap T = \emptyset$ iff $m < j \leq n$. Now it follows that

$$\begin{aligned}
&\sum_{j=1}^{n} (y_j - y_{j-1}) w_{S_j}(T) \\
&= \sum_{j=1}^{m} (y_j - y_{j-1}) [|S_j \cap T| - 1] \\
&= \sum_{j=1}^{m} (y_j - y_{j-1}) \left[\sum_{k=j}^{m} e_k^T \right] - \sum_{j=1}^{m} (y_j - y_{j-1}) \\
&= \sum_{j=1}^{m} \sum_{k=j}^{m} (y_j - y_{j-1}) e_k^T - (y_m - y_0) \\
&= \sum_{k=1}^{m} \sum_{j=1}^{k} (y_j - y_{j-1}) e_k^T - y_m \\
&= \sum_{k=1}^{m} y_k e_k^T - y_m \\
&= \sum_{k \in T} y_k - \max_{k \in T} y_k = w_y(T).
\end{aligned}$$

We conclude that (8) holds. This completes the proof of the theorem. □

From the fact that the class of semiconvex games is a convex cone containing the savings games w_S, $S \subseteq N$, we conclude that the cone $\mathbf{W}^N$ is included in SC^N. Without going into details we note that the games in $\mathbf{W}^N$ possess a large core because of the convexity property for these games (cf. Sharkey 1982).

Next, we present a subclass of semiconvex games with a unique core allocation. Let the class $\mathbf{H}^N$ of games be defined by

$$\mathbf{H}^N = \{v \in \mathbf{G}^N : (v(\{i\}))_{i \in N} \text{ is a (unique) core allocation of } v\}.$$

According to (4) each core allocation is majorized by the marginal contribution vector; in particular, $v \in \mathbf{H}^N$ yields that $b_i^v \geq v(\{i\})$ for all $i \in N$, and from this we deduce that for all $i \in N$ and all $S \subseteq N$ with $i \in S$ we must have $b^v(S \setminus \{i\}) + v(\{i\}) \geq \sum_{j \in S} v(\{j\}) \geq v(S)$. Thus, $v \in \mathbf{SC}^N$. So, $v \in \mathbf{H}^N$ implies $v \in \mathbf{SC}^N$.

So far we established that the class of semiconvex games contains the cones $\mathbf{W}^N$ and $\mathbf{H}^N$, and containing games with a non-empty core. It furthermore contains the game $\hat{w}$ defined by

$$\hat{w}(S) = \begin{cases} -1 & \text{if } S \subseteq N,\ |S| \geq |N|-1, \\ 0 & \text{otherwise.} \end{cases}$$

since it is a non-positive game, 0-normalized, i.e. $v(\{i\}) = 0$ for $i \in N$, and with marginal contribution allocation $b^{\hat{w}}$ equal to the zero allocation. Notice that the core of $\hat{w}$ must be empty.

Due to the fact that $\mathbf{SC}^N$ itself is a convex cone, we conclude that the inclusion $\mathbf{SC}^N \supseteq \mathbf{H}^N + \mathbf{W}^N + \text{Cone}(\{\hat{w}\})$ holds. According to the main theorem, the inverse inclusion is also valid.

Theorem 3.2. $\mathbf{SC}^N = \mathbf{H}^N + \mathbf{W}^N + \text{Cone}(\{\hat{w}\})$.

Proof. Let $v \in \mathbf{SC}^N$. We distinguish two cases.

Case one. Suppose that the game v is 0-normalized, i.e., $v(\{i\}) = 0$ for all $i \in N$. Then the semiconvexity of v yields that $b_i^v \geq v(\{i\}) = 0$ for all $i \in N$. From $b^v \geq 0$ and Theorem 3.1 we derive that the savings game w_{b^v} is contained in the cone $\mathbf{W}^N$. Consider the game

$$w = v - w_{b^v} - \alpha \hat{w},$$

where α denotes the scalar $b^v(N) - v(N) - \max_{j \in N} b_j^v$. We assert that $w(S) \leq 0$ for all $S \subseteq N$. Indeed,

$$\begin{aligned}
w(\{i\}) &= v(\{i\}) = 0 \quad \text{for all } i \in N, \\
w(N) &= v(N) - b^v(N) + \max_{j \in N} b_j^v + \alpha = 0, \\
w(N \setminus \{i\}) &= v(N \setminus \{i\}) - b^v(N \setminus \{i\}) + \max_{j \in N \setminus \{i\}} b_j^v + \alpha \\
&= \max_{j \in N \setminus \{i\}} b_j^v - \max_{j \in N} b_j^v \\
&\leq 0 \quad \text{for all } i \in N, \quad \text{and} \\
w(S) &= v(S) - b^v(S) + \max_{j \in S} b_j^v \\
&\leq 0 \quad \text{for all } S \subset N \text{ with } |S| < |N|-1,
\end{aligned}$$

where the last inequality results from the 0-normalizedness and the semiconvexity of v. Hence, $w(S) \leq 0$ for all $S \subseteq N$. Together with $w(N) = 0$ and $w(\{i\}) = 0$ for all $i \in N$, this implies that the zero allocation is a core allocation of w and, therefore, $w \in \mathbf{H}^N$. Further, the semiconvexity of v yields that $\alpha \geq 0$. Thus, $\alpha \hat{w} \in \text{Cone}(\{\hat{w}\})$. We conclude that $v = w + w_{b^v} + \alpha \hat{w} \in \mathbf{H}^N + \mathbf{W}^N + \text{Cone}(\{\hat{w}\})$.

Case two. Evidently, the additive game $y \in \mathbf{G}^N$ associated with the allocation $y = (v(\{i\}))_{i \in N}$ also belongs to the class $\mathbf{H}^N$. Furthermore, it is straightforward to verify that the semiconvexity of v implies the semiconvexity of the game $v - y$. By applying case one to the semiconvex 0-normalized game $v - y$, we obtain that $v - y \in \mathbf{H}^N + \mathbf{W}^N + \text{Cone}(\{\hat{w}\})$. From this and $y \in \mathbf{H}^N$, we conclude that $v = v - y + y \in \mathbf{H}^N + \mathbf{W}^N + \text{Cone}(\{\hat{w}\})$ for all $v \in \mathbf{SC}^N$. □

With the aid of the theorem we are able to present a generating set for $\mathbf{SC}^N$. By its definition, the cone $\mathbf{W}^N$ is generated by the set $\{w_S : S \subseteq N,\ |S| \geq 2\}$. A generating set for the cone $\mathbf{H}^N$ can be found by noting that $\mathbf{H}^N$ is the algebraic sum of the class $\mathbf{A}^N$ of additive games and the class

$$\{v \in \mathbf{G}^N : v(N) = 0,\ v(\{i\}) = 0 \text{ for all } i \in N \text{ and } v(S) \leq 0 \text{ for all other } S \subset N\}.$$

which is actually the class $\mathbf{H}_0^N$ of 0-normalised games in $\mathbf{H}^N$. It is obvious that $\mathbf{H}_0^N$ is generated by the set $\{-\mathbb{1}_S : S \subset N,\ 2 \leq |S| \leq |N|-1\}$ of games, where the *unity* game $\mathbb{1}_S \in \mathbf{G}^N$ is given by

$$\mathbb{1}_S(T) = 1 \quad \text{if } T = S \quad \text{and}$$

$$\mathbb{1}_S(T) = 0 \quad \text{for all } T \neq S.$$

Corollary 3.3. *A generating set for the class* $\mathbf{SC}^N$ *of semiconvex games is formed by the* $2^{|N|+1} - 1$ *games*

$u_i,\ -u_i,\quad i \in N,$

$-\mathbb{1}_S,\quad S \subset N$ *with* $2 \leq |S| \leq |N|-1,$

$w_S,\quad S \subseteq N$ *with* $2 \leq |S| \leq |N|,$ *and the game* $\hat{w}$.

The class $\mathbf{SC}^N$ does not contain all games with a non-empty core. To show the interrelation of $\mathbf{SC}^N$ with the class of games with a non-empty core consider the decomposition for a game v with core allocation y into the non-negative linear combination of the games $u_{\{i\}}$, $-u_{\{i\}}$, $i \in N$, and $-\mathbb{1}_S$, $\emptyset \neq S \subset N$, as follows (cf. Spinetto 1974):

$$v = \sum_{i \in N} y_i u_{\{i\}} + \sum_{\emptyset \neq S \subset N} \left(\sum_{i \in S} y_i - v(S) \right) (-\mathbb{1}_S).$$

Thus, the mentioned games generate the class of games with a non-empty core. Only the games $-\mathbb{1}_{\{i\}}$, $i \in N$, in this decomposition have not been mentioned in the corollary above and, indeed, these games are clearly not semiconvex.

4. The class of 0-normalized semiconvex games

Except for the additive games, the generating games in the above corollary are 0-normalized. In fact they form a generating set of extreme directions of the class $\mathbf{SC}_0^N$ of 0-normalized semiconvex games:

Theorem 4.1. *The* $2^{|N|+1} - 2|N| - 1$ *games*

$-\mathbb{1}_S,\quad S \subset N$ *with* $2 \leq |S| \leq |N|-1,$
$w_S,\quad S \subseteq N$ *with* $2 \leq |S| \leq |N|,$ *and the game* $\hat{w}$

form a generating set of the class SC_0^N *of 0-normalized semiconvex games. Furthermore, they are extreme directions in* SC_0^N.

Proof. Observe that each game in $\mathbf{W}^N$ is 0-normalized. Also, $-\hat{w} \in SC_0^N$. We, thus, have $\mathbf{H}_0^N + \mathbf{W}^N + \text{Cone}(\{-\hat{w}\}) \subseteq SC_0^N$. Consider the decomposition of a 0-normalized semiconvex game v as $w + w_{b^v} - \alpha\hat{w}$ in the proof of Theorem 3.2. The 0-normalizedness of the games v, w_{b^v} and $-\alpha\hat{w}$ implies the 0-normalizedness of $w \in \mathbf{H}^N$ and, thus, $\mathbf{H}_0^N + \mathbf{W}^N + \text{Cone}\{-\hat{w}\} \supseteq SC_0^N$. Let W denote the set of games mentioned in the theorem. We conclude that W is a generating set of SC_0^N.

Let us prove the extremeness of the games in W.

The extremeness of $-\hat{w}$ and $-\mathbb{1}_S$, with $S \subset N$, $1 < |S| < |N| - 1$, follows from the fact that for each of these games w there exists a coalition S with $w(S) = -1$ and such that its value with respect to the other games in W is non-negative; therefore, w cannot be represented as a non-negative linear combination of the other games in W implying its extremality.

Suppose $-\mathbb{1}_{N\backslash\{i\}}$ equals a non-negative combination, say $\sum_{w \in W} \alpha^w w$, of elements of W. Using the additivity property of the marginal contribution allocation ($b^{\alpha v - w} = \alpha b^v + b^w$ *for all* v, $w \in \mathbf{G}^N$ and $\alpha \in \mathbb{R}$), $b^{-\mathbb{1}_S} = 0$, $2 \le |S| < |N| - 1$, and $b^{\hat{w}} = 0$, we have

$$\mathbb{1}^{\{i\}} = b^{-\mathbb{1}_{N\backslash\{i\}}} = \sum_{w \in W} \alpha^w b^w$$
$$= \sum_{j \in N} \alpha^{-\mathbb{1}_{N\backslash\{j\}}} \mathbb{1}^{\{j\}} + \sum_{S \subseteq N, |S| > 1} \alpha^{w_S} \mathbb{1}^S$$

and this equality only holds in case $\alpha^{-\mathbb{1}_{N\backslash\{i\}}} = 1$, $\alpha^{-\mathbb{1}_{N\backslash\{j\}}} = 0$ for all $j \in N\backslash\{i\}$, and $\alpha^{w_S} = 0$ for each coalition S with $|S| > 1$. Then also $\alpha^{-\mathbb{1}_S} = 0$ for each $S \subset N$ with $1 < |S| < |N| - 1$. Therefore, $-\mathbb{1}_{N\backslash\{i\}} \notin \text{Cone}(W\backslash\{-\mathbb{1}_{N\backslash\{i\}}\})$, implying its extremality.

Now let w_T be a non-negative combination, say $\sum_{w \in W} \alpha^w w$, of elements of W, with $T \subseteq N$, $1 < |T|$, arbitrary. From

$$\mathbb{1}^T = b^{w_T} = \sum_{w \in W} \alpha^w b^w$$
$$= \sum_{j \in N} \alpha^{-\mathbb{1}_{N\backslash\{j\}}} \mathbb{1}^{\{j\}} + \sum_{S \subseteq N, |S| > 1} \alpha^{w_S} \mathbb{1}^S$$

it follows that $\alpha^w = 0$ for the games $w = -\mathbb{1}_{N\backslash\{j\}}$, with $j \in N\backslash T$, and $w = w_S$, for coalitions S, $1 < |S|$, such that $S\backslash T$ is non-empty, i.e.,

$$1 = \alpha^{-\mathbb{1}_{N\backslash\{i\}}} + \sum_{S \subseteq T, |S| > 1, i \in S} \alpha^{w_S}, \quad \text{for each } i \in T. \tag{9}$$

Adding the $|T|$ equalities of (9) we obtain

$$|T| = \sum_{j \in T} \alpha^{-\mathbb{1}_{N\backslash\{j\}}} + \sum_{S \subseteq T, |S| > 1} \alpha^{w_S} |S|. \tag{10}$$

Furthermore,

$$|T| - 1 = w_T(N) = \sum_{S \subseteq T, |S| > 1} \alpha^{w_S}(|S| - 1) - \alpha^{-\hat{w}}$$
$$= \sum_{S \subseteq T, |S| > 1} \alpha^{w_S} |S| - \sum_{S \subseteq T, |S| > 1} \alpha^{w_S} - \alpha^{-\hat{w}}. \tag{11}$$

Combining (10) and (11) we obtain

$$\sum_{j \in T} \alpha^{-\mathbb{1}_{N\backslash\{j\}}} + \sum_{S \subseteq T, |S| > 1} \alpha^{w_S} + \alpha^{-\hat{w}} = 1$$

In combination with (9) this is only possible if $\alpha^{-\mathbb{1}_{N\backslash\{j\}}} = 0$, for $j \in T$, $\alpha^{w_T} = 1$, $\alpha^{w_S} = 0$ for $S \subset T$, $|S| > 1$, and $\alpha^{-\hat{w}} = 0$. It follows now that $\alpha^{-\mathbb{1}_S} = 0$ for all $S \subseteq N$, $|S| > 1$, and we conclude that the game w_T must be extreme. □

Notice that only the games $-\mathbb{1}_{N\backslash\{i\}}$, $i \in N$, w_S, $S \subseteq N$ with $2 \le |S| \le |N|$, and $\hat{w}$ fulfil (6). These games are, therefore, extreme in the cone of 0-normalized games which obey (6).

Furthermore, the games w_S, $S \subseteq N$ with $2 \le |S| \le |N|$, are convex implying that these games are extreme directions in the cone of 0-normalized convex games.

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