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SPECTRAL CONDITIONS IMPLIED BY OBSERVABILITY*

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Abstract. It is well known that a finite-dimensional output space implies limitations on the systems properties, like observability and detectability. In this paper we extend this result for infinite-dimensional output spaces, under the condition that the output operator is relatively compact. We show that if this holds, and the system is exactly observable in finite-time, then the inverse of the infinitesimal generator must be compact. By means of an example we show that this result does not hold for exact observability in infinite-time. Using the Hautus test, we obtain spectral properties of the generator for this case. A consequence of this result is that if the system is exponentially detectable, then the unstable part of the spectrum consists of only point spectrum with finite multiplicity.

Key words. observability, relative compact output operator, Hautus test, stabilizability

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1. Introduction.

Observability and its dual notion controllability are important system theoretic properties. However, showing that a given system processes these properties can be a nontrivial task; see, for instance, the books of [10, 12, 23] and the references therein. Hence it can be very useful to have simple tests for (lack of) observability/controllability. Under the assumption that the output operator is relatively compact we derive necessary conditions for exact and final state observability. This paper fits in a long tradition of necessary conditions for exact observability/controllability. In 1975, Triggiani [21] showed that exact controllability is not possible when the input operator is compact. Two years later he proved the same result for compact semigroups [22]. Thus if the range of the input operator is finite-dimensional, then the system will never be exactly controllable provided the input operator is bounded. Since the one-dimensional wave equation is exactly controllable by boundary control [14], it is clear that this theorem does not hold for unbounded input operators with finite-dimensional range. However, having a finite-dimensional range, exact controllability gives conditions on the system operator; see [5, 6, 17]. Using Weyl characterization of the essential spectrum, in [3] it was shown that exact controllability is impossible when the input operator is relatively compact and the self-adjoint system operator has essential spectrum. For the dual notion of exact observability, we extend these results in two ways. To explain and formulate this, we first have to introduce some notation.

In this paper A denotes the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X. The domain of A is denoted by $D(A)$. By C we denote the linear operator from the domain of A to the Hilbert space Y. The operator C is

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assumed to be relatively compact with respect to A, which is equivalent to assuming that $C(rI - A)^{-1}$ is a compact operator from X to Y for some (or any) r in the resolvent set of A.

We associate to the operators A and C the system $\Sigma(A, -, C)$ as

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{1}$$

$$y(t) = Cx(t). \tag{2}$$

Since A generates the strongly continuous semigroup $(T(t))_{t\geq 0}$, we have that the first equation possesses the unique solution $x(t) = T(t)x_0$. For $x_0 \in D(A)$, the output $y(t)$ is given by $CT(t)x_0$. If this map can be extended to a bounded map from X to $L^2((0, t_f); Y)$, then C is said to be an *admissible* output operator for the semigroup $(T(t))_{t\geq 0}$. We denote this extended map by $\mathcal{O}$. Thus if C is admissible, then there exists an $M_f > 0$ such that for all $x_0 \in X$

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt =: \|\mathcal{O}x_0\|^2_{L^2(0,t_f)} \leq M_f \|x_0\|^2. \tag{3}$$

Using the semigroup property it is easy to show that the boundedness of the observability map is independent of t_f, i.e., if (3) holds, then for any $t_f > 0$, there exists an $\tilde{M}_f > 0$ such that

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt \leq \tilde{M}_f \|x_0\|^2.$$

The system $\Sigma(A, -, C)$ is *exactly observable in finite-time* if there exist a $t_f > 0$ and an $m_f > 0$ such that

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt \geq m_f \|x_0\|^2, \qquad x_0 \in D(A). \tag{4}$$

Note that for this definition we did not assume that C is admissible.

Our first result shows that admissibility and exact observability imply that relative compactness of the output operator is equivalent to compactness of the inverse of A.

THEOREM 1.1. *If C is an admissible output operator and if $\Sigma(A, -, C)$ is exactly observable in finite-time, then $C(rI - A)^{-1}$ is compact if and only if $(rI - A)^{-1}$ is compact.*

Having a compact inverse implies that A can only have point spectrum. If the output space is finite-dimensional, then $C(rI - A)^{-1}$ is compact if and only if it is bounded. For finite-dimensional output spaces and a diagonal A, admissibility and exact observability imply that the spectrum of A, $\{\lambda_n\}$ satisfies (see [17])

$$\sum_n \frac{\mathrm{Re}(\lambda_n)}{1 + |\lambda_n|^2} < \infty.$$

This stronger property of the spectrum no longer holds if the output space is infinite-dimensional. For instance, let A be the generator of a group, and let it have a compact resolvent. Then the identity operator is relatively compact, and the system $\Sigma(A, -, I)$ is exactly observable. Thus, in general, no extra information concerning the spectrum and eigenvectors is possible.

It is well known that if A generates an unitary group, then A is skew-adjoint. Thus for these generators exact observability by a relatively compact output operator is possible only when A has an orthonormal basis of eigenvectors. From [4] it is known that any generator A of a group is (similar to) a bounded perturbation of a skew-adjoint operator. Weyl's theorem implies that if the resolvent of this generator is compact, so is the resolvent of the skew-adjoint operator. By the previous remark, we know that this possesses an orthonormal basis of eigenvectors. Under very mild conditions, this implies that A will also have a Riesz basis of eigenvectors; see [24, 26].

We remark that the above theorem does not hold if the system $\Sigma(A, -, C)$ is only e xactly observable on infinite-time, i.e., when $t_f = \infty$ is (4); see Example 4.2.

It is known that exact observability is a very strong property for a system. Weaker properties are final state observability or exponential detectability. For finite-dimensional systems observability and detectability can be characterized via the Hautus test on some domain in $\mathbb{C}$. For our system (1) we can easily define the Hautus test.

Definition 1.2. *Let Ω be a domain in $\mathbb{C}$. The system $\Sigma(A, -, C)$ satisfies the* Hautus test on Ω *if for all $s \in \Omega$ there exists an $m_s > 0$ such that for all $x \in D(A)$ the following inequality holds:*

$$\|(sI - A)x\|^2 + \|Cx\|^2 \geq m_s \|x\|^2. \tag{5}$$

The Hautus test for infinite-dimensional systems was introduced in [19], where $m_s = \frac{M}{\mathrm{Re}(s)}$. In general this test is not equivalent to exact observability [7], but there are many situations of practical interest for which the equivalence holds; see [6, 8, 13, 25]. It is not hard to show (see Theorem 3.2) that final state observability and exponential detectability imply that (5) holds for $\Omega = \mathbb{C}$ and $\Omega \supset \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$, respectively.

The Hautus test on Ω together with the relative compactness of C implies conditions on the spectrum of A inside Ω. Recall that $\lambda \in \mathbb{C}$ is an element of the approximate point spectrum if there exists a sequence $x_n \in D(A)$, with $\|x_n\| = 1$, and $(\lambda I - A)x_n \to 0$.

Theorem 1.3. *Let C be relatively compact. If $\Sigma(A, -, C)$ satisfies the Hautus test on Ω, then any point $\lambda \in \Omega$ in the approximate point spectrum of A lies in the point spectrum. Moreover, it is an eigenvalue with finite multiplicity. Furthermore, for any $\lambda \in \Omega$ the range of $(\lambda I - A)$ is closed.*

There are some easy consequences of this result.

Since any point on the boundary of the spectrum lies in the approximate point spectrum (see [15, Theorem 0.7]), we see that the theorem implies that the boundary of the spectrum (in Ω) must be part of the point spectrum.

If $\Sigma(A, -, C)$ is exactly observable and C is relatively compact, then by Theorem 1.1 A^{-1} is compact, and so the spectrum of A is countable and cannot contain residual spectrum. Since the Hautus test is weaker than the condition of exact observability, it is a natural question whether these properties of the spectrum still hold. Theorem 1.5 of [8] gives that the Hautus test does not imply that the residual spectrum is empty; see also Example 4.3. Furthermore, we have that the countability of the spectrum does not need to hold either; see Examples 4.2 and 4.3.

In Theorem 1.3 we have that the boundary of the spectrum can only be point spectrum. However, this does not imply that the spectrum is isolated; see Example 4.2. If the spectrum can be decomposed into two parts, then the bounded part is isolated.

Theorem 1.4. *Let the system $\Sigma(A, -, C)$ satisfy the Hautus test on Ω, and let C be relatively compact. If the spectrum of A can be decomposed into two parts which* *are separable by a closed Jordan curve Γ and $\Gamma \subset \Omega$, then the spectrum inside the Jordan curve consists of only finitely many eigenvalues with finite multiplicity.*

This theorem implies that generators A with an accumulation point in their spectrum cannot be final state observable. An example of such a system is a homogeneous beam with viscous damping; see [1].

Since controllability and observability are dual notions, it is clear that results proved in this paper for observability/detectability of the system $\Sigma(A, -, C)$ imply similar results for the controllability/stabilizability of the system $\Sigma(A^*, C^*, -)$.

2. Compactness of CA^{-1} and A^{-1}. The operator C is relatively bounded with respect to A if there exists positive constants α and β such that for all $x_0 \in D(A)$

$$\|Cx_0\| \leq \alpha\|x_0\| + \beta\|Ax_0\|. \tag{6}$$

The infinum over all $\beta > 0$, such that (6) holds for some α, is called the *relative bound* of C; see [9]. It is easy to see that C is relatively bounded with respect to A if and only if it is relatively bounded with respect to $(rI - A)$. Hence, without loss of generality, we may assume that A is invertible.

LEMMA 2.1. *Let A^{-1} be a compact operator on X, and let C be a relatively bounded operator with relative bound zero; then CA^{-1} is a compact operator from X to Y.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Since A^{-1} is a compact operator, there exits a subsequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ such that $A^{-1}\tilde{x}_n$ converges. Choose an $\varepsilon > 0$. Since the relative bound is zero, there exists an $\alpha > 0$ such that for all $x \in D(A)$

$$\|Cx\| \leq \alpha\|x\| + \varepsilon\|Ax\|.$$

Taking $x = A^{-1}\tilde{x}_n - A^{-1}\tilde{x}_m$, we find

$$\|CA^{-1}\tilde{x}_n - CA^{-1}\tilde{x}_m\| \leq \alpha\|A^{-1}(\tilde{x}_n - \tilde{x}_m)\| + \varepsilon\|\tilde{x}_n - \tilde{x}_m\|. \tag{7}$$

Since $A^{-1}\tilde{x}_n$ is converging, there exists N_ε such that, for all $n, m > N_\varepsilon$, we have $\alpha\|A^{-1}(\tilde{x}_n - \tilde{x}_m)\| \leq \varepsilon$. Combining this with (7) gives, for $n, m > N_\varepsilon$,

$$\|CA^{-1}\tilde{x}_n - CA^{-1}\tilde{x}_m\| \leq \varepsilon + 2M\varepsilon, \tag{8}$$

where $M = \sup_n \|\tilde{x}_n\|$. Thus $CA^{-1}\tilde{x}_n$ is a Cauchy sequence in the Hilbert space Y, and thus this sequence converges. Concluding, we see that for every bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ the sequences $\{CA^{-1}x_n\}_{n\in\mathbb{N}}$ has a converging subsequence, and thus CA^{-1} is a compact operator from X to Y. $\square$

Remark 2.2. If the relative bound is not equal to zero, then the above result does not hold. A simple counter example is $C = \nu A$, $\nu > 0$. For this choice $CA^{-1} = \nu I$, which is a noncompact operator on an infinite-dimensional space.

COROLLARY 2.3. *If $\|C(sI - A)^{-1}\| \to 0$ as $s \to \infty$, then the relative bound of C is zero, and thus CA^{-1} is compact if A^{-1} is compact.*

Proof. We need only show that the relative bound is zero; the other assertion follows from the previous lemma. For $x \in D(A)$, we have

$$Cx = sC(sI - A)^{-1}x - C(sI - A)^{-1}Ax.$$

Thus

$$\|Cx\| \leq \|sC(sI - A)^{-1}\|\|x\| + \|C(sI - A)^{-1}\|\|Ax\|.$$

Since $\|C(sI-A)^{-1}\| \to 0$ as $s \to \infty$, we see that the relative bound of C is zero. ∎

Under the assumption that C is admissible and $\Sigma(A,-,C)$ is exactly observable in finite-time, the converse of Lemma 2.1 holds. To prove this we need the following lemma.

LEMMA 2.4. *If C is admissible, i.e.,* (3) *holds, then for* $0 \le a \le b \le t_f$ *the following inequality holds:*

$$2\|CA^{-1}T(\cdot)x_0\|_{L^2(a,b)} \le (b-a)\sqrt{M_f}\|x_0\| + \sqrt{(b-a)^2 M_f\|x_0\|^2 + 4\sqrt{b-a}\sqrt{M_f}\|A^{-1}x_0\|\|CA^{-1}T(b)x_0\|}, \tag{9}$$

where $\|f(\cdot)\|_{L^2(a,b)} = \sqrt{\int_a^b \|f(t)\|^2 dt}$.

Proof. Integrating by parts gives the following equality:

$$\int_a^b \|CA^{-1}T(t)x_0\|^2 dt = \left\langle \int_a^b CA^{-1}T(\tau)x_0 d\tau, CA^{-1}T(b)x_0 \right\rangle - \int_a^b \left\langle \int_a^t CA^{-1}T(\tau)x_0 d\tau, CT(t)x_0 \right\rangle dt. \tag{10}$$

Furthermore, using Cauchy–Schwarz and (3), we find that

$$\left\| \int_a^b CA^{-1}T(t)x_0 dt \right\| \le \int_a^b \|CA^{-1}T(t)x_0\| dt \le \sqrt{b-a}\sqrt{M_f}\|A^{-1}x_0\|. \tag{11}$$

Since the L^2-norm of the convolution of f and g is less than or equal to the L^1-norm of f times the L^2-norm of g (see, e.g., [2, Lemma A.6.6]), we have

$$\left\| \int_a^t CA^{-1}T(\tau)x_0 d\tau \right\|_{L^2(a,b)} \le (b-a)\|CA^{-1}T(\cdot)x_0\|_{L^2(a,b)}. \tag{12}$$

Substituting (11) and (12) in (10), we find that

$$\begin{aligned}
\|CA^{-1}T(\cdot)x_0\|^2_{L^2(a,b)} &= \int_a^b \|CA^{-1}T(t)x_0\|^2 dt \\
&\le \sqrt{b-a}\sqrt{M_f}\|A^{-1}x_0\|\|CA^{-1}T(b)x_0\| \\
&\quad + \left\| \int_a^t CA^{-1}T(\tau)x_0 d\tau \right\|_{L^2(a,b)} \|CT(\cdot)x_0\|_{L^2(a,b)} \\
&\le \sqrt{b-a}\sqrt{M_f}\|A^{-1}x_0\|\|CA^{-1}T(b)x_0\| \\
&\quad + (b-a)\|CA^{-1}T(\cdot)x_0\|_{L^2(a,b)}\sqrt{M_f}\|x_0\|.
\end{aligned} \tag{13}$$

The above is a quadratic inequality in the L^2-norm of $CA^{-1}T(\cdot)x_0$, and thus this norm has to be less than the positive zero of the corresponding quadratic equality. This gives

$$\|CA^{-1}T(\cdot)x_0\|_{L^2(a,b)} \le \frac{p+\sqrt{p^2+4q}}{2},$$

where $p = (b-a)\sqrt{M_f}\|x_0\|$ and $q = \sqrt{b-a}\sqrt{M_f}\|A^{-1}x_0\|\|CA^{-1}T(b)x_0\|$. Substituting these expressions in the above inequality gives (9). ∎

Now we can prove Theorem 1.1. It is easy to see if $\Sigma(A,-,C)$ is admissible and/or exactly observable in finite-time, then the same holds for $\Sigma(A+rI,-,C)$, $r \in \mathbb{C}$. Hence, without loss of generality, we may assume that A is boundedly invertible. Thus we prove the theorem for $r=0$.

Proof of Theorem 1.1. The if-part follows from Corollary 2.3, since the admissibility implies that $\|C(sI-A)^{-1}\| \leq \frac{M}{\sqrt{s}}$ for s real and sufficiently large; see [23, Theorem 4.3.7].

The proof of the reverse implication consists of two steps. In the first step we show that the mapping $\mathcal{O}_{-1} := x_0 \mapsto CA^{-1}T(\cdot)x_0$ is a compact linear operator from X to $L^2((0,t_f);Y)$. In the second step we prove the compactness of A^{-1}.

Step 1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence with bound M_1. We show that there exists a subsequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ such that $CA^{-1}T(\cdot)\tilde{x}_n$ converges in $L^2((0,t_f);Y)$.

By induction, we first show that there exist (sub)sequences $\{x_{n,K}\}_{n\in\mathbb{N}}$ such that $\{x_{n,K}\}_{n\in\mathbb{N}} \subset \{x_{n,K-1}\}_{n\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$ and

$$\int_0^{t_f} \|CA^{-1}T(t)(x_{n,K}-x_{m,K})\|^2 dt \leq \frac{M}{K}, \qquad n,m \geq 1, \tag{14}$$

with M independent of K.

Let $K \in \mathbb{N}$ and suppose that we have constructed the subsequence $\{x_{n,K-1}\}_{n\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$. We define $\varepsilon_K = t_f/K$, and $t_k = k\varepsilon_K$, $k=0,\dots,K$. Since CA^{-1} is compact, the operators $CA^{-1}T(t_k), k=0,\dots,K$, are compact as well. Hence it is possible to choose a subsequence $\{x_{n,K}\}_{n\in\mathbb{N}}$ of $\{x_{n,K-1}\}$ such that for all $n,m \geq 1$

$$\|CA^{-1}T(t_k)(x_{n,K}-x_{m,K})\| \leq \varepsilon_K^{3/2}. \tag{15}$$

Substituting this in (9), we find that

$$\begin{aligned} \|CA^{-1}T(\cdot)(x_{n,K}-x_{m,K})\|_{L^2(t_{k-1},t_k)} \leq{}& \varepsilon_K\sqrt{M_f}M_1 \\ &+\sqrt{\varepsilon_K^2 M_f M_1^2 + \varepsilon_K^2\sqrt{M_f}2\|A^{-1}\|M_1}. \end{aligned} \tag{16}$$

This implies that there exists an M_2 independent of K such that

$$\|CA^{-1}T(\cdot)(x_{n,K}-x_{m,K})\|_{L^2(t_{k-1},t_k)} \leq M_2\varepsilon_K. \tag{17}$$

Hence we see that for all $n,m \geq 1$ there holds

$$\begin{aligned} \int_0^{t_f} \|CA^{-1}T(t)(x_{n,K}-x_{m,K})\|^2 dt &= \sum_{k=1}^{K}\int_{t_{k-1}}^{t_k} \|CA^{-1}T(t)(x_{n,K}-x_{m,K})\|^2 dt \\ &\leq \sum_{k=1}^{K} M_2^2\varepsilon_K^2 = \frac{M_2^2 t_f^2}{K}. \end{aligned} \tag{18}$$

This proves (14).

Next we choose the subsequence $\{\tilde{x}_n\}$ as $\tilde{x}_n = x_{n,n}$. Let $\varepsilon > 0$; then there exists a K such that $1/K \leq \varepsilon$. Furthermore, by the construction of our subsequences, for $n \geq K$, there exists a n_K such that $\tilde{x}_n = x_{n_K,K}$. Hence for $n,m \geq K$, we have by (14) that

$$\int_0^{t_f} \|CA^{-1}T(t)(\tilde{x}_n-\tilde{x}_m)\|^2 dt \leq \frac{M}{K} \leq M\varepsilon.$$

Thus $CA^{-1}T(\cdot)\tilde{x}_n$ is a Cauchy sequence in the Hilbert space $L^2((0,t_f);Y)$, and hence is converging.

Concluding, we see that the operator $\mathcal{O}_{-1} := x_0 \mapsto CA^{-1}T(\cdot)x_0$ is a compact operator from X to $L^2((0,t_f);Y)$.

Step 2. By the exact observability, we know that the mapping $\mathcal{O} : X \mapsto L^2((0,t_f);Y)$ is left-invertible; see (3) and (4). Thus there exists a bounded operator Q such that

$$Q\mathcal{O} = I \qquad \text{on } X. \tag{19}$$

The operator $\mathcal{O}A^{-1}$ equals $\mathcal{O}_{-1}$, which is compact. Thus with (19)

$$Q\mathcal{O}_{-1} = Q\mathcal{O}A^{-1} = A^{-1},$$

where on the left-hand side is the product of a bounded and a compact operator. Thus A^{-1} is compact. $\square$

From the above proof, one can see that we have shown that if C is admissible and relatively compact, then the mapping $\mathcal{O}_{-1} := x_0 \mapsto CA^{-1}T(\cdot)x_0$ is a compact operator from X to $L^2((0,t_f);Y)$. The exact observability is needed to transfer this compactness to the compactness of A^{-1}.

3. Hautus test. In the previous section we showed that if $\Sigma(A,-,C)$ is exactly observable and if C is admissible, then the relative compactness of C implies that A^{-1} is compact. The standard result in functional analysis gives that the spectrum of A can only consist of point spectrum of finite multiplicity and does not contain an accumulation point. However, it is known that exact observability is a very strong property. Weaker properties are final state observable or exponential detectable. If the output space is finite-dimensional, then it is known that these weaker properties imply conditions on the spectrum of A, such as pure (unstable) point spectrum; see [18]. It is interesting to know whether the same properties on the spectrum hold if C is relatively compact. We will prove this by using the Hautus test; see Definition 1.2.

In Theorem 3.2 we show that the Hautus is a necessary condition for several system theoretic properties. Exact observability is defined in the introduction, we define *final state observability* as the existence of a $t_f > 0$ and an $m_f > 0$ such that

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt \geq m_f \|T(t_f)x_0\|^2, \qquad x_0 \in D(A). \tag{20}$$

We also define the following (weak) notion of detectability; see [20, Definition 7.4.2]. To formulate this we need the concept of an operator- and of a system node. For the precise definition, we refer the reader to [20, Definition 4.7.2].

An operator node S is a closed, densely defined mapping from $D(S) \subset X \oplus Y$ to $X \oplus Y$. Thus it can be written as $S = \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right]$. One of the properties of an operator node is that the *main operator* A_S defined by $A_S x = S_1 \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ on $D(A_S) = \{x \in X \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in D(S)\}$ is densely defined and has a nonempty resolvent set. If S is a system node, then A_S generates a C_0-semigroup on X. To any operator node we can define the *observation operator* C_S on $D(A_S)$ as $C_S x = S_2 \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$. It is also possible to define a B_S operator. Therefore, operator nodes are normally not presented as $S = \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right]$, but as $S = \left[\begin{smallmatrix} A_S \& B_S \\ C_S \& D_S \end{smallmatrix}\right]$.

DEFINITION 3.1. *The system* $\Sigma(A,-,C)$ *is* spectrally detectable *if there exist a system node* S *on* (Y,X,Y) *and an operator node* S^I *on* (Y,X,Y) *satisfying the* *following:*

1. $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *with main operator* A *and observation operator* C.
2. $S^I = \left[\begin{smallmatrix} [A\&B]^I \\ [C\&D]^I \end{smallmatrix}\right]$ *with domain* $D(S^I)$.
3. $M := \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 \\ C\&D \end{smallmatrix}\right]$ *maps* $D(S)$ *continuously onto* $D(S^I)$. *It is invertible on* $X \oplus Y$ *with inverse* $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ [C\&D]^I \end{smallmatrix}\right]$ *and* $S^I = SM$.
4. *The spectrum of* A^I *lies in the open left half-plane.*

This definition is an extension of detectability by a bounded output injection; i.e., there exists a bounded K such that $A - KC$ has only spectrum in the open left half-plane. In that case the conditions in Definition 3.1 hold by choosing

$$S = \begin{bmatrix} A & K \\ C & 0 \end{bmatrix}, \quad D(S) = D(A) \oplus Y, \quad \text{and}$$
$$S^I = \begin{bmatrix} A - KC & K \\ C & 0 \end{bmatrix}, \quad D(S^I) = D(A) \oplus Y.$$

THEOREM 3.2. *For the system* $\Sigma(A, -, C)$ *we have the following:*

1. *If* $\Sigma(A, -, C)$ *is exactly observable in finite-time, then the Hautus test holds on* $\mathbb{C}$.
2. *If* $\Sigma(A, -, C)$ *is final state observable, then the Hautus test holds on* $\mathbb{C}$.
3. *If* $\Sigma(A, -, C)$ *is spectrally detectable, then the Hautus test holds on the resolvent set of* A^I, *and thus on an open set including the closed right half-plane.*

Proof. 1. For the first part we refer to [19] for admissible C's. For nonadmissible output operators it follows from the second part.

2. If the Hautus test would not hold in $s_0 \in \mathbb{C}$, then there would exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in D(A)$, $\|x_n\| = 1$, $(s_0 I - A)x_n \to 0$, and $Cx_n \to 0$. For any $x \in X$, the following equality holds:

$$T(t)x = e^{s_0 t}x - (s_0 I - A)\int_0^t e^{s_0(t-\tau)}T(\tau)x d\tau. \tag{21}$$

Defining $z_n = (s_0 I - A)x_n$ implies that

$$T(t)x_n = e^{s_0 t}x_n - \int_0^t e^{s_0(t-\tau)}T(\tau)z_n d\tau. \tag{22}$$

From this we see that there exists an $M_2 > 0$ such that

$$\|T(t_f)x_n - e^{s_0 t_f}x_n\| \le M_2\|z_n\|.$$

Since $z_n \to 0$ and $\|x_n\| = 1$, we find that $\|T(t_f)x_n\| \to e^{\mathrm{Re}(s_0)t_f}$. In particular,

$$\inf_n \|T(t_f)x_n\| > 0. \tag{23}$$

Using (22) once more, we find

$$\begin{aligned}\int_0^{t_f} \|CT(t)x_n\|^2 dt \le 2\int_0^{t_f} |e^{s_0 t}|^2 dt\|Cx_n\|^2 \\ + 2\int_0^{t_f}\left\|C\int_0^t e^{s_0(t-\tau)}T(\tau)z_n d\tau\right\|^2 dt.\end{aligned} \tag{24}$$

Using the equality (21) with $x = z_n$, we find that

$$\int_0^t e^{-s_0\tau}T(\tau)z_n d\tau = e^{-s_0 t}A^{-1}T(t)z_n - A^{-1}z_n + s_0 A^{-1}\int_0^t e^{-s_0\tau}T(\tau)z_n d\tau.$$

Substituting this in (24) gives

$$\begin{aligned}\int_0^{t_f} \|CT(t)x_n\|^2 dt &\leq 2\int_0^{t_f} |e^{s_0 t}|^2 dt \|Cx_n\|^2 \\ &+ 2\int_0^{t_f} \left\| CA^{-1}T(t)z_n - CA^{-1}e^{s_0 t}z_n + s_0 CA^{-1}\int_0^t e^{s_0(t-\tau)}T(\tau)z_n d\tau \right\|^2 dt.\end{aligned}$$

Since $z_n \to 0$, $Cx_n \to 0$, and CA^{-1} is bounded, this converges to zero. However, from (23), we see that $T(t_f)x_n$ stays bounded away from zero. Hence the system cannot be final state observable. Note that if C is admissible, then the second term in (24) tends to zero.

3. By Lemma 7.4.4 of [20] we have that $s \in \rho(A^I)$, the resolvent set of A^I, if and only if $Q_s := \left(\left[\begin{smallmatrix} sI & 0 \\ 0 & I \end{smallmatrix}\right] - \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] \right)$ is invertible. For $x \in D(A)$, we have that $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in D(S)$, and thus

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = Q_s^{-1}Q_s \begin{bmatrix} x \\ 0 \end{bmatrix} = Q_s^{-1} \begin{bmatrix} (sI-A)x \\ -Cx \end{bmatrix}.$$

Thus

$$\|x\|^2 = \left\| Q_s^{-1} \begin{bmatrix} (sI-A)x \\ -Cx \end{bmatrix} \right\|^2 \leq \|Q_s^{-1}\|^2 \left(\|(sI-A)x\|^2 + \|Cx\|^2 \right). \tag{25}$$

Hence for all s in the resolvent set of A^I, the Hautus test holds. ∎

Next we prove that the Hautus test on Ω gives conditions on the spectrum of A which lies in Ω. For this we need the following lemma, which is inspired by Proposition 2.2 of [3].

LEMMA 3.3. *Let C be relatively compact, and let $\Sigma(A, -, C)$ satisfy the Hautus test on Ω. Moreover, let $\lambda \in \Omega$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that*

- *$\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence with values in $D(A)$, i.e., for all n we have that $x_n \in D(A)$ and $\|x_n\| \leq M_1$ for some M_1 independent of n, and*
- *$Ax_n - \lambda x_n \to z$.*

Then $\{x_n\}_{n\in\mathbb{N}}$ has a converging subsequence.

Proof. We first prove that $\{Cx_n\}_{n\in\mathbb{N}}$ possesses a converging subsequence.

Let r be in the resolvent set of A. Since $C(rI - A)^{-1}$ is compact, and $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, we have that $\{C(rI - A)^{-1}x_n\}_{n\in\mathbb{N}}$ has a converging subsequence. Hence there exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{C(rI - A)^{-1}x_n\}_{n\in\mathbb{N}}$ is converging along this subsequence. We call this sequence $\{\tilde{x}_k\}_{k\in\mathbb{N}}$.

For this sequence, we find that

$$\begin{aligned} C\tilde{x}_k - C\tilde{x}_\ell &= C(rI-A)^{-1}(rI-A)(\tilde{x}_k - \tilde{x}_\ell) \\ &= (r-\lambda)C(rI-A)^{-1}(\tilde{x}_k - \tilde{x}_\ell) + C(rI-A)^{-1}(\lambda I - A)(\tilde{x}_k - \tilde{x}_\ell). \end{aligned}$$

For $k, \ell \to \infty$, the first term converges to zero by the choice of the subsequence, whereas the second term converges to zero since $(A - \lambda I)x_n$ converges to z. Thus $\{C\tilde{x}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and hence converges. This proves the assertion.

Now we show that $\{\tilde{x}_k\}_{k\in\mathbb{N}}$ is converging.

We choose x in (5) to be $\tilde{x}_k - \tilde{x}_\ell$, and $s = \lambda$. Hence we obtain

$$\|(\lambda I - A)\tilde{x}_k - (\lambda I - A)\tilde{x}_\ell\|^2 + \|C\tilde{x}_k - C\tilde{x}_\ell\|^2 \geq m_\lambda \|\tilde{x}_k - \tilde{x}_\ell\|^2. \tag{26}$$

Since the left-hand side converges to zero, we find that the sequence $\{\tilde{x}_k\}_{k\in\mathbb{N}}$ is converging. So $\{x_n\}_{k\in\mathbb{N}}$ has a converging subsequence. This provides the proof. ∎

With this lemma, we obtain Theorem 1.3 as the first consequence of the Hautus test.

Proof of Theorem 1.3. Since λ lies in the approximate point spectrum, for all $n \in \mathbb{N}$ there exists a $x_n \in D(A)$ with norm one such that

$$\|(A - \lambda I)x_n\| \leq 1/n.$$

By the previous lemma, we conclude that this sequence must have a (strongly) converging subsequence which we denote by $\{\tilde{x}_n, n \in \mathbb{N}\}$. Hence $\tilde{x}_n \to \tilde{x}_\lambda$ for $n \to \infty$. Since

$$\|A\tilde{x}_n - A\tilde{x}_m\| \leq \|\lambda(\tilde{x}_n - \tilde{x}_m)\| + 1/n + 1/m,$$

we see that $A\tilde{x}_n$ is converging as well. Since the operator is closed, we conclude that $\tilde{x}_\lambda \in D(A)$ and $A\tilde{x}_\lambda = \lambda\tilde{x}_\lambda$. In other words, λ is an eigenvalue.

If the dimension of the kernel of $A - \lambda I$ is infinite-dimensional, then it is possible to find a sequence $\{x_n\}$ such that $\|x_n\| = 1$, $Ax_n = \lambda x_n$, and $\|x_n - x_m\| \geq 1$. By Lemma 3.3 this is impossible, and so the dimension of this kernel must be finite.

It remains to show that the range of $\lambda I - A$ is closed. We note that, without loss of generality, we may assume that λ is not an eigenvalue. If so, then we may choose as state space X modulo the eigenspace corresponding to λ.

Let $\{z_n\}_{n\in\mathbb{N}}$ be a converging sequence in the range of $\lambda I - A$, i.e., $z_n = (\lambda I - A)x_n$ for some $x_n \in D(A)$, and $z_n \to z_\infty$. We first claim that $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence. So suppose that $\{x_n\}_{n\in\mathbb{N}}$ is not a bounded sequence. Since $\{x_n\}_{n\in\mathbb{N}}$ is unbounded, the sequence $\{x_{n,2}\}_{n\in\mathbb{N}}$ defined as $x_{n,2} = \frac{x_n}{\|x_n\|}$ is a bounded sequence, and $(\lambda I - A)x_{n,2}$ converges to zero. By Lemma 3.3, we have that $\{x_{n,2}\}_{n\in\mathbb{N}}$ has a converging subsequence which we denote by $\{\tilde{x}_{n,2}\}_{n\in\mathbb{N}}$. So $\tilde{x}_{n,2} \to x_\infty$ and $(\lambda I - A)\tilde{x}_{n,2} \to 0$. Since A is a closed operator, we conclude that $(\lambda I - A)x_\infty = 0$. Thus x_∞ is an eigenvector corresponding to the eigenvalue λ. By assumption λ is not an eigenvalue, and so this vector must be zero. This is a contradiction to the fact that $\|\tilde{x}_{n,2}\| = 1$ and $\tilde{x}_{n,2} \to x_\infty$. Hence $\{x_n\}_{n\in\mathbb{N}}$ must be a bounded sequence.

Since the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded, Lemma 3.3 implies that there exists a subsequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ with $\tilde{x}_n \to x_\infty$. Combining this with $(\lambda I - A)\tilde{x}_n \to z_\infty$ and the fact that A is a closed operator, we find that $(\lambda I - A)x_\infty = z_\infty$. Hence z_∞ lies in the range of $(\lambda I - A)$. Thus the range is closed. ∎

Remark 3.4. If A is skew-adjoint and the imaginary axis is contained in Ω, then Theorem 1.3 implies that A cannot have essential spectrum; see also [3].

Furthermore, we see that for this class of generators there cannot be an accumulation point in the spectrum. Let λ be an accumulation point in the spectrum. Since the spectrum of A equals the boundary of the spectrum, we have by Theorem 1.3 that there are eigenvalues $\lambda_n \to \lambda$. Let x_n be an eigenvector of norm one associated to the eigenvalue λ_n. So the sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies the conditions of Lemma 3.3. However, since the eigenvectors with different eigenvalues are orthogonal, we have that

$\|x_n - x_m\| = \sqrt{2}$, $n \neq m$, and so $\{x_n\}_{n\in\mathbb{N}}$ does not have a converging subsequence. Concluding, there cannot be an accumulation point in the spectrum. $\square$

Proof of Theorem 1.4. Assume that the spectrum of A is the union of two parts, σ^+ and σ^-, such that Γ can be drawn so as to enclose an open set containing σ^+ in its interior and σ^- in its exterior. This splitting of the spectrum leads naturally to a splitting of the state space as $X = X^+ \oplus X^-$; see [2, Lemma 2.5.7]. The generator restricted to X^+ is bounded, and we denote it by A^+. The spectrum of this restricted operator equals σ^+. We define C^+ as $C^+ = C|_{X^+}$. Since C is relatively compact with respect to A, we find that C^+ is relatively compact with respect to A^+. Furthermore, since $\Sigma(A, -, C)$ satisfies the Hautus test on Ω, it is easy to see that $\Sigma(A^+, -, C^+)$ also satisfies the Hautus test on Ω. Combining this with the fact that the spectrum of A^+ lies in the interior of $\Gamma \subset \Omega$, we find that $\Sigma(A^+, -, C^+)$ satisfies the Hautus test on $\mathbb{C}$. Since A^+ and hence C^+ are bounded, Theorem 1.7 of [19] gives that $\Sigma(A^+, -, C^+)$ is exactly observable. Theorem 1.1 gives that $(rI - A^+)^{-1}$ is compact. This implies that

$$I_{X^+} = (rI - A^+)(rI - A^+)^{-1}$$

is compact as well, since A^+ is bounded. The identity is compact only when the space is finite-dimensional. Thus the range of the spectral projection associated with Γ is finite-dimensional, which proves the assertion. $\square$

4. Examples. Many PDEs on a bounded domain have a compact resolvent. This is a consequence of the Sobolev embedding theorem. However, there are some examples for which this does not hold. We take the following example from [16].

Example 4.1. Let $\mathfrak{O} \subset \mathbb{R}^2$ be a bounded open set with smooth boundary Γ. This boundary consists of two parts Γ_0 and Γ_1 with $\Gamma_0 \cap \Gamma_1 = \emptyset$. The model of the vibration of a thin elastic plate is given by

$$\begin{aligned}
&w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w = 0 && \text{in } \mathfrak{O} \times [0,\infty),\\
&w = \frac{\partial w}{\partial \nu} = 0 && \text{on } \Gamma_0 \times [0,\infty),\\
&\Delta w + (1-\mu)B_1 w = u_1 && \text{on } \Gamma_1 \times [0,\infty),\\
&\frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau} - \gamma\frac{\partial w_{tt}}{\partial \nu} + \frac{\partial^2 w_t}{\partial \tau^2} = u_2 && \text{on } \Gamma_1 \times [0,\infty),
\end{aligned} \tag{27}$$

where $\nu = (\nu_1, \nu_2)$ and $\tau = (\tau_1, \tau_2)$ denote the unit normal and the unit tangent vector, respectively. Furthermore, $\gamma > 0$ and $\mu \in (0, \frac{1}{2})$. The boundary operators are defined as

$$B_1 w = 2\nu_1\nu_2 \frac{\partial^2 w}{\partial \xi_1 \partial \xi_2} - \nu_1^2 \frac{\partial^2 w}{\partial \xi_2^2} - \nu_2^2 \frac{\partial^2 w}{\partial \xi_1^2}, \tag{28}$$

$$B_2 w = (\nu_1^2 - \nu_2^2)\frac{\partial^2 w}{\partial \xi_1 \partial \xi_2} + \nu_1\nu_2 \left(\frac{\partial^2 w}{\partial \xi_2^2} - \frac{\partial^2 w}{\partial \xi_1^2}\right), \tag{29}$$

where (ξ_1, ξ_2) denotes the spatial coordinate. The controls u_1 and u_2 are chosen as

$$\frac{du_1}{dt} = -u_1 - \frac{\partial w_t}{\partial \nu} \qquad \text{on } \Gamma_1 \times [0,\infty), \tag{30}$$

$$\frac{du_2}{dt} = -u_2 + w_t \qquad \text{on } \Gamma_1 \times [0,\infty). \tag{31}$$

Writing (27)–(31) as one abstract differential equation of first order gives a generator of a strongly continuous semigroup; see [16]. The inverse of the infinitesimal generator is noncompact. The noncompactness is caused by the fact that the infinitesimal generator of the subsystem (30)–(31) is minus the identity, and thus has noncompact inverse on the infinite-dimensional space $L^2(\Gamma_1)$; see [16, Proposition 2.2]. If the inputs are set to zero, then the inverse of the infinitesimal generator corresponding to the system (27)–(29) is compact; see, e.g., [11].

If we define the output operator as

$$Cw = w_t|_{\Gamma_1}, \tag{32}$$

then this is relatively compact from the domain of the generator to $L^2(\Gamma_1)$; see the proof of Proposition 2.2 in [16]. Hence we see that with this output operator the system cannot be exactly observable in finite-time.

In the following example we show that Theorem 1.1 does not hold if the system $\Sigma(A,-,C)$ is exactly observable in infinite-time. The system $\Sigma(A,-,C)$ is exactly observable in infinite-time if there exists an $m_\infty > 0$ such that for all $x_0 \in D(A)$

$$\int_0^\infty \|CT(t)x_0\|^2 dt \geq m_\infty \|x_0\|^2. \tag{33}$$

It is easy to show that if the semigroup is exponentially stable, then exact observability in infinite-time is equivalent to exact observability for some finite-time. This does not hold, in general, as can be seen in the following example. However, the main result in this example is that Theorem 1.1 does not hold if we replace exact observability in finite-time by exact observability in infinite-time.

Example 4.2. As the Hilbert space X, we take $X = L^2(0,\infty)$. Furthermore, we take

$$Af = \frac{df}{d\xi}, \tag{34}$$

with domain

(35)

$$D(A) = \left\{ f \in L^2(0,\infty) \mid f \text{ is absolutely continuous (abs. cont.), and } \frac{df}{d\xi} \in L^2(0,\infty) \right\}.$$

As an output operator, we take

$$Cf = f(0), \qquad f \in D(A). \tag{36}$$

It is well known that A is the infinitesimal generator of the left-shift, i.e.,

$$(T(t)f)(\xi) = f(t+\xi). \tag{37}$$

Using this we have that $CT(t)f = f(t)$. Thus

$$\int_0^\infty \|CT(t)f\|^2 dt = \int_0^\infty \|f(t)\|^2 dt = \|f\|^2_{L^2(0,\infty)}. \tag{38}$$

This proves that C is admissible, and that $\Sigma(A,-,C)$ is exactly observable in infinite-time. Using (37), it is easy to see that $\Sigma(A,-,C)$ is not exactly observable in finite-time. Since C is relatively bounded with respect to A and has one-dimensional range, we have that C is relatively compact. The inverse, $(rI-A)^{-1}$, is given by

$$\left((rI-A)^{-1}f\right)(\xi) = \int_\xi^\infty e^{r(\xi-\tau)} f(\tau) d\tau, \tag{39}$$

which is noncompact. Hence Theorem 1.1 does not hold if exact observability in finite-time is replaced by exact observability in infinite-time.

As mentioned before, exact observability is related to the Hautus test. For this example, we show that the Hautus test does not hold on $\mathbb{C}$, but only on $\mathbb{C}^- \cup \mathbb{C}^+$, where $\mathbb{C}^- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}, \mathbb{C}^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$.

Since the spectrum of A is the closure of $\mathbb{C}^-$, the Hautus test holds trivially on $\mathbb{C}^+$. For $s = r + i\omega \in \mathbb{C}^-$, we have that

$$\|(sI - A)f\|^2 = \|(i\omega I - A)f\|^2 + r^2\|f\|^2 + r\|f(0)\|^2. \tag{40}$$

Thus

$$\|(sI - A)f\|^2 - \mathrm{Re}(s)\|Cf\|^2 \geq \mathrm{Re}(s)^2\|f\|^2.$$

Since the real part of s is negative, we see that the Hautus test (5) holds. It remains to show that the Hautus test does not hold for s on the imaginary axis. Fix $\omega \in \mathbb{R}$ and define for $r < 0$, $f_r(\xi) = \sqrt{-2r}e^{s\xi}$, $s = r + i\omega$. It is easy to see that $\|f_r\| = 1$, and $Cf_r \to 0$ if $r \to 0$. Furthermore, from (40), we have that

$$0 = \|(i\omega I - A)f_r\|^2 + r^2 - 2r^2.$$

Thus $\|(i\omega I - A)f_r\|^2 \to 0$ for $r \to 0$. Combining this with $\|f_r\| = 1$ and $Cf_r \to 0$ if $r \to 0$ gives that the Hautus cannot hold for $s = i\omega$.

Since the Hautus test holds on $\mathbb{C}^-$, we can apply Theorem 1.3 for this set. This theorem gives that any point in the approximate point spectrum will be an eigenvalue with finite multiplicity. For $s \in \mathbb{C}^-$, it is easy to see that $e^{s\xi}$ is the eigenfunction of A. Note that we cannot apply Theorem 1.4 because we cannot split the spectrum.

For the above example, the spectrum contained in $\mathbb{C}^-$ consisted only of point spectrum. In [8] an example was given of a strongly stable semigroup for which the Hautus test holds with $C = 0$. Without the assumption of strong stability, such an example is easy to construct.

Example 4.3. In this example we take A to be the dual of the generator from the previous example, i.e.,

$$Af = -\frac{df}{d\xi}, \tag{41}$$

with domain

$$D(A) = \left\{ f \in L^2(0,\infty) \mid f \text{ is abs. cont.}, \frac{df}{d\xi} \in L^2(0,\infty) \text{ and } f(0) = 0 \right\}.$$

Similar to (40), we have that for $s = r + i\omega \in \mathbb{C}^-$ there holds

$$\|(sI - A)f\|^2 = \|(i\omega I - A)f\|^2 + r^2\|f\|^2. \tag{42}$$

Thus for $C = 0$, the Hautus test is satisfied on $\mathbb{C}^-$. From the above equation, we also conclude that any $s \in \mathbb{C}^-$ cannot be an eigenvalue of A. It is well known that the spectrum of A in the open left half-plane is contained in the residual spectrum. Hence in this example the point spectrum is empty, but the residual spectrum is not.

Under the assumption that A satisfies the Hautus test (for some relatively compact C), we see from Theorem 1.3 that the spectrum of A (contained in Ω) has some properties in common with the spectrum of a compact operator, such as the property that an element in the point spectrum has finite multiplicity and $sI - A$ has closed range. However, from the previous two examples we see that there can be differences. For instance, the Fredholm alternative does not need to hold for A.

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