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Efficiency and Optimality Properties of a Class of k-Sample Rank Tests Against Trend

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Abstract: A class of k-sample rank tests is considered for testing the hypothesis H_0: $F_i(x) = F(x)$, $i \leqslant k$ against the hypothesis $H_1: F_i(x) = F(x - \theta_i)$, $\theta_1 \leqslant \theta_2 \leqslant \ldots \leqslant \theta_k$, $\theta_1 \neq \theta_k$. These tests are based on a rank statistic $\underline{S} = \sum_i d_i(\vec{n}) \sum_l a_n(\underline{R}_{i,l}, \gamma)$ in which $\vec{n} = (n_1, \ldots, n_k)$. If $n^{-1} n_{i,n} \to \xi_i$ and $n^{1/2}\theta_{i,n} \to \theta_i$ it is shown that Eff $((\vec{\xi}, \vec{d}, \gamma) \mid (F, \vec{\theta})) =$ Eff $(\gamma \mid F)$ Eff $(\vec{\xi} \mid \vec{\theta})$ Eff$_{\vec{\xi}}$ $(\vec{d} \mid \vec{\theta})$. For given $\vec{\xi}$ the 'minimax' efficiency weight-vector $\vec{d}_0(\vec{\xi})$ is derived with respect to $\Theta = \{\vec{\theta} \mid -1 = \theta_1 \leqslant \ldots \leqslant \theta_k = 1\}$ and also the Bayes vector $\vec{d}(\vec{\xi}, \tau)$ with respect to a d.f. τ on Θ. The properties of these tests are investigated. Further an allied class of tests is considered based on a statistic $\underline{W} = \sum\sum_{h<i} d_{h,i}(\vec{n}) \underline{W}_{h,i}$, where $\underline{W}_{h,i}$ is a rank statistic for the samples taken of $\underline{x}_h$ and $\underline{x}_i$.

1. Introduction and Summary

By means of $n = n_1 + \ldots + n_k$ completely independent observations $x_{i,l}$, $l \leqslant n_i$, $i \leqslant k$ taken of k variables $\underline{x}_1, \ldots, \underline{x}_k$ with distribution functions $F_i(x) = F(x - \theta_i)$, $i \leqslant k$ we want to test the hypothesis

$$H_0: \theta_1 = \ldots = \theta_k ,$$

against the alternative hypothesis

$$H_1: \theta_1 \leqslant \theta_2 \leqslant \ldots \leqslant \theta_k , \qquad \theta_1 \neq \theta_k .$$

The class of tests considered are defined by means of a statistic of the structure

$$S(\vec{n}, \vec{d}(\vec{n}), \gamma; \vec{\underline{R}}) := \sum_{i=1}^{k} d_i(\vec{n}) \sum_{l=1}^{n_i} a_n(\underline{R}_{i,l}, \gamma), \tag{1.1}$$

in which $R_{i,l}$ is the rank number of observation $x_{i,l}$ obtained by arranging all n observations according to increasing magnitude, $d_i(\vec{n})$ the weight of sample i, γ a score function and $a_n(s, \gamma)$, $s \leqslant n$ the scores which are defined by

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$$a_n(s, \gamma) := E\gamma(\underline{U}^{(s)}), \qquad s \leqslant n, \tag{1.2}$$

$\underline{U}^{(1)}, \ldots, \underline{U}^{(n)}$ being the order statistics of a random sample of size n from a uniform distribution on [0, 1].

We assume that γ satisfies the condition

$$\Gamma: \int_0^1 \gamma(u)\,du = 0, \qquad 0 < \int_0^1 \gamma^2(u)\,du < \infty. \tag{1.3}$$

The score function γ_F corresponding to the d.f. considered is given by [cf. *Hájek/Šidák*, p. 19]

$$\gamma_F(u) = -\frac{f'}{f}(F^{-1}(u)), \qquad 0 < u < 1, \tag{1.4}$$

in which f is the probability density function.

If f is absolutely continuous and f' absolutely integrable then γ_F satisfies the first condition in (1.3).

If $\gamma \in \Gamma$ then $\sum_s a_n(s, \gamma) = n \int_0^1 \gamma(u)\,du = 0$ and $E(\underline{S} \mid H_0) = 0$.

For convenience we introduce the following notation

$$\sigma^2(\gamma) := \int_0^1 \gamma^2(u)\,du, \tag{1.5a}$$

$$\left.\begin{aligned} \mu_{\vec{n}}(\vec{d}) &:= \sum_i \frac{n_i}{n} d_i, \\ \sigma^2_{\vec{n}}(\vec{d}) &:= \sum_i \frac{n_i}{n} d_i^2 - \Big(\sum_i \frac{n_i}{n} d_i\Big)^2, \end{aligned}\right\} \tag{1.5b}$$

$$\left.\begin{aligned} \mu_{\vec{\xi}}(\vec{d}) &:= \sum_i \xi_i d_i, \qquad 0 \leqslant \xi_i \leqslant 1, \sum_i \xi_i = 1, \\ \sigma^2_{\vec{\xi}}(\vec{d}) &:= \sum_i \xi_i d_i^2 - \Big(\sum_i \xi_i d_i\Big)^2. \end{aligned}\right\} \tag{1.5c}$$

Then [cf. *Hájek/Šidák*, p. 163]

$$\operatorname{var}(\underline{S} \mid H_0) = n\,\sigma^2_{\vec{n}}(\vec{d}(\vec{n}))\,\sigma^2(\gamma). \tag{1.6}$$

If we assume that $\sigma^2_{\vec{n}}(\vec{d}(\vec{n})) > 0$ and $\gamma \in \Gamma$ then we can consider instead of $\underline{S}$ the standardized variable

$$\underline{S}_n^* := \{\operatorname{var}(\underline{S}_n \mid H_0)\}^{-1/2}\,\underline{S}_n, \tag{1.7}$$

which has mean zero and variance one under H_0.

We now assume that following conditions are satisfied for $n \to \infty$

$$\left.\begin{array}{ll} k \text{ does not depend on } n, & \\ n^{-1} n_{i,n} \to \xi_i, & i \leqslant k, \\ d_i(\vec{n}_n) \to d_i(\vec{\xi}), & i \leqslant k, \\ \sigma^2_{\vec{\xi}}(\vec{d}(\vec{\xi})) > 0. & \end{array}\right\} \tag{1.8}$$

Then the variable $\underline{S}_n^*$ is asymptotically equal (with probability one) to the variable

$$T(\vec{\xi}, \vec{d}(\vec{\xi}), \gamma; \vec{n}, \vec{\underline{R}}_n) := (n\, \sigma^2_{\vec{\xi}}(\vec{d}(\vec{\xi}))\ \sigma^2(\gamma))^{-1/2} \cdot \sum_{i=1}^{k} d_i(\vec{\xi}) \sum_{l=1}^{n_{i,n}} a_n(\underline{R}_{i,l}, \gamma). \tag{1.9}$$

Thus if we want to investigate the asymptotic properties of tests based on test-statistics $\underline{S}$ defined by (1.1) and the conditions (1.8) are satisfied then this is equivalent to investigating the asymptotic properties of sequences of tests consisting of critical regions

$$Z_{\alpha,n} := \{\omega \mid T(\vec{\xi}, \vec{d}(\vec{\xi}), \gamma; \vec{n}, \vec{R}_n(\omega)) > \xi_{1-\alpha}\}, \tag{1.10}$$

where $\xi_{1-\alpha} = \phi^{-1}(1-\alpha)$, $\phi(u)$ being the normal distribution function. For convenience we denote the foregoing sequence of tests by $(\vec{\xi}, \vec{d}(\vec{\xi}), \gamma)$.

From the results of *Hájek/Šidák* [1967, p. 227] we immediately obtain the following

Theorem 1: Under condition (1.8) the asymptotic power of a procedure $(\vec{\xi}, \vec{d}, \gamma)$, $\gamma \in \Gamma$, for testing the hypothesis H_0 against a sequence of alternatives defined by F and $\vec{\theta}_n$, for which $\gamma_F \in \Gamma$ and

$$n^{1/2}\, \vec{\theta}_n \to \vec{\theta}, \tag{1.11}$$

is equal to

$$1 - \phi(\xi_{1-\alpha} - \rho(\gamma, \gamma_F)\, \rho_{\vec{\xi}}(\vec{d}, \vec{\theta})\, \sigma(\gamma_F)\, \sigma_{\vec{\xi}}(\vec{\theta})), \tag{1.12}$$

where

$$\rho(\gamma, \gamma_F) := \int_0^1 \gamma(u)\, \gamma_F(u)\, du \tag{1.13}$$

and

$$\left.\begin{array}{l} \rho_{\vec{\xi}}(\vec{d}, \vec{\theta}) := \dfrac{\operatorname{cov}_{\vec{\xi}}(\vec{d}, \vec{\theta})}{\sigma_{\vec{\xi}}(\vec{d}) \cdot \sigma_{\vec{\xi}}(\vec{\theta})}, \\ \text{where} \\ \operatorname{cov}_{\vec{\xi}}(\vec{d}, \vec{\theta}) := \sum_{i=1}^{k} \xi_i\, (d_i - \mu_{\vec{\xi}}(\vec{d}))\, (\theta_i - \mu_{\vec{\xi}}(\vec{\theta})). \end{array}\right\} \tag{1.14}$$

Remark:

1. This theorem contains the property that the variable $\underline{T}$ defined by (1.9) is asymptotically normally distributed under H_0.
2. The test considered is invariant with respect to the location parameter θ. Thus without loss of generality we may consider the sequence $\{\vec{\theta}_n\}$ for which (1.11) holds instead of $n^{1/2}(\vec{\theta}_n - \theta \cdot \vec{1}) \to \vec{\theta}$, where θ is the common value of $\theta_1, \ldots, \theta_k$ under H_0.

From (1.12) the known property follows that for each $\vec{\xi}$ the asymptotic most powerful test against $(F, \vec{\theta})$ is obtained by taking $\gamma = \gamma_F$ and $\vec{d} = \vec{\theta}$. This test is also a locally most powerful rank test [cf. *Lehmann*].

Defining for given $\vec{\xi}$ the efficiency of a procedure $(\vec{\xi}, \vec{d}, \gamma)$ with respect to an alternative $(F, \vec{\theta})$ as the fraction of the number of observations the asymptotic most powerful test $(\vec{\xi}, \vec{\theta}, \gamma_F)$ needs to reach the same asymptotic power as the test considered, then it follows from (1.12) that

$$\mathrm{Eff}_{\vec{\xi}}((\vec{d}, \gamma) \mid (F, \vec{\theta})) = \mathrm{Eff}\,(\gamma \mid F) \cdot \mathrm{Eff}_{\vec{\xi}}(\vec{d} \mid \vec{\theta}), \tag{1.15}$$

where

$$\mathrm{Eff}\,(\gamma \mid F) := \rho^2(\gamma, \gamma_F) \tag{1.16}$$

and

$$\mathrm{Eff}_{\vec{\xi}}(\vec{d} \mid \vec{\theta}) := \rho^2_{\vec{\xi}}(\vec{d}, \vec{\theta}). \tag{1.17}$$

If we now take into consideration the possibility of designing the experiment in such a way as to increase the power it follows that for each $(F, \vec{\theta})$ the asymptotic most powerful test is obtained if we take

$$\left.\begin{aligned} \gamma &= \gamma_F, \\ \vec{\xi} &= \vec{\xi}_0 := (1/2, 0, \ldots, 0, 1/2), \\ d &= d_0 := (-1, 0, \ldots, 0, 1). \end{aligned}\right\} \tag{1.18}$$

It follows that with respect to this optimal procedure

$$\mathrm{Eff}\,((\vec{\xi}, \vec{d}, \gamma) \mid (F, \vec{\theta})) = \mathrm{Eff}\,(\gamma \mid F) \cdot \mathrm{Eff}\,(\vec{\xi} \mid \vec{\theta}) \cdot \mathrm{Eff}_{\vec{\xi}}(\vec{d} \mid \vec{\theta}), \tag{1.19}$$

where

$$\mathrm{Eff}\,(\vec{\xi} \mid \vec{\theta}) := \frac{4}{(\theta_k - \theta_1)^2} \cdot \sigma^2_{\vec{\xi}}(\vec{\theta}). \tag{1.20}$$

Thus we can speak about the efficiency of the score function γ with respect to F, the efficiency of the design $\vec{\xi}$ with respect to $\vec{\theta}$ and, given the design $\vec{\xi}$, the efficiency of the weight-vector $\vec{d}$ with respect to $\vec{\theta}$.

Further we have

$$\text{Eff}\,((\vec{\xi}, \vec{d}) \mid \vec{\theta}) = \text{Eff}\,(\vec{\xi} \mid \vec{\theta}) \cdot \text{Eff}_{\vec{\xi}}\,(\vec{d} \mid \vec{\theta}). \tag{1.21}$$

As the efficiency of $(\vec{\xi}, \vec{d})$ with respect to $\vec{\theta}$ is invariant for a linear transformation of $\vec{\theta}$ we assume $\vec{\theta} \in \Theta$, where

$$\Theta := \{\vec{\theta} \mid -1 = \theta_1 \leqslant \theta_2 \leqslant \ldots \leqslant \theta_k = 1\}. \tag{1.22}$$

We remark that the optimal procedure $(\vec{\xi}_0, \vec{d}_0, \gamma_F)$ only regards the samples taken of $\underline{x}_1$ and $\underline{x}_k$, thus the efficiencies of a procedure $(\vec{\xi}, \vec{d})$ for different vectors $\vec{\theta} \in \Theta$ are comparable.

In the sequel we shall consider optimality problems with respect to the "part" $(\vec{\xi}, \vec{d})$. For the "part" γ analogous problems can be considered. We shall pay special attention to the choice of the weight-vector $\vec{d}$ for a given non-optimal design $\vec{\xi}$.

If $\vec{\theta}$ is known then $\vec{d}$ will be taken equal to $\vec{\theta}$. But if $\vec{\theta}$ is not known then we must choose $\vec{d}$ in some optimal way and it may be expected that the optimal vector $\vec{d}$ will depend on $\vec{\xi}$.

In the following section we derive for given $\vec{\xi}$, $\xi_1 > 0$, $\xi_k > 0$ the "minimax" weight-vector $\vec{d}_0\,(\vec{\xi})$ for which the minimum of $\text{Eff}\,((\vec{\xi}, \vec{d}) \mid \vec{\theta})$ with respect to Θ is maximal (cf. (2.1)). It appears that the efficiency of $(\vec{\xi}, \vec{d}_0\,(\vec{\xi}))$ does not depend on $\vec{\theta}$ and that it is equal to the efficiency of the corresponding procedure which takes only into consideration the samples taken of $\underline{x}_1$ and $\underline{x}_k$ (cf. (2.2)). Also the Bayes-vector $\vec{d}\,(\vec{\xi}, \tau)$ and the Bayes-efficiency $\text{Eff}\,(\vec{\xi}, \tau)$ with respect to a distribution function τ on Θ can be obtained. It appears that $\text{Eff}\,(\vec{\xi}, \tau)$ is equal to the largest eigenvalue of the positive definite matrix (2.11) and that $\vec{d}\,(\vec{\xi}, \tau)$ immediately follows from the corresponding eigenvector (cf. (2.9)).

In section 3 we consider an allied class of tests based on onesided critical regions defined by means of a variable

$$W(\vec{n}, \{d_{h,i}\,(\vec{n})\}, \gamma; \underline{R}) := \sum_{h<i}\sum d_{h,i}\,(\vec{n})\, W_{h,i}\,(\gamma; \underline{\vec{R}}), \tag{1.23}$$

where

$$W_{h,i}\,(\gamma; \underline{\vec{R}}) := (n_h + n_i + 1) \sum_{l=1}^{n_i} a_{n_h+n_i}(\underline{R}_{i,l}^{(h,i)}, \gamma), \tag{1.24}$$

$R_{i,l}^{(h,i)}$, $l \leqslant n_i$ being the rank numbers of the observations $x_{i,l}$, $l \leqslant n_i$ if the two samples taken of $\underline{x}_h$ and $\underline{x}_i$ are arranged according to increasing magnitude.

For $n \to \infty$ and under appropriate conditions the variable $n^{-3/2}\,\underline{W}^{(n)}$ is under H_0 as well as under contiguity alternatives asymptotically equivalent with a statistic $n^{-1/2}\underline{S}_n$ of the structure (1.1) with a weight vector $\vec{d}^{(W)}(\vec{\xi})$ given by (3.17). From this equivalence we immediately obtain property (3.19) for the asymptotic efficiency of the weight-functions $\{d_{h,i}\,(\vec{n})\}$ for a given design $\vec{\xi}$ against a contiguity alternative $\vec{\theta}$. This efficiency will be investigated for some tests formerly introduced by the author [cf. *Terpstra*, 1952, 1955].

2. The Minimax and the Bayes Weight-Vector

We will prove the following

Theorem 2: For a given design $\vec{\xi}$, $\xi_1 > 0$, $\xi_k > 0$ the minimax weight-vector $\vec{d}_0\ (\vec{\xi})$ is given by

$$\vec{d}_0\ (\vec{\xi}) = \left(-1, \frac{\xi_k - \xi_1}{\xi_k + \xi_1}, \ldots, \frac{\xi_k - \xi_1}{\xi_k + \xi_1}, 1 \right) \tag{2.1}$$

and

$$\text{Eff}\ ((\vec{\xi}, \vec{d}_0\ (\vec{\xi})) \mid \vec{\theta}) = \frac{4\xi_1\ \xi_k}{\xi_1 + \xi_k}, \qquad \vec{\theta} \in \Theta, \tag{2.2}$$

this efficiency being equal to that of the corresponding procedure which is only based on the two samples taken of $\underline{x}_1$ and $\underline{x}_k$.

Remark: This theorem is in concordance with property (3.3) shown by *Koziol/Reid* [1977] from which it follows that under H_0 as well as under any contiguity alternative $\vec{\theta}$

$$T\,(\vec{\xi}, \vec{d}_0\ (\vec{\xi}), \gamma; \vec{n}_n, \vec{\underline{R}}_n) - (n^3 \xi_1 \xi_k\ (\xi_1 + \xi_k)\ \sigma^2\ (\gamma))^{-1/2}\ \underline{W}_{1,k}^{(n)}\} \xrightarrow{P} 0, \tag{2.3}$$

where $\underline{W}_{1,k}^{(n)}$ is defined by (1.24).

Proof: The latter part of the theorem immediately follows by remarking that

$$\text{Eff} \left\{ \left(\left(\frac{\xi_1}{\xi_1 + \xi_k}, \frac{\xi_k}{\xi_1 + \xi_k} \right), (-1,1) \right) \mid (-1,1) \right\} = \frac{4\xi_1 \xi_k}{(\xi_1 + \xi_k)^2}. \tag{2.4}$$

To prove the first part we remark that

$$\inf_{\vec{\theta}}\ \text{Eff}\ ((\vec{\xi}, \vec{d}) \mid \vec{\theta}) = \inf_{\vec{\theta}}\ \rho^2_{\vec{\xi}}(\vec{d}, \vec{\theta})\ \sigma^2_{\vec{\xi}}(\vec{\theta}) \leqslant \inf_{\vec{\theta}} \sigma^2_{\vec{\xi}}(\vec{\theta}) = \sigma^2_{\vec{\xi}}(\vec{d}_0\ (\vec{\xi})), \tag{2.5}$$

while

$$\text{Eff}\ ((\vec{\xi}, \vec{d}) \mid \vec{d}_0\ (\vec{\xi})) < \text{Eff}\ ((\vec{\xi}, \vec{d}_0\ (\vec{\xi})) \mid \vec{d}_0\ (\vec{\xi})) = \sigma^2_{\vec{\xi}}(\vec{d}_0\ (\vec{\xi})), \quad \vec{d} \neq \vec{d}_0\ (\vec{\xi}). \tag{2.6}$$

We remark that (cf. (1.14) and (1.5c))

$$\text{cov}_{\vec{\xi}}(\vec{d}_0\ (\vec{\xi}), \vec{\theta}) = \sigma^2_{\vec{\xi}}(\vec{d}_0\ (\vec{\xi})), \qquad \vec{\theta} \in \Theta, \tag{2.7}$$

consequently

$$\text{Eff}\ ((\vec{\xi}, \vec{d}_0\ (\vec{\xi})) \mid \vec{\theta}) = \sigma^2_{\vec{\xi}}(\vec{d}_0\ (\vec{\xi})), \qquad \vec{\theta} \in \Theta. \tag{2.8}$$

From (2.6) and (2.8) it follows that $\vec{d}_0\ (\vec{\xi})$ is the minimax-weight-vector and that it is an equalizer vector.

Theorem 3: For a given design $\vec{\xi}$, $\xi_i > 0$, $i \leqslant k$ and a given apriori distribution τ on Θ the Bayes-vector $\vec{d}(\vec{\xi}, \tau)$ and the Bayes-efficiency Eff $(\vec{\xi}, \tau)$ are given by

$$\vec{d}\,(\vec{\xi}, \tau) = \left(\frac{u_1\,(\vec{\xi}, \tau)}{\sqrt{\xi_1}}, \ldots, \frac{u_k\,(\vec{\xi}, \tau)}{\sqrt{\xi_k}} \right) \tag{2.9}$$

and

$$\text{Eff}\,(\vec{\xi}, \tau) = \lambda\,(\vec{\xi}, \tau), \tag{2.10}$$

where $\lambda\,(\vec{\xi}, \tau)$ is the largest eigenvalue and $\vec{u}\,(\vec{\xi}, \tau)$ the corresponding eigenvector of the matrix

$$\| \sqrt{\xi_i\, \xi_j} \cdot E_\tau\,(\underline{\theta}_i - \mu_{\vec{\xi}}(\vec{\underline{\theta}}))\,(\underline{\theta}_j - \mu_{\vec{\xi}}(\vec{\underline{\theta}}))\, \| \,. \tag{2.11}$$

Proof: For each $\vec{d}$ with $\sigma^2_{\vec{\xi}}(\vec{d}) > 0$

$$E_\tau\ \{\text{Eff}\,((\vec{\xi}, \vec{d}) \mid \vec{\underline{\theta}})\} = E_\tau\ \frac{\text{cov}^2_{\vec{\xi}}\,(\vec{d}, \vec{\underline{\theta}})}{\sigma^2_{\vec{\xi}}\,(\vec{d})}\,. \tag{2.12}$$

As the right member is invariant for a linear transformation of $\vec{d}$, we may assume that

$$\sigma^2_{\vec{\xi}}\,(\vec{d}) = \sum_i \xi_i\, d_i^2 = 1. \tag{2.13}$$

Defining $\tilde{\theta}_{i,\vec{\xi}} := \theta_i - \mu_{\vec{\xi}}(\vec{\theta})$ we have

$$\text{cov}_{\vec{\xi}}\,(\vec{d}, \vec{\theta}) = \sum_i \xi_i\, d_i\, \tilde{\theta}_{i,\vec{\xi}}\,. \tag{2.14}$$

Defining $\vec{u}$ by

$$u_i := \sqrt{\xi_i} d_i, \qquad i = 1, \ldots, k, \tag{2.15}$$

we have

$$E_\tau\,\{\text{Eff}\,((\vec{\xi}, \vec{d}) \mid \vec{\underline{\theta}})\} = \sum_i \sum_j u_i\, u_j\, c_{i,j}\,(\vec{\xi}, \tau) \tag{2.16}$$

where

$$c_{i,j}\,(\vec{\xi}, \tau) := \sqrt{\xi_i\, \xi_j}\, E_\tau\, \underline{\tilde{\theta}}_{i,\vec{\xi}}\, \underline{\tilde{\theta}}_{j,\vec{\xi}}\,. \tag{2.17}$$

The Bayes-vector corresponds with the vector $\vec{u}\,(\vec{\xi}, \tau)$ that maximizes the right member of (2.16) under the constraint (cf. (2.13) and (2.15))

$$\sum_{i=1}^{k} u_i^2 = 1, \tag{2.18}$$

thus the Bayes-efficiency is equal to the largest eigenvalue $\lambda\,(\vec{\xi}, \tau)$ of the matrix $\|c_{i,j}\,(\vec{\xi}, \tau)\|$ and $\vec{u}\,(\vec{\xi}, \tau)$ is equal to the corresponding eigenvector.

Corollary 1: The following inequality holds

$$\text{Eff}\,(\vec{\xi}, \tau) \geqslant E_\tau \text{Eff}\,((\vec{\xi}, E_\tau \underline{\vec{\theta}}) \mid \underline{\vec{\theta}}) \geqslant \text{Eff}\,((\vec{\xi}, E_\tau \underline{\vec{\theta}}) \mid E_\tau \underline{\vec{\theta}}) . \tag{2.19}$$

Proof: Using the inequality $E\underline{z}^2 = \text{var}\,\underline{z} + (E\underline{z})^2$, we obtain

$$\begin{aligned}\text{Eff}\,(\vec{\xi}, \tau) &= \sup_{\vec{d}}[\text{var}_\tau \{\rho_{\vec{\xi}}(\vec{d}, \underline{\vec{\theta}})\, \sigma_{\vec{\xi}}(\underline{\vec{\theta}})\} + \rho^2_{\vec{\xi}}(\vec{d}, E_\tau \underline{\vec{\theta}})\, \sigma^2_{\vec{\xi}}(E_\tau \underline{\vec{\theta}})] \geqslant \\ &\geqslant \sup_{\vec{d}} \rho^2_{\vec{\xi}}(\vec{d}, E_\tau \underline{\vec{\theta}})\, \sigma^2_{\vec{\xi}}(E_\tau \underline{\vec{\theta}}) = \sup_{\vec{d}} \text{Eff}\,((\vec{\xi}, \vec{d}) \mid E_\tau \underline{\vec{\theta}}) = \\ &= \text{Eff}\,((\vec{\xi}, E_\tau \underline{\vec{\theta}}) \mid E_\tau \underline{\vec{\theta}}) = \sigma^2_{\vec{\xi}}(E_\tau \underline{\vec{\theta}}) .\end{aligned} \tag{2.20}$$

3. An Allied Class of Tests

We consider the class of tests which are based on a statistic $\underline{W}$ as defined by (1.23) and (1.24). We shall investigate the asymptotic properties of these tests under the condition that for $n \to \infty$

$$\left.\begin{array}{ll} k \text{ does not depend on } n, & \\ n^{-1} n_{i,n} \to \xi_i , & i \leqslant k, \\ d_{h,i}(\vec{n}_n) \to d_{h,i}(\vec{\xi}), & h < i, \quad h, i \leqslant k. \end{array}\right\} \tag{3.1}$$

Defining

$$\underline{W}_i := (n+1) \sum_{l=1}^{n_i} a_n (\underline{R}_{i,l}, \gamma), \tag{3.2}$$

we use the property shown by *Koziol/Reid* [1977] that under H_0 and any contiguity alternative $\vec{\theta}$

$$n^{-3/2} \{\underline{W}^{(n)}_{h,i} - (\xi_h \underline{W}^{(n)}_i - \xi_i \underline{W}^{(n)}_h)\} \xrightarrow{P} 0. \tag{3.3}$$

Further they proved that under the foregoing conditions

$$\text{var}\,(n^{-3/2} \underline{W}^{(n)}_{h,i}) \to \xi_h \xi_i (\xi_h + \xi_i)\, \sigma^2(\gamma) \tag{3.4}$$

and

$$\left.\begin{array}{l} \text{cov}\,(n^{-3/2} \underline{W}^{(n)}_{h,i}, n^{-3/2} \underline{W}^{(n)}_{h,j}) \to \xi_h \xi_i \xi_j \sigma^2(\gamma) \\ \text{cov}\,(n^{-3/2} \underline{W}^{(n)}_{h,i}, n^{-3/2} \underline{W}^{(n)}_{i,j}) \to -\xi_h \xi_i \xi_j \sigma^2(\gamma) \\ \text{cov}\,(n^{-3/2} \underline{W}^{(n)}_{h,i}, n^{-3/2} \underline{W}^{(n)}_{j,l}) \to 0, \neq (h, i, j, l). \end{array}\right\} \tag{3.5}$$

Remark: For the special case that γ is equal to the logistic score function

$$\gamma_L(u) = 2u - 1, \qquad 0 \leqslant u \leqslant 1, \tag{3.6}$$

property (3.3) has been independently obtained by the author. Then $\underline{W}_{h,i}$ and $\underline{W}_i$ are equal to the variables

$$\underline{U}_{h,i} := \sum_{1}^{n_h} l \sum_{1}^{n_i} l' \operatorname{sgn}(\underline{x}_{i,l'} - \underline{x}_{h,l}) \tag{3.7}$$

respectively

$$\underline{U}_i := \sum_{h \neq i} \underline{U}_{h,i} \,. \tag{3.8}$$

For the variables $\underline{U}_{h,i}$, $h < i$, $h, i \leqslant k$ we have [cf. *Terpstra*, 1954]

$$\operatorname{var}(\underline{U}_{h,i} \mid H_0) = \frac{1}{3} n_h\, n_i\, (n_h + n_i + 1) \tag{3.9}$$

and

$$\left.\begin{aligned} &\operatorname{cov}(\underline{U}_{h,i}, \underline{U}_{h,j} \mid H_0) = \frac{1}{3} n_h\, n_i\, n_j\,, \\ &\operatorname{cov}(\underline{U}_{h,i}, \underline{U}_{i,j} \mid H_0) = -\frac{1}{3} n_h\, n_i\, n_j\,, \\ &\operatorname{cov}(\underline{U}_{h,i}, \underline{U}_{j,l} \mid H_0) = 0, \qquad \neq (h, i, j, l), \end{aligned}\right\} \tag{3.10}$$

from which it follows that under H_0 and consequently also under any contiguity alternative $\vec{\theta}$

$$n^{-3} \cdot \operatorname{var}(\xi_h\, \underline{U}_{i,j}^{(n)} + \xi_i\, \underline{U}_{j,h}^{(n)} + \xi_j\, \underline{U}_{h,i}^{(n)}) \xrightarrow{P} 0 \,. \tag{3.11}$$

Denoting the sample consisting of all observations not taken of $\underline{x}_h$ and $\underline{x}_i$ by $(\overline{h,i})$, we have

$$\underline{U}_{(\overline{h,i}),i} = \underline{U}_i - \underline{U}_{h,i} \,. \tag{3.12}$$

Using this property and applying (3.11) to the three samples h, i and $(\overline{h,i})$ we immediately obtain (3.3).

From (3.1), (3.4) and (3.5) it follows that under H_0 and any contiguity alternative

$$\operatorname{var}(n^{-3/2}\, \underline{W}^{(n)}) \to \sigma_W^2 \,, \tag{3.13}$$

where

$$\begin{aligned} \sigma_W^2 := [\sum_{h<i}\sum d_{h,i}^2(\vec{\xi})\, \xi_h\, \xi_i\, (\xi_h + \xi_i) + 2 \sum_{h<i<j}\sum\sum \{d_{h,i}(\vec{\xi})\, d_{h,j}(\vec{\xi}) + \\ + d_{h,j}(\vec{\xi})\, d_{i,j}(\vec{\xi}) - d_{h,i}(\vec{\xi})\, d_{i,j}(\vec{\xi})\}\, \xi_h\, \xi_i\, \xi_j] \cdot \sigma^2(\gamma) \,. \end{aligned} \tag{3.14}$$

We now consider a sequence of tests consisting of critical regions

$$Z_{\alpha,n}^{(W)} := \{\omega \mid n^{-3/2}\, W^{(n)}\, (\vec{R}_n\, (\omega)) > \xi_{1-\alpha} \cdot \sigma_w\}, \tag{3.15}$$

where $\xi_{1-\alpha} = \phi^{-1}(1-\alpha)$, ϕ being the normal distribution function.

Now from (1.23), (1.24), (3.2) and (3.3) it follows that under H_0 as well as under any contiguity alternative $\vec{\theta}$.

$$n^{-3/2}\, (W^{(n)} - (n+1) \sum_{i=1}^{k} d_i^{(w)}\, (\vec{\xi}) \sum_{l=1}^{n_i} a_n\, (\underline{R}_{i,l}, \gamma)) \xrightarrow{P} 0, \tag{3.16}$$

where

$$d_i^{(w)}\, (\vec{\xi}) := \sum_{h<i} \xi_h\, d_{h,i}\, (\vec{\xi}) - \sum_{h>i} \xi_h\, d_{i,h}\, (\vec{\xi})\,. \tag{3.17}$$

From (3.16), (1.9) and Theorem 1 it follows that under the condition

$$\sigma^2_{\vec{\xi}}\, (\vec{d}^{(w)}\, (\vec{\xi})) > 0 \tag{3.18}$$

the sequence of tests $\{Z_{\alpha,n}^{(W)}\}$ defined by (3.15) and the procedure $(\vec{\xi}, \vec{d}^{(W)}\, (\vec{\xi}), \gamma)$ defined by (1.10) and (3.17) have the same asymptotic size α under H_0 and the same asymptotic power against any contiguity alternative $\vec{\theta}$.

From the foregoing properties it immediately follows that

$$\mathrm{Eff}_{\vec{\xi}}\, (\{d_{h,i}\, (\vec{\xi})\} \mid \vec{\theta}\,) = \mathrm{Eff}_{\vec{\xi}}\, (\vec{d}^{(w)}\, (\vec{\xi}) \mid \vec{\theta}\,)\,, \tag{3.19}$$

which is given by (3.17), (1.17) and (1.14).

Some special cases:

First we consider the statistic

$$\underline{W}_1 := \sum_{h<i}\sum \underline{U}_{h,i}\,, \tag{3.20}$$

where $\underline{U}_{h,i}$ is defined by (3.7).

The variable $\underline{W}_1$ is related to *Kendall's* rank correlation statistic $\underline{S}$ when ties of the sizes $n_1, \ldots, n_k$ are present in one ranking [cf. *Terpstra*, 1952]. For this statistic $d_{h,i}\, (\vec{\xi}) = 1$ and it follows that

$$d_i^{(W_1)}\, (\vec{\xi}) = 2\, (\xi_1 + \ldots + \xi_{i-1}) + \xi_i - 1\,. \tag{3.21}$$

Thus $\vec{d}^{(W_1)}\, (\vec{\xi})$ satisfies the condition that for all $\vec{\xi}$

$$d_1\, (\vec{\xi}) \leqslant d_2\, (\vec{\xi}) \leqslant \ldots \leqslant d_k\, (\vec{\xi}),\ d_1\, (\vec{\xi}) < d_k\, (\vec{\xi}), \tag{3.22}$$

which means that for all $\vec{\xi}$ the test based on $\underline{W}_1$ is admissible as $\mathrm{Eff}_{\vec{\xi}}(\vec{d}\, (\vec{\xi}) \mid \vec{\theta}) = 1$ if $\vec{\theta} = \vec{d}\, (\vec{\xi})$, while $\vec{d}\, (\vec{\xi}) \in \Theta$.

We also consider the statistic

$$\underline{W}_2 := n^2 \sum_{h<i}\sum\, (n_h\, n_i)^{-1}\, \underline{U}_{h,i}\,, \tag{3.23}$$

for which [cf. *Terpstra*, 1955]

$$n^{-2}\, E\, \underline{W}_2 = \sum_{h<i}\sum \{2P\,[\underline{x}_h < \underline{x}_i] - 1\}. \tag{3.24}$$

For this statistic $d_{h,i}(\vec{\xi}) = (\xi_h\, \xi_i)^{-1}$ and it follows that

$$d_i^{(W_2)}(\vec{\xi}) = (\xi_i)^{-1}\,(2i - (k+1)). \tag{3.25}$$

The test based on $\underline{W}_2$ is not admissible for each $\vec{\xi}$ as the necessary condition (3.22) does not hold for each $\vec{\xi}$.

References

Hájek, J., and *Z. Šidák*: Theory of Rank Tests. Prague 1967.

Koziol, J.A., and *N. Reid*: On the asymptotic equivalence of two ranking methods for *k*-sample linear rank statistics. Ann. Statist. **5**, 1977, 1099–1106.

Lehmann, E.L.: Testing Statistical Hypotheses. New York 1959.

Terpstra, T.J.: The asymptotic normality and consistency of Kendall's test against trend when ties are present in one ranking. Nederl. Akad. Wetensch. Indag. Math. **14**, 1952, 327–333.

– : A nonparametric test for the problem of *k*-samples. Nederl. Akad. Wetensch. Indag. Math. **16**, 1954, 505–512.

– : A generalization of Kendall's rank correlation statistic. Nederl. Akad. Wetensch. Indag. Math. 17, 1955, 690–696 and **18**, 59–66.

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