

Tutte sets in graphs II: The complexity of finding maximum Tutte sets[☆]

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Abstract

A well-known formula of Tutte and Berge expresses the size of a maximum matching in a graph G in terms of what is usually called the deficiency. A subset X of $V(G)$ for which this deficiency is attained is called a Tutte set of G . While much is known about maximum matchings, less is known about the structure of Tutte sets. We explored the structural aspects of Tutte sets in another paper. Here, we consider the algorithmic complexity of finding Tutte sets in a graph. We first give two polynomial algorithms for finding a maximal Tutte set. We then consider the complexity of finding a maximum Tutte set, and show it is NP-hard for general graphs, as well as for several interesting restricted classes such as planar graphs. By contrast, we show we can find maximum Tutte sets in polynomial time for graphs of level 0 or 1, elementary graphs, and 1-tough graphs.

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1. Introduction

In this paper we consider only simple graphs. Our terminology will be standard. Good references for any undefined terms are [9,13].

Given a graph G , define the *deficiency* of G , denoted by $\text{def}(G)$, as the number of vertices unmatched in a maximum matching of G . Thus the size of a maximum matching in G may be expressed as $1/2(|V(G)| - \text{def}(G))$ edges.

Let $\omega(G)$ (resp., $\omega_0(G)$, $\omega_e(G)$) denote the number of components (resp., odd, even components) of G . An important result in matching theory is due to Tutte [12].

Theorem 1.1 (Tutte's theorem). *A graph G has a perfect matching if and only if $\omega_0(G - X) \leq |X|$ for all $X \subseteq V(G)$.*

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In 1958, Berge [7] extended Tutte's theorem to give the exact size of a maximum matching in a graph G . In particular, he proved that $\text{def}(G) = \max_{X \subseteq V(G)} \{\omega_0(G - X) - |X|\}$, where the maximum is taken over all proper subsets of $V(G)$. Thus we have:

Theorem 1.2 (Tutte–Berge formula). *The maximum size of a matching in a graph G is $1/2(|V(G)| - \max_{X \subseteq V(G)} \{\omega_0(G - X) - |X|\})$.*

Motivated by the above formula, we define a *Tutte set* in G to be a subset $X \subseteq V(G)$ such that $\omega_0(G - X) - |X| = \text{def}(G)$. Another standard term for Tutte set in the literature is *barrier* (see [11]).

In [5], we studied the structure of maximal Tutte sets in graphs. In this note we consider the algorithmic complexity of finding maximal and maximum Tutte sets in graphs.

We begin with some necessary definitions and theorems from [5].

Let G be a graph. The Edmonds–Gallai decomposition of G is the partition $D_G \cup A_G \cup C_G$ of $V(G)$ given by

- $D_G = \{v \in V(G) \mid \text{some maximum matching in } G \text{ fails to match } v\}$,
- $A_G = \{u \in V(G) - D_G \mid u \text{ is adjacent to a vertex in } D_G\}$,
- $C_G = V(G) - D_G - A_G$.

In what follows, we omit the subscript G , if understood.

In particular, if G contains a perfect matching, then $D = A = \emptyset$, and $G[C] = G$. The Edmonds–Gallai decomposition of a graph can be obtained efficiently by using Edmonds' matching algorithm [8].

Before stating the Edmonds–Gallai structure theorem, we need the following definitions. A graph H is said to be *factor-critical* if deleting any vertex from H results in a graph with a perfect matching. Such a matching in H is called *near-perfect*. The primary importance of the Edmonds–Gallai decomposition is contained in the following theorem.

Theorem 1.3 (Edmonds–Gallai structure theorem). *Let G be a graph and $D \cup A \cup C$ be the Edmonds–Gallai decomposition of G . Then A is a Tutte set, $G[D]$ is the union of the odd components of $G - A$, each of which is factor-critical, and $G[C]$ is the union of the even components of $G - A$. Moreover, any maximum matching in G consists of*

- a perfect matching in $G[C]$;
- a near-perfect matching in every (odd) component of $G[D]$;
- an edge joining v to some vertex in D , for every $v \in A$.

The Edmonds–Gallai decomposition of G is closely related to the structure of maximal Tutte sets in G . Indeed [11], the set A is the intersection of all the maximal Tutte sets in G , and no vertex in the set D can occur in any Tutte set of G . In fact, we have (cf. [5, Theorem 3.5])

Theorem 1.4. *Let G be a graph and $X \subseteq V(G)$. Then X is a maximal Tutte set in G if and only if $X = A \cup Z$, where Z is a maximal Tutte set in $G[C]$.*

Since $G[C]$ always contains a perfect matching [11], this shows that finding maximal Tutte sets in G reduces to finding maximal Tutte sets in graphs which contain a perfect matching. In the sequel, therefore, we will focus on the complexity of finding a maximal Tutte set in a graph with a perfect matching.

In [5], we found that the study of maximal Tutte sets in a graph G with a perfect matching is greatly facilitated by introducing a related graph $D(G)$. When G contains a perfect matching, we define $D(G)$ as follows: $V(D(G)) = V(G)$, and $E(D(G)) = \{(x, y) \mid G - \{x, y\} \text{ contains a perfect matching}\}$. We call a graph H a D -graph if $H = D(G)$ for some graph G .

There is a useful alternative definition of $E(D(G))$. Let M be a perfect matching in G . We denote by $P_M[x, y]$ an M -alternating-path in G joining x and y , which begins and ends with an edge in M . Similarly, we denote by $P_M(x, y)$ an M -alternating-path in G joining x and y , which begins and ends with an edge not in M ; the M -alternating-paths $P_M[x, y)$ and $P_M(x, y]$ are defined analogously. By a theorem of Berge [6], $(x, y) \in E(D(G))$ if and only if there exists a path $P_M[x, y]$ in G . Clearly, this definition of $E(D(G))$ is independent of the choice of the perfect matching M .

A key result for this paper is the following (cf. [5, Theorem 3.4]).

Theorem 1.5. *Let G be a graph with a perfect matching and let $X \subseteq V(G)$. Then X is a maximal Tutte set in G if and only if X is a maximal independent set in $D(G)$.*

Let M be a perfect matching in G and $x \in V(G)$. We denote by x' the vertex in $V(G)$ that is matched to x under M . We note that if $(x, y) \in E(G) - M$, then $(x', y) \in E(D(G))$ since G contains the 3-path $P_M[x', y] = (x', x, y, y')$. Thus, G is isomorphic to a spanning subgraph of $D(G)$ via the mapping from $V(G)$ to $V(D(G))$ given by $x \rightarrow x'$. We denote this fact by $G \preceq D(G)$.

We define the iterated D -graphs of G recursively as follows: $D^0(G) = G$ and $D^k(G) = D(D^{k-1}(G))$ for $k \geq 1$. Since $V(D(G)) = V(G)$ and $G \preceq D(G)$, it follows that for any graph G , there exists an integer $l \geq 0$ such that $D^l(G) \cong D^{l+1}(G)$. We call the smallest such integer l the *level* of G and denote it by $\text{level}(G)$. In [5], we prove the following unexpected result.

Theorem 1.6. *For any graph G , $\text{level}(G) \leq 2$.*

The graphs G of level 0 (i.e., with $G \cong D(G)$) will be of special interest. In [5], we characterize such graphs using the following definition. Let G be a graph with a perfect matching M . We say G has the C_4 -property if and only if whenever G contains a path $P_M(x, y)$ of length 3, G also contains the edge (x, y) .

In [5] we prove the following; for convenience, we include the proof here.

Theorem 1.7. *Let G be a graph with a perfect matching. Then G has level 0 if and only if G satisfies the C_4 -property.*

Proof. Let M be a perfect matching in G .

(\Rightarrow) Suppose G contains a path $P_M(x, y)$ of length 3. If $[x, y] \in M$ we are done. Else $[x', x]oP_M(x, y)o[y, y']$ is a path $P_M[x', y']$ in G of length 5, and $(x', y') \in E(D(G))$. Since $G \cong D(G)$ via the mapping $x \rightarrow x'$, it follows that $(x, y) \in E(G)$. Thus G has the C_4 -property.

(\Leftarrow) Let $(u', v') \in E(D(G))$. It suffices to show that $(u, v) \in E(G)$. Since $(u', v') \in E(D(G))$, there exists a path $P_M[u', v']$ in G , and thus a path $P_M(u, v) = (u, a_1, a'_1, a_2, a'_2, \dots, a_r, a'_r, v)$ in G . Since G has the C_4 -property, we obtain by a simple iterative argument that G contains the edges $(u, a_2), (u, a_3), \dots, (u, a_r), (u, v)$. \square

2. Main results

We now summarize the remainder of this paper. In Section 2.1, we give two efficient algorithms to construct maximal Tutte sets in graphs. In Section 2.2, we show that it is NP-complete to find maximum Tutte sets in general graphs, and that it remains NP-complete for the class of planar graphs, k -connected graphs, and triangle-free graphs. By contrast, we show in Section 2.3 that maximum Tutte sets can be found in polynomial time for graphs having level 0 or 1, elementary graphs, and 1-tough graphs. We conclude in Section 3 with a short discussion of some open questions.

2.1. Efficient algorithms for maximal Tutte sets

We now present two efficient algorithms to find a maximal Tutte set in a graph.

The first algorithm uses Theorems 1.4 and 1.5. Given G , find the Edmonds–Gallai decomposition $D \cup A \cup C$ of $V(G)$. Construct $D(G[C])$, and find a maximal independent set Z in $D(G[C])$. By Theorems 1.4 and 1.5, Z is a maximal Tutte set in $G[C]$ and $A \cup Z$ is a maximal Tutte set in G .

The second algorithm is based directly on the Edmonds–Gallai structure theorem.

Algorithm (Maximal Tutte set in G).

Let $D_0 \cup A_0 \cup C_0$ be the Edmonds–Gallai decomposition of $V(G)$;

$X := A_0$; $G_0 := G[C_0]$; $i := 0$;

while $C_i \neq \emptyset$ **do**

 let v_i be an arbitrary vertex in C_i ;

 let $D_{i+1} \cup A_{i+1} \cup C_{i+1}$ be the Edmonds–Gallai decomposition of $V(G_i - v_i)$;

$X := X \cup A_{i+1} \cup \{v_i\}$; $G_{i+1} := G[C_i]$; $i := i + 1$;

return X

To see that the algorithm is correct, note that $E(C_i, D_i) = \emptyset$ (by definition) and $D_{i+1} \subseteq C_i$. Thus $E(D_i, D_j) = \emptyset$ for $i \neq j$. Since the components of $G[D_i]$, $i \geq 0$, are factor-critical by the Edmonds–Gallai structure theorem, it follows that all the components of $G - X = G[D_0 \cup D_1 \cup \dots]$ are also factor-critical, and so X is a maximal Tutte set in G .

It can be shown that the set X returned by the second algorithm will be a maximum Tutte set of G if and only if $v_i \in C_i$ is always selected to be a vertex which occurs in a maximum Tutte set of $G[C_i]$. But as we are about to see, selecting such a vertex is almost certainly intractable.

2.2. Finding a maximum Tutte set is NP-complete

Consider the following decision problem.

MAX TUTTE SET:

Instance. Graph G , integer $k \geq 0$.

Question. Does G contain a Tutte set X with $|X| \geq k$?

Theorem 2.1. *MAX TUTTE SET is NP-complete.*

Proof. Clearly MAX TUTTE SET \in NP, and we only show it is NP-hard. We will use a polynomial reduction from the following well-known NP-complete problem [9].

INDEPENDENT SET:

Instance. Graph G , integer $k \geq 0$.

Question. Is $\alpha(G) \geq k$?

Let (H, k) be any instance of INDEPENDENT SET, where $V(H) = \{v_1, v_2, \dots, v_n\}$. Form the graph H' by attaching a graph S_i to each vertex v_i , where S_i consists of a $C_4 : v_i, a_{i,1}, a_{i,2}, a_{i,3}, v_i$ with one chord $(a_{i,1}, a_{i,3})$. One easily checks that H' contains a perfect matching. Concerning $D(H')$, note that

1. $D(S_i)$ is complete, except for the edge $(a_{i,1}, a_{i,3})$.
2. $V(H)$ is an independent set in $D(H')$.
3. If $(v_i, v_j) \in E(H)$, there is a complete bipartite join between $D(S_i) - v_i$ and $D(S_j) - v_j$ in $D(H')$.
4. If $(v_i, v_j) \notin E(H)$, there are no edges in $D(H')$ joining $D(S_i)$ and $D(S_j)$.

Let $I = \{v_1, \dots, v_\alpha\}$ be a maximum independent set in H . Using the above observations it is easy to see that a maximum independent set in $D(H')$ consists of $\{a_{i,1}, a_{i,3} \mid 1 \leq i \leq \alpha\} \cup \{v_i \mid \alpha + 1 \leq i \leq n\}$. Letting $m(H')$ denote the cardinality of a maximum Tutte set in H' , we have by Theorem 1.5 that

$$\begin{aligned} m(H') &= \alpha(D(H')) \\ &= 2\alpha(H) + (|V(H)| - \alpha(H)) \\ &= |V(H)| + \alpha(H). \end{aligned}$$

This completes the polynomial reduction. \square

It is interesting to consider the complexity of MAX TUTTE SET for k -connected graphs where $k \geq 2$. We note that trivially INDEPENDENT SET remains NP-complete for connected graphs.

Theorem 2.2. *MAX TUTTE SET is NP-complete for k -connected graphs for any $k \geq 2$.*

Proof. The proof follows the same lines as the proof of Theorem 2.1. Given H we now construct H' as follows. For every $v_i \in V(H)$, we attach the graph S_i as shown in Fig. 1. In Fig. 1, each circle represents a set of k independent vertices, where $V_i \in S_i$ is identified with $v_i \in H$. The double edge connecting sets of vertices represents a complete bipartite join. To complete the construction of H' , we connect V_i to V_j by a complete bipartite join if and only if $(v_i, v_j) \in E(H)$.

It is easy to see that if H is connected, then H' is k -connected. Since each S_i has a perfect matching, so does H' . It can be shown that no perfect matching in H' can contain an edge joining a vertex of V_i to a vertex of V_j . Note that $D(H')$ has properties analogous to 1–4 in Theorem 2.1. Since $\{x_i\} \cup \bigcup_{k=0}^3 A_{i,2k+1}$ forms a maximum independent

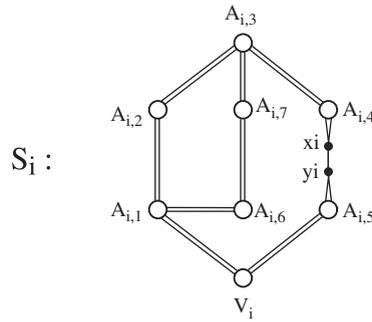


Fig. 1. Each circle represents k independent vertices, and each double edge represents a complete bipartite join.

set in $D(S_i)$, and $\{y_i\} \cup V_i$ are also independent in $D(S_i)$, we conclude that a maximum independent set in $D(H')$ is $\{x_i\} \cup \bigcup_{k=0}^3 A_{i,2k+1} | 1 \leq i \leq \alpha \} \cup \{y_i, V_i\} | \alpha + 1 \leq i \leq n$. Hence

$$\begin{aligned} m(H') &= \alpha(D(H')) \\ &= (4k + 1)\alpha(H) + (k + 1)(|V(H)| - \alpha(H)) \\ &= (k + 1)|V(H)| + 3k\alpha(H), \end{aligned}$$

completing the polynomial reduction. \square

Now suppose that H is a 2-connected planar graph with a perfect matching M , and let S_i be as in Fig. 1 with $k = 1$. Construct H' as in the proof of Theorem 2.2, but add an edge between vertices $A_{i,4}$ and $A_{j,4}$ precisely if $(v_i, v_j) \in M$. The resulting graph H' will be planar and 2-connected. Moreover, $D(H')$ has the same properties as $D(H')$ in the proof of Theorem 2.2, with the exception that now $\{y_i\} \cup V_i \cup A_{i,4}$ are independent in $D(S_i)$. Hence we have

$$\begin{aligned} m(H') &= \alpha(D(H')) \\ &= 5\alpha(H) + 3(|V(H)| - \alpha(H)) \\ &= 3|V(H)| + 2\alpha(H). \end{aligned}$$

Since INDEPENDENT SET is NP-complete for the class of 2-connected planar graphs with a perfect matching [10], we have the following result.

Theorem 2.3. MAX TUTTE SET is NP-complete for the class of 2-connected planar graphs.

In the proof of Theorem 2.2, note that if H is triangle-free, then so is H' . Since INDEPENDENT SET remains NP-complete for triangle-free graphs [9], we have:

Theorem 2.4. MAX TUTTE SET is NP-complete for triangle-free graphs.

2.3. Classes of graphs for which MAX TUTTE SET can be solved in polynomial time

In contrast to the NP-completeness results of Section 2.2, we now consider several interesting classes of graphs in which maximum Tutte sets can be found in polynomial time.

2.3.1. Graphs with level 0 or 1

We will prove:

Theorem 2.5. MAX TUTTE SET $\in P$ for the class of graphs with level 0 or 1.

In order to prove this, we first require:

Lemma 2.6. INDEPENDENT SET $\in P$ for the class of level 0 graphs.

Proof. Let G be a graph with $\text{level}(G) = 0$, and let M be a perfect matching in G . Let I_0 be an independent set in G , and let $S_0 = I_0 \cup I'_0$, where I'_0 consists of the mates of the vertices of I_0 under M . Note that S_0 can be partitioned into $|I_0|$ sets, each of which induces a clique whose vertices are perfectly matched under M .

Suppose now that $I \subseteq S \subseteq V(G)$, where

1. I is independent in G ;
2. S can be partitioned into $|I|$ sets, each of which induces a clique whose vertices are perfectly matched under M .

If $S = V(G)$, then clearly I is a maximum independent set in G . Otherwise, there exists an edge $(v, v') \in M$ with $v, v' \notin S$. If v and v' are adjacent, respectively, to distinct vertices $x, y \in I$, then by the C_4 -property, x must be adjacent to y , contradicting the independence of I . Thus, there cannot be two independent edges between two vertices in I and $\{v, v'\}$.

We now consider two cases, indicating in each how to redefine I and S so that $|S|$ increases, and 1 and 2 still hold. By iterating this procedure, we eventually obtain $S = V(G)$, at which point I will be a maximum independent set in G .

Case 1: v (resp., v') is not adjacent to any vertex in I . Redefine I to be $I \cup \{v\}$ (resp., $I \cup \{v'\}$), and S to be $S \cup \{v, v'\}$. It is easy to see that 1 and 2 still hold.

Case 2: There exists a unique vertex $x \in I$ such that $(x, v), (x, v') \in E(G)$. Let K be the even clique in the partition of $V(S)$ that contains x , and let y be any vertex in $K - x$. Of course, x and y are matched in some perfect matching in K . If y were adjacent to a vertex $z \in I - x$, then by the C_4 -property, v and v' are each adjacent to z , contradicting the uniqueness of $x \in I$ as a neighbor of both v and v' . Thus, $I_y = (I - x) \cup \{y\}$ is an independent set in G for any $y \in K - x$, with $|I_y| = |I|$.

Since $(x, v), (x, v') \in E(G)$, y is the only vertex in I_y that might be adjacent to either v or v' . If v (resp., v') is not adjacent to y , then redefine I to be $I_y \cup \{v\}$ (resp., $I_y \cup \{v'\}$), and S to be $S \cup \{v, v'\}$, observing that 1 and 2 still hold. But if v, v' are each adjacent to each $y \in K$, then $K \cup \{v, v'\}$ is an even clique. Leaving I unchanged, but redefining S to be $S \cup \{v, v'\}$, we see that 1 and 2 still hold.

This proves Lemma 2.6. \square

Proof of Theorem 2.5. Let M be a perfect matching in G . If $\text{level}(G) = 0$, let I be a maximum independent set in G . By Lemma 2.6, we can obtain I in polynomial time. Since $G \cong D(G)$ and $G \preceq D(G)$, we have that $I' = \{x' \mid x \in I\}$ is a maximum independent set in $D(G)$. But then by Theorem 1.5, I' is a maximum Tutte set in G .

If $\text{level}(G) = 1$, then $\text{level}(D(G)) = 0$. By Lemma 2.6, we can construct a maximum independent set I in $D(G)$ in polynomial time. By Theorem 1.5, I is a maximum Tutte set in G . \square

2.3.2. Elementary graphs

A graph G is called *elementary* if it contains a perfect matching and if the edges which occur in at least one perfect matching in G induce a connected subgraph. A substantial study of elementary graphs and their properties is given in [11], where the following result is proved.

Theorem 2.7. *Let G be a graph with a perfect matching. Then G is elementary if and only if G satisfies any of the following conditions:*

- (i) *the maximal Tutte sets in G form a partition of $V(G)$;*
- (ii) *$C_{G-x} = \emptyset$, for all $x \in V(G)$;*
- (iii) *for any non-empty Tutte set $X \subseteq V(G)$, $G - X$ has only odd components.*

The following is also proved in [11] (cf. Theorem 5.2.2(b)).

Theorem 2.8. *Let G be an elementary graph with $x, y \in V(G)$. Then $G - \{x, y\}$ has a perfect matching if and only if x and y occur in different maximal Tutte sets in G .*

Theorems 2.7 (i) and 2.8, together with the definition of $D(G)$, immediately give the following.

Theorem 2.9. *A graph G is elementary if and only if $D(G)$ is a complete multipartite graph.*

Since finding a maximum independent set in a complete multipartite graph is trivial, Theorem 1.5 immediately yields the following result.

Theorem 2.10. *MAX TUTTE SET $\in P$ for the class of elementary graphs.*

2.3.3. 1-tough graphs

A graph G is called 1-tough if $\omega(G - X) \leq |X|$ for all non-empty $X \subseteq V(G)$. We wish to consider the complexity of finding maximum Tutte sets in 1-tough graphs. To this end, we now prove two theorems.

Theorem 2.11. *If G is 1-tough on an odd number of vertices, then G is factor-critical.*

Proof. Suppose G is 1-tough on an odd number of vertices, but not factor-critical. Then there exists $v \in V(G)$ such that $G' = G - v$ has no perfect matching. Thus there exists $X' \subseteq V(G')$ with $\omega_0(G' - X') = |X'| + 1 + k$, where $k \geq 0$. Setting $X = X' \cup \{v\}$, we have

$$\begin{aligned} \omega(G - X) &= \omega_0(G - X) + \omega_e(G - X) \\ &= \omega_0(G' - X') + \omega_e(G' - X') \\ &= |X'| + 1 + k + \omega_e(G' - X') \\ &= |X| + k + \omega_e(G' - X') \\ &\geq |X| \geq 1. \end{aligned}$$

Since G is 1-tough, $k = \omega_e(G' - X') = 0$; otherwise $\omega(G - X) > |X| \geq 1$, a contradiction. Letting $H_1, \dots, H_{|X'|+1}$ denote the odd components of $G' - X'$, we find $|V(G)| = 1 + |X'| + \sum_{i=1}^{|X'|+1} |V(H_i)|$ is even, a contradiction. \square

Theorem 2.12. *If G is 1-tough on an even number of vertices, then G is elementary.*

Proof. Clearly G contains a perfect matching. If G is not elementary, then by Theorem 2.7(iii), G would contain a non-empty Tutte set X such that $G - X$ contains one or more even components. Since X is a Tutte set, $G - X$ contains at least $|X|$ odd components as well. Thus $\omega(G - X) \geq |X| + 1 > |X| \geq 1$, and G would not be 1-tough, a contradiction. \square

From Theorems 2.11 and 2.12 we have the following.

Theorem 2.13. *MAX TUTTE SET $\in P$ for the class of 1-tough graphs.*

Proof. Let G be a 1-tough graph on n vertices. If n is odd, then G is factor-critical by Theorem 2.11, and thus the only Tutte set in G is the empty set. If n is even, then G is elementary by Theorem 2.12, and we can find a maximum Tutte set in G in polynomial time by Theorem 2.10. \square

Corollary 2.14. *MAX TUTTE SET $\in P$ for the following classes of graphs:*

- (a) hamiltonian graphs,
- (b) 2-connected claw-free graphs,
- (c) k -regular, k -edge-connected graphs for any $k \geq 1$.

Proof. It is well known that hamiltonian graphs and 2-connected claw-free graphs are 1-tough. It is also easy to show that k -regular, k -edge-connected graphs are 1-tough for any $k \geq 1$. \square

3. Open problems

We conclude with several open problems:

1. We showed that MAX TUTTE SET can be solved in polynomial time for graphs of level 0 or 1, elementary graphs, and 1-tough graphs. Are there other interesting classes of graphs for which MAX TUTTE SET can be solved in polynomial time?

2. We know that MAX TUTTE SET is NP-complete for 2-connected planar graphs (Theorem 2.3) and polynomial for 4-connected planar graphs, since they are hamiltonian (Corollary 2.14). What is the complexity of MAX TUTTE SET for 3-connected planar graphs?
3. By Theorem 2.13, MAX TUTTE SET can be solved in polynomial time for 1-tough graphs, and hence for planar 1-tough graphs. Given $\varepsilon > 0$, is MAX TUTTE SET polynomial for planar $(1 - \varepsilon)$ -tough graphs? We strongly believe that MAX TUTTE SET is NP-complete for $(1 - \varepsilon)$ -tough general graphs. A possible approach to proving this is to first note that INDEPENDENT SET remains NP-complete for the class of hamiltonian graphs. If H in the proof of Theorem 2.2 is hamiltonian, the resulting graph H' appears to have toughness $(k + 1)/(k + 2)$, with tough set $V_1 \cup \{y_1\}$.
4. The class of D -graphs has been useful in our study of Tutte sets. But it remains an open problem whether level 1 D -graphs can be recognized in polynomial time (the problem is uninteresting for level 0 or 2 graphs, of course). This recognition problem becomes trivial for the class of bipartite graphs, since it was proved in [5] that a bipartite graph G is a D -graph if and only if $\text{level}(G) = 0$, and thus there are no bipartite level 1 D -graphs.

Note added in proof. It has been brought to the authors' attention that a linear time algorithm to construct a maximal Tutte set in a graph was given independently in [4], where a "splitter" is an independent set in $D(G)$. An independent proof of Theorem 1.6 was also given in [4]. Additional results related to those in this paper can be found in [1–3].

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References

- [1] M. Bartha, M. Kresz, Isolating the families of soliton graphs, *Pure Math. Appl.* 13 (2002) 49–62.
- [2] M. Bartha, M. Kresz, Structuring the elementary components of graphs having a perfect internal matching, *Theoret. Comput. Sci.* 299 (2003) 179–210.
- [3] M. Bartha, M. Kresz, Tutte type theorems in graphs having a perfect internal matching, *Inform. Process. Lett.* 91 (2004) 277–284.
- [4] M. Bartha, M. Kresz, Splitters and barriers in open graphs having a perfect internal matching, Preprint.
- [5] D. Bauer, H.J. Broersma, A. Morgana, E. Schmeichel, Tutte sets in graphs I: maximal Tutte sets and D -graphs, *J. Graph Theory*, in press.
- [6] C. Berge, Two theorems in graph theory, *Proc. Nat. Acad. Sci. USA* 43 (1957) 842–844.
- [7] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. (Paris)* 247 (1958) 258–259.
- [8] J.R. Edmonds, Maximum matching and a polyhedron with 0,1-vertices, *J. Res. Nat. Bur. Standards* 69B (1965) 125–130.
- [9] M.R. Garey, D.S. Johnson, *Computers and Intractability*, Freeman, San Francisco, CA, 1979.
- [10] M.R. Garey, D.S. Johnson, L. Stockmeyer, Some simplified NP-complete graph problems, *Theoret. Comput. Sci.* 1 (1976) 237–267.
- [11] L. Lovász, M.D. Plummer, *Matching theory*, *Ann. Discrete Math.* 29, (1986).
- [12] W.T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* 22 (1947) 107–111.
- [13] D.B. West, *Introduction to Graph Theory*, second ed., Prentice-Hall, Upper Saddle River, NJ, 2001.