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On Skew-symmetric Preconditioning for Strongly Non-symmetric Linear Systems

To solve iteratively linear system $Au = b$ with large sparse strongly non-symmetric matrix A we propose preconditioning $\hat{A}\hat{u} = \hat{b}$, $\hat{A} = (I + \tau L_1)^{-1} A (I + \tau U_1)^{-1}$, $\tau > 0$ where respectively lower and upper triangular matrices L_1 and U_1 are so that $L_1 + U_1 = 1/2(A - A^*)$. Such preconditioning technique may be treated as a variant of ILU-factorization, and we call it MSSILU — Modified Skew-Symmetric ILU.

We investigate and optimize (with respect to τ) convergence of preconditioned Richardson method (RM) of the following special form: $\hat{x}^{m+1} = (I - \tau\hat{A})\hat{x}^m + \tau\hat{b}$, $m \geq 0$, where τ is the same as in $\hat{A}$. For this method we give an estimate for rate of convergence in relevant Euclidean norm for the case of positive real matrix A.

Numerical experiments have included solving linear systems arising from 5-point FD approximation of convection–diffusion equation with dominated convection by MSSILU+RM, MSSILU+GMRES(2) and MSSILU+GMRES(10).

1. MSSILU — Modified Skew–Symmetric ILU factorization

Solving the system of the linear algebraic equations

$$Au = b \tag{1}$$

with large sparse strongly non-symmetric matrix A we propose to apply an iterative process to the following preconditioned system

$$\hat{A}\hat{u} = \hat{b}, \quad \hat{A} = (I + \tau L_1)^{-1} A (I + \tau U_1)^{-1}, \quad \hat{u} = (I + \tau U_1)u, \quad \hat{b} = (I + \tau L_1)^{-1} b\,, \tag{2}$$

where I is the identity matrix, L_1 and U_1 are respectively lower and upper triangular parts of matrix $A_1 = 1/2(A - A^*)$ which is the skew-symmetric component of A, so that $L_1 + U_1 = A_1$, $\tau > 0$ is a scalar parameter. Such preconditioning technique may be treated as a simplifyed variant of ILU-factorization, and we call it MSSILU — Modified Skew-Symmetric ILU. Consider MSSILU+RM (that is the preconditioned Richardson Method) of the form

$$\hat{u}^{m+1} = G\hat{u}^m + \tau\hat{b}, \quad m \geq 0\,, \quad G = I - \tau\hat{A}\,, \tag{3}$$

$$\text{or} \quad B\frac{u^{m+1} - u^m}{\tau} + Au^m = b, \quad B = (I + \tau L_1)(I + \tau U_1)\,, \tag{4}$$

where $B \neq B^*$ is preconditioning matrix and τ is the same as in (2). For (3),(4) we provide with convergence analysis [1] and way to choose the parameter τ optimally in the case when A is positive real matrix, i.e. when $A_0 = 1/2(A + A^*)$ is positive definite matrix. We establish and optimize convergence in the Euclidean norm induced by $B_0 = 1/2(B + B^*)$ that is symmetric part of the preconditioning matrix. It is very important to note that in (3),(4)

$$B_1 = \tau A_1\,, \qquad B_1 = \frac{1}{2}(B - B^*)\,, \qquad A_1 = \frac{1}{2}(A - A^*)\,. \tag{5}$$

Let us agree that vector (operator) norm without lower index is everywhere Euclidean norm. Using the approach developed in [2,1] we prove the following result:

Theorem 1. *Let A be positive real. Then for arbitrary positive real matrix B satisfying* (5) *iterative method of the form* (4) *converges, so that*

$$\|G\|_{B_0} = \left\|B_0^{1/2} G B_0^{-1/2}\right\| < 1\,, \tag{6}$$

$$\textit{as soon as} \qquad (B_0 u, u) > \frac{\tau}{2}(A_0 u, u)\ \big(>0\big)\,, \qquad \forall u \in \mathbb{C}^n\,. \tag{7}$$

In assumption that spectrum of A is such that $\mathrm{sp}A_0 \subseteq [\gamma_1;\gamma_2]$, $\gamma_1 > 0$ and $\rho(A_1) = 2\gamma_3$ we may evaluate spectrum of B as

$$\mathrm{sp}B_0 \subseteq [1-\gamma_3^2\tau^2;1]\,, \quad \rho(B_1) = 2\tau\gamma_3 \tag{8}$$

(where B_0 is positive definite for $\tau < \gamma_3^{-1}$) and then reformulate Theorem 1 in more constructive manner:

Theorem 2. *Let A and B be positive real. Then MSSILU+RM* (3),(4) *converges (that means* (6) *holds) as soon as τ satisfy the constraint*

$$0 < \tau < \hat{\tau} = \left(\sqrt{\gamma_2^2 + 16\gamma_3^2} - \gamma_2\right) \Big/ 4\gamma_3^2\,. \tag{9}$$

Minimization of $\|G\|_{B_0}$ reveals optimal value $\tilde{\tau}$ to be very close to $\hat{\tau}$, and though $\tilde{\tau}$ is not available in explicit analytical form we estimate the convergence rate of MSSILU+RM (3),(4):

Theorem 3. *Let A be positive real. Then MSSILU+RM* (3),(4) *converges for the optimal value $\tilde{\tau} \in (0;\hat{\tau})$,*

$$\|e^m\|_{B_0} < \rho_0^m \|e^0\|_{B_0}, \quad \rho_0 \le 1 - \gamma_1 \left(\sqrt{\gamma_2^2 + 16\gamma_3^2} - \gamma_2\right) \Big/ 4\gamma_3^2\,. \tag{10}$$

where e^m is the error vector on iteration m. The iteration number $\hat{m}$ needed to achieve the prescribed accuracy ϵ is of the form $\hat{m} < m_0(\epsilon)$, $m_0(\epsilon) = \ln\epsilon / \ln\rho_0$ (here $\ln x = \log_e x$).

2. Practical choice of the parameter. Numerical experiments

We show how to avoid in practice the necessity to know spectral bounds for A. For strongly non-symmetric linear systems (where $\gamma_2 \ll \gamma_3$) one may settle for a knowing γ_3 only (as it occurs, $\tilde{\tau} \approx \gamma_3^{-1}$). Moreover one may try to substitute γ_3 by $\|L_1\|_\infty = \max_i \sum_j |l_{1i,j}|$ since $\gamma_3 \le \|L_1\| \approx \|L_1\|_\infty$. One may verify that in fact $\|L_1\|_\infty \ge \gamma_3$, so that $\|L_1\|_\infty^{-1} \le \tilde{\tau}$ and practical choice $\tau = \|L\|_\infty^{-1}$ may be too underestimated. Various numerical tests shows that in matrix $I + \tau L_1$ the part of the "diagonal dominant" rows (i.e. rows i for which $\tau \sum_j |l_{1i,j}| < 1$) observed for the fastest convergence is 0.6–0.8. This heuristics gives convenient way for defining optimal value $\tilde{\tau}$ in practice.

Numerical experiments included solving linear systems derived from the 5-point FD discretization of the steady convection – diffusion equation

$$Pe^{-1}\Delta u + 0.5\left[(v_1 u)_x + v_1 u_x + (v_2 u)_y + v_2 u_y\right] = 0 \tag{11}$$

on the unit square with homogeneous Dirichlet boundary conditions. The Peclet number Pe was taken 10^3, 10^4 and 10^5. We compared performance of MSSILU+RM, MSSILU+GMRES(2) and MSSILU+GMRES(10) for the model problems. For the most "recalcitrant" problem (11) with $Pe = 10^5$ and $v_1 = \sin 2\pi x$, $v_2 = -2\pi y \cos 2\pi x$ MSSILU+GMRES(10) requires 275 iterations (restarts), MSSILU+RM requires 2389 iterations (respectively 727 s and 273 s of IBM PC 486/DX2-66 CPU time) on grid 63×63 (iterations were performed until $\|r^m\|/\|r_0\| \le 10^{-6}$). For the same problem on the coarser grid 31×31 we observed 767 iterations (restarts) (379 s) for MSSILU+GMRES(10) and 7098 iterations (152 s) for MSSILU+RM.

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3. References

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