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Feedback Decomposition of Nonlinear Control Systems

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Abstract—**By using the recently developed (differential) geometric approach to nonlinear systems, a feedback decomposition for nonlinear control systems is derived.**

I. Introduction

Consider a control system of the form

$$\dot{x} = A(x) + \sum_{i=1}^{m} B_i(x) u_i \tag{1.1a}$$

$$z_i = H_i(x), \qquad i = 1, \cdots, m \tag{1.1b}$$

where x are local coordinates of a smooth n-dimensional manifold $M, A, B_1, \cdots, B_m$ are smooth vector fields on M, and H_i: $M \to N_i$ is a smooth output map from M to a smooth $p_i-(p_i \geqslant 1)$ dimensional manifold N_i for $i = 1, \cdots, m$. We assume that each H_i, $i = 1, \cdots, m$, is a surjective submersion. Furthermore, we will assume that the system (1.1a) is strongly accessible (see [12]).

In this note we will study the *static state feedback noninteracting control problem*. That is (see [4]), we seek a control law of the form

$$u = \alpha(x) + \beta(x) v \tag{1.2}$$

where α: $M \to \mathbb{R}^m$, β: $M \to \mathbb{R}^{m \times m}$ are smooth maps, $\beta(x) = (\beta_{ij}(x))$ is nonsingular for all x in M, and $v = (v_1, \cdots, v_m)^t \in \mathbb{R}^m$. Let $\tilde{A}(x) = A(x) + \sum_{i=1}^m B_i(x)\alpha_i(x)$ and $\tilde{B}_i(x) = \sum_{j=1}^m B_j(x)\beta_{ji}(x)$. Then, in suitable local coordinates the modified dynamics $\dot{x} = \tilde{A}(x) + \sum_{i=1}^m \tilde{B}_i(x) v_i$ should read

$$\left\{ \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_m \end{pmatrix} = \begin{pmatrix} \tilde{A}_1(x_1) \\ \tilde{A}_2(x_2) \\ \vdots \\ \tilde{A}_m(x_m) \end{pmatrix} + \begin{pmatrix} \tilde{B}_1(x_1) & & & \bigcirc \\ & \tilde{B}_2(x_2) & & \\ & & \ddots & \\ \bigcirc & & & \tilde{B}_m(x_m) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \right. \tag{1.3a}$$

$$\begin{cases} z_1 = H_1(x_1) \\ \vdots \\ z_m = H_m(x_m) \end{cases} \tag{1.3b}$$

where $x = (x_1, \cdots, x_m)$ with each x_i and z_i possibly being a vector. For linear systems, the above problem—the restricted decoupling problem (RDP)—has been solved under the additional assumption that the set of outputs is "complete"; i.e., $\cap_{i=1}^m \ker H_i = 0$ (see [13]). In the solution we present here, we use as key tools the so-called (regular) controllability distributions introduced in [8]. In this way, our approach completely fits in the systematic work on the generalization of Wonham's geometric approach to linear systems (see, e.g., [3]–[10]). We note that a *parallel decomposition* as in (1.3a) has been studied in [11]. We also note that similar results are derived in [4] and in a different style in [1]. The main purpose of this note is to show that the solution of the nonlinear RDP can also be derived by directly generalizing the theory of [13].

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II. Problem Formulation

Recall the following definitions (see [3]–[9]).

Definition 2.1: An involutive distribution D of fixed dimension, on M, is *controlled invariant* for the system (1.1a) if there exists a feedback of the form (1.2) such that the modified dynamics $\dot{x} = \tilde{A}(x) + \sum_{i=1}^m \tilde{B}_i(x) v_i$ leave D invariant; i.e.,

$$[\tilde{A}, D] \subset D$$
$$[\tilde{B}_i, D] \subset D, \qquad i = 1, \cdots, m.$$

Definition 2.2: An involutive distribution D of fixed dimension, on M, is a *regular controllability distribution* of the system (1.1a) if it is controlled invariant for the system, and moreover,

$$D = \text{involutive closure of } \{ ad_{\tilde{A}}^k \tilde{B}_i | k \in \mathbb{N}, i \in I \}$$

for a certain subset $I \subset \{1, \cdots, m\}$.

Instead of the above notion of controlled invariance, it is sufficient to use a somewhat weaker concept.

Definition 2.3: An involutive distribution D of fixed dimension, on M, is *locally controlled invariant* for the system (1.1a) if locally around each point $x_0 \in M$ there exists a feedback of the form (1.2) such that the modified dynamics $\dot{x} = \tilde{A}(x) + \sum_{i=1}^m \tilde{B}_i(x) v_i$ leave D invariant.

Similarly, one defines a local version of Definition 2.2: the regular local controllability distributions.

In considering the static state feedback noninteracting control problem, we seek regular local controllability distributions $R_1, \cdots, R_m$ defined by

$$R_i := \text{involutive closure of } \{ ad_{\tilde{A}}^k \tilde{B}_i | k \in \mathbb{N} \} \tag{2.1}$$

where $\tilde{A}$ and $\tilde{B}_i$ are as in (1.3a), $i = 1, \cdots, m$.

Remark: In the local coordinates of (1.3a) we see that $R_i = \text{span}\{\partial / \partial x_i\}$, and clearly each distribution R_i satisfies $[\tilde{A}, R_i] \subset R_i$ and $[\tilde{B}_j, R_i] \subset R_i$, $j = 1, \cdots, m$, $i = 1, \cdots, m$.

Assuming (2.1), we see that

$$R_i \subset \ker H_{j*} =: K_j \qquad j \neq i, \ i, j = 1, \cdots, m \tag{2.2}$$

which means exactly that $v_j(\cdot)$ does not affect the output $z_i(\cdot)$, for $j \neq i$. Second, we have the nonlinear version of *output controllability*; that is,

$$H_{i*}(R_i) = TN_i \qquad i = 1, \cdots, m. \tag{2.3}$$

This follows from the fact that the system (1.1a) is strongly accessible, so that (1.3a) is also strongly accessible. But then each of the systems $\dot{x}_i = \tilde{A}_i(x_i) + \tilde{B}_i(x_i) v_i$ is strongly accessible, and by the fact that the map H_i is a surjective submersion we see that the set of reachable output values has nonempty interior in N_i for all $i = 1, \ldots, m$.

Thus, the static state feedback noninteracting control problem can be stated as follows.

Given the system (1.1a) (1.1b), find (if possible) a local feedback law of the form (1.2) such that (2.2) and (2.3) hold for the distributions R_i defined by (2.1). Now, as in the linear case, there is a compatibility problem (see [13]). Clearly, if we have controlled invariant distributions $D_1, \ldots, D_m$, then by no means does it follow that there exists a local feedback (1.2) which leaves each of them invariant. Therefore, we make the following assumption:

$$\bigcap_{i=1}^{m} \ker H_{i*} = 0 \tag{2.4}$$

which means that the map

$$H: M \to N_1 \oplus N_2 \oplus \cdots \oplus N_m, \quad H(x) = (H_1(x), \cdots, H_m(x))$$

is locally injective.

III. Main Theorem

Define R_i^* : = supremal regular local controllability distribution in

$$\bigcap_{j \neq i} \ker H_{j*}, \qquad i = 1, \cdots, m.$$

Remark: R_i^* is well defined (see [6], [8]), but the dimension probably is not fixed.

Theorem 3.1: Under the assumption (2.4) and the assumption that each R_i^* has fixed dimension, $i = 1, \cdots, m$, the static state feedback noninteracting control problem is solvable in a local fashion if and only if

$$R_i^* + K_i = TM. \tag{3.1}$$

Proof: Assume (3.1) holds, then (2.2) and (2.3) are true for R_i^*. We show next that the $\hat{K}_i := \cap_{j \neq i} \ker H_{j*}$, $i = 1, \cdots, m$, are independent. Indeed,

$$\hat{K}_i \cap \sum_{j \neq i} \hat{K}_j = \left(\bigcap_{r \neq i} \ker H_{r*} \right) \cap \sum_{j \neq i} \left(\bigcap_{s \neq j} \ker H_{s*} \right)$$
$$\subset \left(\bigcap_{r \neq i} \ker H_{r*} \right) \cap \ker H_{i*} = \bigcap_{r=1}^{m} \ker H_{r*} = 0.$$

Since $R_i^* \subset \hat{K}_i$, $i = 1, \cdots, m$, it follows that the R_i^* are independent. In the next step, we will show that the R_i^* are compatible; i.e., there is a local feedback (1.2) which leaves each of the distributions R_i^* invariant. From (3.1) we see that for each $i = 1, \cdots, m$, $R_i^* \neq 0$. For if $R_i^* = 0$ for an $i \in \{1, \cdots, m\}$, then $K_i = TM$, which means that $z_i = H_i(x)$ is constant. Therefore, we know by the independence of the R_i^* that locally there exist independent vector fields $0 \neq \bar{B}_i$ with $\bar{B}_i \in R_i^* \cap \operatorname{span}\{B_1, \cdots, B_m\}$, $i = 1, \cdots, m$. So, $\operatorname{span}\{B_1, \cdots, B_m\} = \operatorname{span}\{\bar{B}_1, \cdots, \bar{B}_m\}$. We also have that $\dim R_i^* \geqslant p_i$ (by assumption R_i^* has fixed dimension), and thus from the independency of the R_i^*, we have $R_1^* = + \cdots + R_m^* = TM$. Thus, the distributions $R_1^*, \cdots, R_m^*$ are *simultaneously integrable* (see [11, Definition 3.1 and Lemma 3.1]). So locally around each point $x_0 \in M$ there exist coordinates such that $R_i^* = \operatorname{span}\{\partial/\partial x_i\}$, $i = 1, \cdots, m$, with each x_i possibly being a vector. Now, from the fact that the distributions R_i^* are locally controlled invariant we have

$$\left[\bar{B}_i, R_j^*\right] \subset R_j^* + \operatorname{span}\{\bar{B}_1, \cdots, \bar{B}_m\}, \qquad i = 1, \cdots, m \tag{3.2a}$$
$$\left[A, R_j^*\right] \subset R_j^* + \operatorname{span}\{\bar{B}_1, \cdots, \bar{B}_m\} \tag{3.2a}$$

for all $j = 1, \cdots, m$.

From (3.2a) we see that

$$\left[\bar{B}_1, R_2^* + \cdots + R_m^*\right] \subset R_2^* + \cdots + R_m^* + \operatorname{span}\{\bar{B}_1, \cdots, \bar{B}_m\}$$
$$= R_2^* + \cdots + R_m^* + \operatorname{span}\{\bar{B}_1\} \tag{3.3}$$

where the last equality follows from the fact that $\bar{B}_i \in R_i^*$, $i = 1, \cdots, m$. Note also that the distribution $R_2^* + \cdots + R_m^*$ is involutive, cf. [11]. Now, from (3.3) and [5], [7] it follows that there locally exists a vector field $\tilde{B}_1$ such that $\operatorname{span}\{\tilde{B}_1\} = \operatorname{span}\{\bar{B}_1\}$ and $[\tilde{B}_1, R_2^* + \cdots + R_m^*] \subset R_2^* + \cdots + R_m^*$. Therefore, in the coordinate system constructed above we have that $\tilde{B}_1(x) = (\tilde{B}_1(x_1), 0, \cdots, 0)^t$.

Similarly, we construct vector fields $\tilde{B}_i$, $i = 2, \cdots, m$, such that $[\tilde{B}_i, R_1^* + \cdots + R_{i-1}^* + R_{i+1}^* + \cdots + R_m^*] \subset R_1^* + \cdots + R_{i-1}^* + R_{i+1}^* + \cdots + R_m^*$ and $\operatorname{span}\{\tilde{B}_i\} = \operatorname{span}\{\bar{B}_i\}$. Thus,

$$\tilde{B}_i(x) = (0, \cdots, 0, \tilde{B}_i(x_i), 0, \cdots, 0)^t.$$

Next, from (3.2b) we see that

$$[A, R_2^* + \cdots + R_m^*] \subset R_2^* + \cdots + R_m^* + \operatorname{span}\{\bar{B}_1\} \tag{3.4}$$

and therefore we can construct a local feedback $u = \bar{B}(x)\alpha_1(x)$ such that $\tilde{A}(x) = A(x) + \bar{B}_1(x)\alpha_1(x)$ satisfies (cf. [3]) $[\tilde{A}, R_2^* + \cdots + R_m^*] \subset R_2^* + \cdots + R_m^*$. Similarly, for the distribution $R_1^* + \cdots + R_{i-1}^* + R_{i+1}^* + \cdots + R_m^*$, we construct a feedback $u = \bar{B}_i(x)\alpha_i(x)$ such that the modified dynamics leave this distribution invariant. Finally, by applying the total feedback $u = \bar{B}_1(x)\alpha_1(x) + \cdots + \bar{B}_m(x)\alpha_m(x)$, we obtain $A(x) = (A_1(x)_1), A_2(x_2), \cdots, A_m(x_m))$. So we have established a local feedback (1.2) such that the modified dynamics are as in (1.3a), and also from (3.1) (1.3b) is satisfied. Furthermore, we note that each system $\dot{x}_i = A_i(x_i) + B_i(x_i)v_i$ is strongly accessible, and we have that

$$R_i^* = \text{involutive closure of } \{ad_{\tilde{A}}^k \tilde{B}_i | k \in \mathbb{N}\}, \qquad i = 1, \cdots, m.$$

Conversely, from the fact that the R_i^* are supremal relative to the condition (2.2) and from (2.3)—which is equivalent to $R_i + K_i = TM$—it follows that (3.1) is necessary. □

IV. Remarks

1) In [11, Lemma 3.1] the distributions $D_1, \cdots, D_L$ should be independent; i.e., for each disjoint subset I_1 and I_2 of $\{1, \cdots, L\}$, one has that $D^{I_1} \cap D^{I_2} = 0$.

2) $[ad_{\tilde{A}}^k \tilde{B}_i, ad_{\tilde{A}}^l \tilde{B}_j] = 0$ for all $k, l \in \mathbb{N}$ and $i \neq j$ (see also, [11]).

3) If the number of output channels is smaller than the number of inputs, the above procedure still works in a slightly modified way. Namely, there are more than one independent vector fields $\bar{B}_i$ in $R_i^* \cap \operatorname{span}\{B_1, \cdots, B_m\}$, and/or there exist some additional input vector fields $\tilde{B}_k$ which do not belong to one of the distributions R_i^*, but—after applying feedback—also have the form $\tilde{B}_k(x) = (\tilde{B}_k^1(x_1), \tilde{B}_k^1(x_2), \cdots, \tilde{B}_k^m(x_m))^t$. These vector fields are superfluous for the whole control synthesis of the system.

4) Each of the systems $\dot{x}_i = \tilde{A}_i(x_i) + \tilde{B}_i(x_i)v_i$, $z_i = H_i(x_i)$ is strongly invertible (see [2]). This has also been clarified in a geometric way in [9], and follows directly from the condition that $R_i^* + K_i = TM$, so R_i^* is not contained in $\ker H_{i*}$. We also note that the situation described in Theorem 3.1 is even more special. Namely, the system $\dot{x}_i = \tilde{A}_i(x_i) + \tilde{B}_i(x_i)v_i$ is strongly invertible with respect to each of the components of the output z_i.

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