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Zero-two law for cosine families

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Abstract. For $(C(t))_{t\geq 0}$ being a strongly continuous cosine family on a Banach space, we show that the estimate $\limsup_{t\to 0^+} \|C(t) - I\| < 2$ implies that $C(t)$ converges to I in the operator norm. This implication has become known as the zero-two law. We further prove that the stronger assumption of $\sup_{t\geq 0} \|C(t) - I\| < 2$ yields that $C(t) = I$ for all $t \geq 0$. For discrete cosine families, the assumption $\sup_{n\in\mathbb{N}} \|C(n) - I\| \leq r < \frac{3}{2}$ yields that $C(n) = I$ for all $n \in \mathbb{N}$. For $r \geq \frac{3}{2}$, this assertion does no longer hold.

1. Introduction

Let $(T(t))_{t\geq 0}$ denote a strongly continuous semigroup on the Banach space X with infinitesimal generator A. It is well known that the inequality

$$\limsup_{t\to 0^+} \|T(t) - I\| < 1, \tag{1.1}$$

implies that the generator A is a bounded operator, see, e.g. [12, Remark 3.1.4] or equivalently that the semigroup is uniformly continuous (at 0), i.e.

$$\limsup_{t\to 0^+} \|T(t) - I\| = 0. \tag{1.2}$$

This has become known as *zero-one* law for semigroups. Surprisingly, the same law holds for general semigroups on semi-normed algebras, i.e. (1.1) implies (1.2), see, e.g. [5]. For a nice overview and related results, we refer the reader to [4].

In this paper, we study the zero-two law for strongly continuous cosine families on a Banach space, i.e. whether

$$\limsup_{t\to 0^+} \|C(t) - I\| < 2 \quad \text{implies that} \quad \limsup_{t\to 0^+} \|C(t) - I\| = 0. \tag{1.3}$$

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This implication is known if the Banach space is UMD, see Fackler [6, Corollary 4.2], hence, in particular for Hilbert spaces. On the other hand, the $0 - 3/2$ law, i.e.

$$\limsup_{t\to 0^+} \|C(t) - I\| < \frac{3}{2} \quad \text{implies that} \quad \limsup_{t\to 0^+} \|C(t) - I\| = 0,$$

holds for cosine families on general Banach spaces as was proved by Arendt in [1, Theorem 1.1 in Three Line Proofs]. The result even holds without assuming that the cosine family is strongly continuous. In the same work, Arendt poses the question whether the zero-two law holds for cosine families, [1, Question 1.2 in Three Line Proofs]. The following theorem answers this question positively for strongly continuous cosine families. For its proof and the definition of a cosine family, we refer to Sect. 2.

THEOREM 1.1. *Let $(C(t))_{t\geq 0}$ be a strongly continuous cosine family on the Banach space X. Then,*

$$\limsup_{t\to 0^+} \|C(t) - I\| < 2, \tag{1.4}$$

implies that $\lim_{t\to 0^+} \|C(t) - I\| = 0$.

By taking $X = \ell^2$ and

$$C(t) = \begin{pmatrix} \cos(t) & 0 & \cdots & \\ 0 & \cos(2t) & 0 & \cdots \\ \vdots & & \ddots & \end{pmatrix},$$

it is easy to see that this result is optimal. Whether one can remove the assumption that the cosine family is strongly continuous remains open.

The zero-one law for semigroups and the zero-two law for cosine families tells something about the behaviour near $t = 0$. Instead of studying the behaviour around zero, we could study the behaviour on the whole time axis. A result dating back to the sixties is the following; for a semigroup the assumption

$$\sup_{t\geq 0} \|T(t) - I\| < 1, \tag{1.5}$$

implies that $T(t) = I$ for all $t \geq 0$, see, e.g. Wallen [13] and Hirschfeld [8]. This seems not to be well known among researchers working in the area of strongly continuous semigroup. The corresponding result for cosine families, i.e.

$$\sup_{t\in\mathbb{R}} \|C(t) - I\| < 2 \quad \text{implies that} \quad C(t) = I \tag{1.6}$$

is hardly studied at all. We prove (1.6) for strongly continuous cosine families on Banach spaces. This result is strongly motivated by the recent work of Bobrowski and Chojnacki. In [3, Theorem 4], they showed that if $r < \frac{1}{2}$, where

$$r = \sup_{t\geq 0} \|C(t) - \cos(at)I\|; \tag{1.7}$$

then, $C(t) = \cos(at)I$ for all $t \geq 0$. They used this to conclude that scalar cosine families are isolated points in the space of bounded strongly continuous cosine families on a fixed Banach space.

Hence, we show that for $a = 0$ the r can be chosen to be 2, provided C is strongly continuous. We remark that by using the proof idea in [1, Theorem 1.1 in Three Line Proofs] the implication

$$\sup_{t\in\mathbb{R}} \|C(t) - I\| < r \quad \text{implies that} \quad C(t) = I$$

holds for $r < \frac{3}{2}$ for any cosine family. While this paper was being revised, we heard that Bobrowski, Chojnacki and Gregosiewicz showed that for $a \neq 0$ the implication

$$\sup_{t\in\mathbb{R}} \|C(t) - \cos(at)I\| < r \quad \text{implies that} \quad C(t) = \cos(at)I \tag{1.8}$$

holds for general cosine families with $r = \frac{8}{3\sqrt{3}}$. This constant is optimal, as can be directly seen by choosing $C(t) = \cos(3at)I$. In [11], we wrongly claimed that $r = 2$ was the optimal constant.

The layout of this paper is as follows. In Sect. 2, we prove the zero-two law for strongly continuous cosine families, i.e. Theorem 1.1 is proved. In Sect. 3, we prove the implication (1.6). Furthermore, we study the corresponding discrete version and show that there the 2 has to be replaced by $\frac{3}{2}$. Finally, we give an elementary alternative proof for strongly continuous semigroups. Throughout the paper, we use standard notation, such as $\sigma(A)$ and $\rho(A)$ for the spectrum and resolvent set of the operator A, respectively. Furthermore, for $\lambda \in \rho(A)$, $R(\lambda, A)$ denotes $(\lambda I - A)^{-1}$.

2. The zero-two law at the origin

In this section, we prove that for a strongly continuous cosine family C on the Banach space X Theorem 1.1 holds; i.e.

$$\limsup_{t\to 0^+} \|C(t) - I\| < 2 \quad \text{implies that} \quad \limsup_{t\to 0^+} \|C(t) - I\| = 0.$$

However, before we do so, we first recall the definition of a strongly continuous cosine family. For more information, we refer to [2] or [7].

DEFINITION 2.1. A family $C = (C(t))_{t\in\mathbb{R}}$ of bounded linear operators on X is called a *cosine family* when the following two conditions hold

1. $C(0) = I$, and
2. For all $t, s \in \mathbb{R}$ there holds

$$2C(t)C(s) = C(t+s) + C(t-s). \tag{2.1}$$

It is defined to be *strongly continuous*, if for all $x \in X$ and all $t \in \mathbb{R}$, we have

$$\lim_{h \to 0} C(t+h)x = C(t)x.$$

Similar as for strongly continuous semigroups, we can define the infinitesimal generator.

DEFINITION 2.2. Let C be a strongly continuous cosine family; then, the *infinitesimal generator A* is defined as

$$Ax = \lim_{t \to 0} \frac{2(T(t)x - x)}{t^2}$$

with its domain consisting of those $x \in X$ for which this limit exists.

This infinitesimal generator is a closed, densely defined operator. For the proof of Theorem 1.1, the following well-known estimates, which can be found in [7, Lemma 5.5 and 5.6], are needed.

LEMMA 2.3. *Let C be a strongly continuous cosine family with generator A. Then, there exists $\omega \geq 0$ and $M \geq 1$ such that*

$$\|C(t)\| \leq Me^{\omega t} \qquad \forall t \geq 0. \tag{2.2}$$

Furthermore, for $\operatorname{Re}\lambda > \omega$ *we have* $\lambda^2 \in \rho(A)$ *and*

$$\|\lambda^2 R(\lambda^2, A)\| \leq M \cdot \frac{|\lambda|}{\operatorname{Re}\lambda - \omega}. \tag{2.3}$$

Hence, the above lemma shows that the spectrum of A must lie within the parabola $\{s \in \mathbb{C} \mid s = \lambda^2 \text{ with } \operatorname{Re}\lambda = \omega\}$. To study the spectral properties of the points within this parabola, we use the following lemma.

LEMMA 2.4. *Let C be a strongly continuous cosine family on the Banach space X and let A be its generator. Then, for $\lambda \in \mathbb{C}$ and $s \in \mathbb{R}$ there holds*

1. *$S(\lambda, s)$ defined by*

$$S(\lambda, s)x = \int_0^s \sinh(\lambda(s-t))C(t)x\, dt, \quad x \in X, \tag{2.4}$$

   is a linear and bounded operator on X and its norm satisfies

$$\|S(\lambda, s)\| \leq \sup_{t \in [0,|s|]} \|C(t)\| \cdot \frac{\sinh(|s| \operatorname{Re}\lambda)}{\operatorname{Re}\lambda}. \tag{2.5}$$

2. *For $x \in X$ we have $S(\lambda, s)x \in D(A)$,*

$$(\lambda^2 I - A)S(\lambda, s)x = \lambda(\cosh(\lambda s)I - C(s))x. \tag{2.6}$$

   Furthermore, $S(\lambda, s)A \subset AS(\lambda, s)$.

3. *The bounded operators* $S(\lambda, s)$ *and* $C(s)x - \cosh(\lambda s)I$ *commute.*
4. *If* $\lambda \neq 0$ *and* $\cosh(\lambda s) \in \rho(C(s))$*, then* $\lambda^2 \in \rho(A)$ *and*

$$\begin{aligned}\|R(\lambda^2, A)\| &\leq \frac{1}{|\lambda|} \cdot \|S(\lambda, s)\| \cdot \|R(\cosh(\lambda s), C(s))\| \\ &\leq \sup_{t\in[0,|s|]} \|C(t)\| \cdot \frac{2|s|e^{|s \operatorname{Re}\lambda|}}{|\lambda|} \cdot \|R(\cosh(\lambda s), C(s))\|. \end{aligned} \tag{2.7}$$

Proof. We begin by showing item 1. Since the cosine family is strongly continuous, the integral in (2.4) is well defined. Hence $S(\lambda, s)$ is well defined and linear. For the estimate (2.5), we consider

$$\begin{aligned}\|S(\lambda, s)x\| &\leq \sup_{t\in[0,|s|]} \|C(t)\| \cdot \|x\| \cdot \int_0^{|s|} |\sinh(\lambda t)|\, dt \\ &= \sup_{t\in[0,|s|]} \|C(t)\| \cdot \|x\| \cdot \frac{1}{2}\int_0^{|s|} |e^{\lambda t} - e^{-\lambda t}|\, dt \\ &\leq \sup_{t\in[0,|s|]} \|C(t)\| \cdot \|x\| \cdot \frac{e^{|s|\operatorname{Re}\lambda} - e^{-|s|\operatorname{Re}\lambda}}{2\operatorname{Re}\lambda}.\end{aligned}$$

By definition, the last fraction equals $\frac{\sinh(|s|\operatorname{Re}\lambda)}{\operatorname{Re}\lambda}$, and so the inequality (2.5) is shown.
Item 2. See [10, Lemma 4].
Item 3. This is clear, since $C(t)$ and $C(s)$ commute for $s, t \in \mathbb{R}$.
Item 4. We define the bounded operator

$$B = \frac{1}{\lambda} S(\lambda, s) R(\cosh(\lambda s), C(s)).$$

By item 2., we see that $(\lambda^2 I - A)B = I$. By item 3., we get that $B = \frac{1}{\lambda} R(\cosh(\lambda s), C(s))S(\lambda, s)$. Thus, again by 2., $B(\lambda^2 I - A)x = x$ for $x \in D(A)$. Hence, $\lambda^2 \in \rho(A)$ and the first inequality of (2.7) follows. By using the power series of the exponential function, it is easy to see that $\frac{\sinh(|s|\operatorname{Re}\lambda)}{\operatorname{Re}\lambda} \leq 2|s|e^{|s\operatorname{Re}\lambda|}$. Combining this with (2.5) gives the second inequality in (2.7). □

With the use of the above lemma, we show that the spectrum of A is contained in the intersection of a ball and a parabola, provided that (1.4) holds, i.e. provided $\limsup_{t\to 0^+} \|C(t) - I\| < 2$.

LEMMA 2.5. *Let* C *be a strongly continuous cosine family on the Banach space* X *with generator* A*. Assume that there exists* $c > 0$ *such that*

$$\limsup_{t\to 0^+} \|C(t) - I\| < c < 2. \tag{2.8}$$

Then, there exists $M_c, r_c > 0$ *and* $\phi_c \in (0, \frac{\pi}{2})$ *such that*

$$\mathcal{R}_c := \left\{\lambda^2 \mid \lambda \in \mathbb{C}, |\lambda| > r_c, |\arg(\lambda)| \in \left(\phi_c, \frac{\pi}{2}\right]\right\} \subset \rho(A), \tag{2.9}$$

and

$$\forall \mu \in \mathcal{R}_c \qquad \|\mu R(\mu, A)\| \leq M_c. \tag{2.10}$$

Proof. First, we note that by (2.8) we have the existence of a $t_0 > 0$ such that $\|C(t) - I\| < c$ for all $t \in [0, t_0)$, and by symmetry, for all $t \in (-t_0, t_0)$. Using the assumption, we find that $\frac{1}{2}\|C(t) - I\| < \frac{c}{2} < 1$, and hence, $I + \frac{1}{2}(C(t) - I) = \frac{1}{2}(C(t) + I)$ is invertible with $\|(C(t) + I)^{-1}\| < \frac{1}{2-c}$ for all $t \in (-t_0, t_0)$. In other words, $-1 \in \rho(C(t))$. By standard spectral theory, it follows that the open ball centred at -1 with radius $\|R(-1, C(t))\|^{-1}$ is included in $\rho(C(t))$. Therefore,

$$B_{\frac{2-c}{2}}(-1) \subset B_{\frac{1}{2\|R(-1,C(t))\|}}(-1) \subset \rho(C(t)) \quad \forall t \in (-t_0, t_0), \tag{2.11}$$

and by the analyticity of the resolvent, we have for $\mu \in B_{\frac{2-c}{2}}(-1)$ and $t \in (-t_0, t_0)$ that

$$\begin{aligned} \|R(\mu, C(t))\| &= \left\| \sum_{n=0}^{\infty} (\mu + 1)^n R(-1, C(t))^{n+1} \right\| \\ &\leq 2\|R(-1, C(t))\| < \frac{2}{2-c}. \end{aligned} \tag{2.12}$$

Since $\cosh(t)$ is entire and $\cosh(i\pi) = -1$, there exists an $\tilde{r} > 0$ such that

$$\cosh(B_{\tilde{r}}(i\pi)) \subset B_{\frac{2-c}{2}}(-1). \tag{2.13}$$

Let $\lambda \in \mathbb{C}$ be such that $|\arg(\lambda)| \leq \frac{\pi}{2}$. We search for $s \in \mathbb{R}$ such that $\lambda s \in B_{\tilde{r}}(i\pi)$. Let $s_\lambda = \frac{\pi \sin(\arg(\lambda))}{|\lambda|}$ be the unique element on the line $\{\lambda s : s \in \mathbb{R}\}$ which is closest to $i\pi$. We have that $|i\pi - \lambda s_\lambda| = \pi \cos(\arg(\lambda))$. Now, choose $\phi_c \in (0, \frac{\pi}{2})$ large enough such that $\pi \cos(\phi_c) < \tilde{r}$ and choose $r_c > 0$ such that $\frac{\pi}{r_c} < t_0$. Then, for all $\lambda^2 \in \mathcal{R}_c$, we have that $\lambda s_\lambda \in B_{\tilde{r}}(i\pi)$ with $s_\lambda \in (-t_0, t_0)$. By (2.13), $\cosh(\lambda s_\lambda) \in B_{\frac{2-c}{2}}(-1)$. Thus,

$$\cosh(\lambda s_\lambda) \in \rho(C(s_\lambda)), \quad \text{and} \quad \|R(\cosh(\lambda s_\lambda), C(s_\lambda))\| \leq \frac{2}{2-c}, \tag{2.14}$$

by (2.11) and (2.12). Therefore, 4. of Lemma 2.4 implies that $\lambda^2 \in \rho(A)$ and

$$\begin{aligned} \|R(\lambda^2, A)\| &\leq \sup_{t\in[0,|s_\lambda|]} \|C(t)\| \cdot \frac{2|s_\lambda| e^{|s_\lambda \operatorname{Re} \lambda|}}{|\lambda|} \cdot \|R(\cosh(\lambda s), C(s_\lambda))\| \\ &\leq \sup_{t\in[0,t_0]} \|C(t)\| \cdot \frac{2\pi e^{\pi}}{|\lambda|^2} \cdot \frac{2}{2-c} \leq \frac{M_c}{|\lambda|^2} \end{aligned}$$

for some M_c only depending on $\sup_{t\in[0,t_0]} \|C(t)\|$ and c. □

Combining the results from Lemmas 2.3 and 2.5 enables us to prove Theorem 1.1. As for semigroups, we can prove a slightly more general result.

THEOREM 2.6 (Zero-two law for cosine families). *Let C be a strongly continuous cosine family on the Banach space X. Denote by A its infinitesimal generator. Then, the following assertions are equivalent*

1. $\limsup_{t\to 0^+} \|C(t) - I\| < 2$;
2. $\limsup_{t\to 0^+} \|C(t) - I\| = 0$;
3. *A is a bounded operator.*

Proof. Trivially the second item implies the first one. If the assertion in item 3 holds, then the corresponding cosine family is given by

$$C(t) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n t^{2n}}{(2n)!}.$$

From this, the property in item 2 is easy to show. Hence, it remains to show that item 1 implies item 3.

Let c be the constant from Eq. (2.8), and let $r_c > 0$, $\phi_c \in [0, \frac{\pi}{2})$ be the constants from Lemma 2.5. By Lemma 2.3, we have that there exists $\omega' > \omega \geq 0$ such that

$$\sup_{\lambda\in R_{\omega'}\cap S_{\phi_c}} \|\lambda^2 R(\lambda^2, A)\| < \infty, \tag{2.15}$$

where $R_{\omega'} = \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \omega'\right\}$ and $S_{\phi_c} = \{\mu \in \mathbb{C} : |\arg\mu| \leq \phi_c\}$. Now, let λ such that $|\lambda| > r_c$ and $|\arg(\lambda)| \in (\phi_c, \frac{\pi}{2}]$. Thus, $\lambda^2 \in \mathcal{R}_c$, see (2.9), and so by Lemma 2.5,

$$\sup_{\lambda^2\in\mathcal{R}_c} \|\lambda^2 R(\lambda^2, A)\| < \infty. \tag{2.16}$$

Let $f(z) = z^2$. It is easy to see that the closure of $\mathbb{C}\setminus\left(\mathcal{R}_c \cup f(R_{\omega'} \cap S_{\phi_c})\right)$ is compact. Thus, (2.15) and (2.16) yield that there exists an $R > 0$ such that the spectrum $\sigma(A)$ lies within the open ball $B_R(0)$ and

$$\sup_{|\mu|>R} \|\mu R(\mu, A)\| < \infty. \tag{2.17}$$

Hence, we have that $\mu \mapsto R(\mu, A)$ has a removable singularity at ∞. Since A is closed, this implies that A is a bounded operator, [9, Theorem I.6.13], and therefore, item 3 is shown. □

3. Similar laws on $\mathbb{R}$ and $\mathbb{N}$

In the previous section, we showed that uniform estimates in a neighbourhood of zero imply additional properties. In this section, we study estimates which hold on $\mathbb{R}$, $(0, \infty)$, $\mathbb{Z}$, or $\mathbb{N}$. For $\mathbb{R}$ and $(0, \infty)$, we show that by applying a scaling trick, the results can be obtained from the already proved laws. The main theorem of this section is the following.

THEOREM 3.1. *The following assertions hold*

1. *For a semigroup T we have that (1.5) implies that $T(t) = I$ for all $t \geq 0$.*
2. *If the strongly continuous cosine family C on the Banach space X satisfies*

$$\sup_{t \geq 0} \|C(t) - I\| = r < 2 \tag{3.1}$$

then $C(t) = I$ for all t.

Proof. Since the proof of the two items is very similar, we concentrate on the second one.

For the Banach space X, we define $\ell^2(\mathbb{N}; X)$ as

$$\ell^2(\mathbb{N}; X) = \left\{ (x_n)_{n \in \mathbb{N}} \mid x_n \in X, \sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty \right\}. \tag{3.2}$$

With the norm

$$\|(x_n)\| = \sqrt{\sum_{n \in \mathbb{N}} \|x_n\|^2},$$

this is a Banach space. On this extended Banach space, we define $C_{\text{ext}}(t)$, $t \in \mathbb{R}$ as

$$C_{\text{ext}}(t)(x_n) = (C(nt)x_n). \tag{3.3}$$

Hence, it is a diagonal operator with scaled versions of C on the diagonal. By a standard argument, it follows that the cosine family C_{ext} is strongly continuous. Now we estimate the distance from this cosine family to the identity on $\ell^2(\mathbb{N}; X)$ for $t \in (0, 1]$.

$$\begin{aligned} \|C_{\text{ext}}(t) - I\|^2 &= \sup_{\|(x_n)\|=1} \|C_{\text{ext}}(t)(x_n) - (x_n)\|^2 \\ &= \sup_{\|(x_n)\|=1} \sum_{n \in \mathbb{N}} \|C(nt)x_n - x_n\|^2 \\ &\leq \sup_{\|(x_n)\|=1} \sum_{n \in \mathbb{N}} r^2 \|x_n\|^2 = r^2, \end{aligned}$$

where we have used (3.1). In particular, this implies that

$$\limsup_{t \to 0^+} \|C_{\text{ext}}(t) - I\| < 2.$$

By Theorem 2.6, we conclude that the infinitesimal generator of C_{ext} is bounded. Since $C_{\text{ext}}(t)$ is a diagonal operator, it is easy to see that its infinitesimal generator A_{ext} is diagonal as well. Furthermore, the n'th diagonal element equals nA. Since n runs to infinity, A_{ext} can only be bounded if $A = 0$. This immediately implies that $C(t) = I$ for all t. □

From the above proof, it is clear that if Theorem 2.6 would hold for non-strongly continuous cosine families, then the strong continuity assumption can be removed from item 2 in the above theorem as well.

We emphasise that for semigroups no continuity assumption was needed. As mentioned in the introduction, this can also be proved using operator algebraic result going back to Wallen [13]. In Sect. 3.2, we present an (also simple) alternative proof. However, first we study the analogue of Theorem 3.1 for discrete cosine families.

3.1. Discrete cosine families

A family of bounded operators $C = (C(n))_{n\in\mathbb{Z}}$ is called a *discrete cosine family* when $C(0) = I$ and (2.1) holds for all $t, s \in \mathbb{Z}$.

THEOREM 3.2. *If a discrete cosine family C on the Banach space X satisfies*

$$\sup_{n\in\mathbb{N}} \|C(n) - I\| = r < \frac{3}{2}, \tag{3.4}$$

then $C(n) = I$ for all n. Furthermore, there exists a discrete cosine family such that $C(n) \neq I$ for all $n \in \mathbb{N}$ and

$$\sup_{n\in\mathbb{N}} \|C(n) - I\| = \frac{3}{2}.$$

Proof. We follow closely the proof in [1]. Using Eq. (2.1), we find for $n \in \mathbb{Z}$ that

$$2\,(C(n) - I)^2 = C(2n) - I - 4(C(n) - I).$$

Hence,

$$4(C(n) - I) = C(2n) - I - 2\,(C(n) - I)^2.$$

Taking norms, we find

$$4\|C(n) - I\| \le \|C(2n) - I\| + 2\|C(n) - I\|^2. \tag{3.5}$$

Let $L := \sup_{n\in\mathbb{N}} \|C(n) - I\|$; then, (3.5) implies that

$$4L \le L + 2L^2$$

In other words, $L = 0$ or $L \ge 3/2$. By assumption, the latter does not hold, and therefore, $L = 0$, or equivalently $C(n) = I, n \ge 0$. This proves the first part of the theorem. To show that the constant 3/2 is sharp, we consider the following scalar discrete cosine family on $X = \mathbb{C}$,

$$C(n) = \cos\left(\frac{2\pi}{3}n\right), \quad n \in \mathbb{Z}.$$

It is easy to see that this family only takes the values 1 and $-\frac{1}{2}$, and thus,

$$\sup_{n\in\mathbb{N}} \|C(n) - I\| = \sup_{n\in\mathbb{N}} \left|\cos\left(\frac{2\pi}{3}n\right) - 1\right| = \frac{3}{2}. \tag{3.6}$$

Hence, we conclude that $\frac{3}{2}$ is the best possible constant in (3.4). □

3.2. An elementary proof for semigroups

We now give an elementary proof of the following result.

THEOREM 3.3. *Let T be a strongly continuous semigroup on the Banach space X, and let A denote its infinitesimal generator. If*

$$r := \sup_{t \geq 0} \|T(t) - I\| < 1, \tag{3.7}$$

then $T(t) = I$ for all $t \geq 0$.

Proof. In general, it holds that

$$T(t)x - x = A \int_0^t T(s)x \, \mathrm{d}s, \quad t > 0, x \in X. \tag{3.8}$$

For $t > 0$, let B_t denote the bounded operator $x \mapsto B_t x := \int_0^t T(s)x\mathrm{d}s$. For $x \in X$,

$$\|x - t^{-1}B_t x\| = \frac{1}{t} \left\| \int_0^t x - T(s)x \, \mathrm{d}s \right\| \leq \frac{1}{t} \int_0^t \|x - T(s)x\| \mathrm{d}s \leq r\|x\|.$$

Thus, since $r < 1$, it follows that $t^{-1}B_t$ is boundedly invertible for all $t > 0$ and

$$\|tB_t^{-1}\| \leq \frac{1}{1-r} \Leftrightarrow \|B_t^{-1}\| \leq \frac{1}{t(1-r)}. \tag{3.9}$$

By (3.8) and (3.7), we have that $\|AB_t\| \leq 1$. Thus,

$$\|A\| \leq \|B_t^{-1}\| \overset{(3.9)}{\leq} \frac{1}{t(1-r)} \quad \forall t > 0; \tag{3.10}$$

hence, $A = 0$ which concludes the proof. □

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REFERENCES

[1] W. Arendt. A $0 - 3/2$ - Law for Cosine Functions. *Ulmer Seminare, Funktionalanalysis und Evolutionsgleichungen*, 17:349–350, 2012.

[2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*, volume 96 of *Monographs in Mathematics*. Second Edition. Birkhäuser Verlag, Basel, 2011.

[3] A. Bobrowski and W. Chojnacki. Isolated points of the set of bounded cosine families, bounded semigroups, and bounded groups on a Banach space. *Studia Mathematica*, 217(3):219–241, 2013.

[4] I. Chalendar, J. Esterle and J.R. Partington. Dichotomy results for norm estimates in operator semigroups. To appear.

[5] J. Esterle. Zero-one and zero-two laws for the behavior of semigroups near the origin. In *Banach algebras and their applications*, volume 363 of *Contemp. Math.*, pages 69–79. Amer. Math. Soc., Providence, RI, 2004.

[6] S. Fackler. Regularity of semigroups via the asymptotic behaviour at zero. *Semigroup Forum*, 87:1-17, 2013.

[7] H. O. Fattorini. Ordinary differential equations in linear topological spaces. I. *J. Differential Equations*, 5:72–105, 1969.

[8] R. A. Hirschfeld. On semi-groups in Banach algebras close to the identity. *Proc. Japan. Acad.*, 44:755, 1968.

[9] T. Kato. Perturbation Theory for Linear Operators. *Classics in Mathematics.* Springer Verlag, Berlin Heidelberg New York, 1980.

[10] B. Nagy. On cosine operator functions in Banach spaces. *Acta Sci. Math. (Szeged)*, 36:281–289, 1974.

[11] F. Schwenninger and H. Zwart. Zero-two law for cosine families. *ArXiv e-prints*: 1402.1304, 2014.

[12] O. Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[13] L. J. Wallen. On the Magnitude of $x^n - 1$ in a Normed Algebra. *Proc. Amer. Soc.*, 18:956, 1967.

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