3D MESH REFINEMENT PROCEDURE USING THE BISECTION AND RIVARA ALGORITHMS WITH MESH QUALITY ASSESSMENT

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ABSTRACT

Mesh refinement procedures for the solution of three dimensional problems are described. The computational domain is represented by an assembly of tetrahedral elements and the mesh refinement is acheived by the bisection and Rivara methods using an explicit mesh density function coupled with an automatic 3D mesh generator.
A couple of benchmark examples is used to compare the performance of both refinement methods in terms of mesh and size qualities, number of generated elements and CPU time consumed.

KEY WORDS

3D mesh generation, mesh refinement, bisection method, Rivara method and mesh quality.

1. INTRODUCTION

The finite element (FE) method has proved to be a very useful tool in numerical analysis. However, a major difficulty has been the assessement of discretisation errors and the design of suitable meshes. Some recent developments have helped to improve this situation. For example, Zienkiewicz and Zhu [1] have introduced a successful adaptive mesh refinement (AMR) strategy based on a simple errror estimation which is reasonably accurate and which can be easily implemented in existing FE codes. It can thus be combined with a full adaptive refinement process or, simply, provide guidance for mesh design which allows the user to reach predetermined standards of accuracy.

To achieve a given accuracy with the least effort, mesh generation procedures are essential. Indeed, in an Adaptive Mesh Refinement context, these should be capable of designing a mesh from the specification of an element size distribution. However, limitations in the availability of robust, versatile and efficient 3D mesh generators have hindred the extension of 2D AMR procedures [1-5] to the 3D case [6-14].

In recent publications [7,8], as a contribution to this subject, we have presented an adaptive 3D mesh refinement procedure using the bisection method rather than a remeshing procedure. This choice is justified by the prohibitive cost of a remeshing process. Moreover, the refinement algorithms have proved to be efficient and cost effective in practice.

In general, the refinement procedures have to overcome three major difficulties. They should be able to
- produce meshes of a desired density,
- generate conforming elements of good quality, and
- avoid the generation of an excessive number of elements (nodes).

It is on these bases that the bisection algorithm and a 3D version of Rivara algorithm are compared in the present work in a mesh refinement process.

First, an overview of the mesh generator used is presented. The bisection and Rivara algorithms are presented in Section 3. The quality issues are then discussed and a couple of benchmark examples given.

2. MESH GENERATION

Various methods of mesh generation are in existence [15-17]. While there are a large number of 2D mesh generators, 3D mesh generators are scarce.
Due to the complexities associated with generating 3D meshes, much work has been devoted to the automation of the procedure [18-24].
In an adaptive mesh refinement context, in which a refinement or a remeshing procedure is used iteratively until the estimated error reaches a user specified value, the problem is further complicated.

In the present work a fully automatic 3D mesh generator called XMAILLE [18] is used.
XMAILLE utilises Constructive Solid Graph as the representation scheme of objects to be meshed.
The geometry of the computational domain is described using a volumetric modeler which uses a library of volumetric primitives.

Each volumetric primitive is meshed with valid finite elements. Continuity between primitives is achieved via neighbourhood relations.
The mesh generator takes as input the geometry and the associated meshing model of the geometric primitives and proceeds by boolean operations between elements [1].
XMAILLE is used to generate initial meshes. The subsequent mesh refinement procedure is next presented.

3. MESH REFINEMENT

In this section, a couple of subdivision algorithms based on the bisection and Rivara methods [25-30], are presented. Other related problems are also discussed.

3.1 3D Bisection algorithm

A typical 3D bisection subdivision strategy is based on the following algorithm :

1. Initialise the refinement information and determine the set E_t of elements to be refined.
2. Sort elements in E_t on the basis of their longest edge length.
3. Bisect elements in E_t and insert the newly created elements (if to be refined) in a set E_r.
4. If E_r is empty stop. Else $E_t \leftarrow E_r$, $E_r \leftarrow \varnothing$ and go to 2.

The initialisation routine (Step 1) consists in reading the refinement information for each element. All elements to be refined according to the error estimator are inserted in a set E_t. These are, henceforth, sorted on the basis of their longest edge length (Step 2), which, to some extent, avoids the deterioration of the quality of the resulting mesh. In Step 3, it is necessary after bisecting each element in E_t, to propagate the bisection to the neighbouring elements for conformity reason.

3.2 3D Rivara algorithm [26]

[1] *Details of the method used in XMAILLE and some examples to illustrate its potential can be found in reference 18.*

A typical 3D Rivara subdivision strategy is based on the following algorithm :

1. Initialise the refinement information and determine the set E_t of elements to be refined.
2. Sort the elements in E_t on the basis of their longest edge length.
3. Bisect elements in E_t and insert the newly created elements (if to be refined) in a set E_r. Mark all non-conforming elements generated in this way.
4. Make conforming all non-conforming elements created in step 3.
5. If E_r is empty stop. Else $E_t \leftarrow E_r$, $E_r \leftarrow \varnothing$ and go to 2.

The first and second steps are analogous to those of the bisection method. In step 3, contrary to the bisection method, elements in E_t are bisected with no propagation to neighbour elements sharing the edge of subdivision. However the non-conforming elements generated in this way are marked in Step 3 and made conforming in Step 4.

Obviously, the difference between the bisection and Rivara algorithms, is the added effort in the later method in marking and treating the non-conforming elements created during the bisection process.
This process is complex in terms of data management and is beyond the scope of the present paper. This will be presented in a forthcoming publication.

Before proceeding further, we make the following additional comments

Refinement information The subdivision data for the newly created elements is updated by an interpolation in which each refinement parameter is weighted by the volume of the element to give a refinement density which, in turn, is assigned to each node by averaging the refinement densities of the elements surrounding every node.

Sorting of elements Elements to be refined are sorted on the basis of their longest edge length. The first element in the sorted list is the element with the longest edge length and is the first candidate for subdivision.
However, it is found that when the mesh to be refined is not of good quality, the ratio : radius of inscribed sphere over the longest edge length is better indicated for the sorting process.

Edge of subdivision It should be noted that the choice of the longest edge may not be unique. In this case, a random selection is not indicated and another criterion based on the exploration of the surrounding elements of each long edge is introduced. The edge with the minimum incident elements not to be refined is selected. This criterion ensures that a minimum number of unnecessary elements are generated in order to maintain the conformity of the mesh.
Finally we note that, in some other cases, edges on the boundary of the domain are to be privileged.

Evaluation of boundary nodes Boundary nodes may be evaluated on the geometry of the model that is driven by the volumetric modeler. After a subdivision iteration, a request to the modeler allows the projection of the boundary nodes of the mesh onto the supporting geometry.
The model that is driven by the modeler is a Constructive Solid Graph in which the volumetrique primitives are referenced by numbers. The projection on the boundaries necessitates a referencing mechanism that establishes a link between the boundary nodes and their geometric support. A node may reference a geometric entity or more if it is at an intersection.
We note that the projection may cause the problem of inverted elements with negative volumes. To avoid this problem it is recommended to adequately refine the surfaces.

Nodes repositionning Elements with poor aspects ratios are smoothed by the polyhedron that encloses a newly created internal node. A relaxation based method is used to move the newly created node to the centroid of the surrounding polyhedron if the new node is in the interior.

Attributes heritage The topological entities (elements, edges, vertices) do refer to attributes or properties (elements size, boundary conditions, material properties, loading, etc). The refinement process maintains each information using a heritage mechanism. The properties are implemented in the form of sets. A topological entity that inherits a property is inserted in a set that bears the same name as that property. The heritage consists in analysing the properties of each created entity that is affected by the subdivision of an element and depending on their nature they will or will not be affected to the newly created entities.

4. MESH QUALITY

To characterise a tetrahedron shape and detect the presence of some common configuration of poorly shaped tetrahedra *e.g.* thin, wedge like, flat and sliver elements, researchers have proposed various measures [2]. For example, Cavendish *et al* [31] characterise a tetrahedron by the ratio of the inscribed sphere radius r to the circumscribed sphere radius R

$$\beta = 3.\frac{r}{R} \tag{1}$$

Baker [32] proposed the combined use of the ratios inscribed sphere radius r to maximum edge length L_{Max}, maximum edge length L_{Max} to circumscribed sphere radius R and minimum edge length L_{Min} to maximum edge length L_{Max}

$$\sigma = 4.898979\frac{r}{L_{Max}} \qquad \omega = 0.612507\frac{L_{Max}}{R} \qquad \tau = 1.\frac{L_{Min}}{L_{Max}} \tag{2}$$

Cougny et al [33] used the four composing facet areas $A_i\,(i = 1,4)$ and the tetrahedron volume V to define the following normalised aspect ratio

$$\kappa = 4.58457^{-4}\,\frac{\sum_{i=1}^{4} A_i^3}{V^4} \tag{3}$$

Dannenlogue and Tanguy [34] found that the ratio involving the average edge length of the six composing edges L_{Avr} and the tetrahedron volume V suffices to characterise a tetrahedron

$$\alpha = 8.479670\frac{V}{L_{Avr}^3} \tag{4}$$

An extension of this measure, in which the average edge length is replaced by the root mean square of the edge lengths L_{RMS}, has recently been proposed by Parthasarathy et al [35]

$$\gamma = 8.4779670\frac{V}{L_{RMS}^3} \tag{5}$$

For a given element, the number of basic compuations is minimum for measures τ, α, γ and maximum for β.

In addition to shape quality measures, in a refinement process, a size criterion is also introduced. The size criterion is defined as the ratio between the actual size L (mean edge length) of a given element i and the desired size $\overline{L}$ and is given by

$$\varphi = L / \overline{L} \tag{6}$$

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[2] Scaled to be 1 for an equilateral tetrahedron.

In order to a have a fairly good picture of the mesh quality the Minimum, Mean and Maximum qualities are introduced

$$Q_{Min} = Min(Q_i) \quad \underset{1 \le i \le N}{} \qquad Q_{Mean} = \frac{1}{N}\sum_{i=1}^{N} Q_i \qquad Q_{Max} = Max(Q_i) \tag{7}$$

where N is the number of tetrahedra and Q_i is the quality measure of the *ith* tetrahedron.

It should be stressed that, in an Adaptive Mesh Refinement context, the existence of badly shaped elements should be related to the estimated error for those elements. Obviously, a badly shaped element with a large estimated error (*i.e.* in a critical region) is more damaging than a badly shaped element with a small estimated error. Therefore, a badly shaped element should be considered as *critical* if its estimated error is larger than the mean estimated error for the mesh under consideration.

5. NUMERICAL EXAMPLES

To illustrate the numerical performance of the refinement strategies used in this work a couple of benchmark examples are now considered. These are compared in terms of number and quality of generated elements and total CPU time consumed (exclusive of mesh quality assessment - *For fair comparison purpose, the quality evaluator acts as a separate routine*).

The quality measure of the resulting meshes is evaluated with respect to the r / L_{Max} and L_{Min} / L_{Max} criteria. We shall consider, as is commonly accepted, that a given mesh is of good shape quality if there exist no elements in the following ranges : r / L_{Max} < 0.4 and L_{Min} / L_{Max}< 0.3 or r / L_{Max} < 0.2. The minimum values of both criteria are also used for comparison purpose.
We shall also consider a mesh to be of good size quality if the ratio $L / \overline{L}$ lays in the interval [2/3,3/2] [37].

5.1 Example 1 : **Uniform mesh refinement of a cube**

We first consider the uniform refinement of a cube. The initial mesh quality is presented in Table 1. A desired element size of 10 is given. The resulting mesh shape and size qualities are presented in Tale 2.

	r / R	r / L_{Max}	V / L_{Avr}^3	L_{Min} / L_{Max}
Min	0.508666	0.508666	0.601921	0.577350
Mean	0.564057	0.608834	0.703934	0.620602
Max	0.597717	0.732051	0.803509	0.707107

Table 1. Initial mesh quality.

This example clearly shows that the use of the bisection method without sorting considerably deteriorates the quality of the final mesh (almost 9% of the elements are in the range r / L_{Max} < 0.4 and approximately 4% in the range L_{Min} / L_{Max} < 0.3).
The analysis of the results also shows that the mesh generated by the Rivara algorithm (with or without sorting) is of a better quality as there are no elements in the range r / L_{Max} < 0.4 and L_{Min} / L_{Max} < 0.3. Nevertheless, the mesh generated using the bisection method with sorting remains acceptable as all generated elements quality is in the range r / L_{Max} > 0.2.
With respect to the 2/3 < $L / \overline{L}$ < 3/2 criterion for mesh size control, it is seen that both algoritms generate elements of the desired mesh size distribution. The range of the $L / \overline{L}$ values is, as expected, more restrained when Rivara algorithm is used.

Finally we note that, the simple bisection generates more nodes than the other algorithms because of the important number of bisections it operates.

	Bisection	Bisection + Sorting	Rivara3D	Rivara3D + Sorting
Iterations	7	7	7	7
Elements	3936	3515	3517	3519
Nodes	933	842	841	841
r / L_{Max} **Min**	0.0499186	0.313694	**0.400783**	0.400783
r / L_{Max} **Mean**	0.572655	0.601893	0.60431	0.604619
r / L_{Max} **Max**	1	1	1	1
L_{Min} / L_{Max} **Min**	0.188982	0.288675	**0.408248**	**0.408248**
L_{Min} / L_{Max} **Mean**	0.535242	0.564946	0.564194	0.564023
L_{Min} / L_{Max} **Max**	1	1	1	1
$L / \overline{L}$ **Min**	0.873464	0.956832	1.10485	1.10485
$L / \overline{L}$ **Mean**	1.22086	1.24317	1.24097	1.24052
$L / \overline{L}$ **Max**	1.46158	1.35316	1.35316	1.35316
r / L_{Max} **< 0.4%**	8.86	0.28	0	0
r / L_{Max} **< 0.2%**	1.57	0	0	0
L_{Min} / L_{Max} **< 0.3%**	3.88	0.08	0	0
2/3 < % ($L / \overline{L}$ **) < 3/2**	100	100	100	100
CPU (s)	140	131	145	150

Table 2. Uniform Refinement of a cube : Results summary.

5.2 Example 2 : **Mesh refinement of a hollow cube**

As a second example, we consider the hollow cube shown in Figure 1.
In this example the mesh density is given by the following explicit function

$$D(r) = h_0 + H\,(1 - e^{-r/\tau}) \tag{8}$$

with

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{9}$$

A corner of the cube is taken as the origine (x_0, y_0, z_0) of the mesh density function. This function allows us to describe a size that is close to $h_0 + H$ as we move away from the origine and that tends to h_0 as we get closer to it. The τ parameter allows us to tune the range of the variation of $D(r)$. A small value of τ means that we try to satisfy a mesh size of h_0 in the vicinity of the origin.

The initial mesh quality is presented in Table 3. The results obtained for different refinement algorithms are presented in Table 4 and the final meshes displayed in Figure 2.
The minimum quality values obtained for this example, show that Rivara algorithm is the one which produces the best quality refined meshes : the minimum values of r / L_{Max} and L_{Min} / L_{Max} are the largest and the number of elements in the range r / L_{Max} < 0.4 is the smallest.

These results also demonstrate that the mesh produced by simple bisection cannot be used because of the presence of an important number of badly shaped elments (more than 12% in the range r / L_{Max} < 0.4 and nearly 4% in the range L_{Min} / L_{Max} < 0.3%).
Once more the use of sorting in conjunction with Rivara algorithm does not improve the mesh quality but rather slightly deteriorates it.

The analysis of the results in terms of size quality, once again, shows that all tested algorithms have generated almost all of the elements in the tolerance range [2/3,3/2]. Moreover, a scrutiny of Table 3, shows that when Riavara algorithm is used, the interval $[(L/\overline{L})_{Min}, (L/\overline{L})_{Max}]$ is more restrained than in the case of other algorithms.

	r/R	r/L_{Max}	V/L^3_{Avr}	L_{Min}/L_{Max}
Min	0.508666	**0.508666**	0.601921	0.577350
Mean	0.568254	**0.604382**	0.704405	0.612247
Max	0.597717	0.732051	0.803509	0.707107

Table 3. Quality of the initial mesh of the hollow cube.

	Bisection	**Bisection + Sorting**	**Rivara3D**	**Rivara3D + Sorting**
Iterations	15	15	15	15
Elements	11640	9917	10117	10168
Nodes	2805	2435	2473	2482
r/L_{Max} **Min**	0.0555519	0.293179	**0.359681**	0.339459
r/L_{Max} **Mean**	0.545894	0.584374	0.5842	0.583953
r/L_{Max} **Max**	1	1	1	1
L_{Min}/L_{Max} **Min**	0.0762493	0.25	**0.353553**	0.288675
L_{Min}/L_{Max} **Mean**	0.514044	0.554054	0.556644	0.557701
L_{Min}/L_{Max} **Max**	1	1	1	1
$L/\overline{L}$ **Min**	0.50402	0.717876	0.800021	0.615981
$L/\overline{L}$ **Mean**	1.20205	1.24002	1.23506	1.23271
$L/\overline{L}$ **Max**	1.49821	1.49821	1.49821	1.49821
r/L_{Max} **< 0.4%**	12.6	1.62	1.12	1.27
r/L_{Max} **< 0.2%**	0.9	0	0	0
L_{Min}/L_{Max} **< 0.3%**	3.89	0.18	0	0
2/3 < % ($L/\overline{L}$ **) < 3 /2**	99.63	100	100	99.99
CPU (s)	687	600	733	747

Table 4. Mesh Refinement of a hollow cube : Results summary.

Figure 1. Density function used in Example 2.

Figure 2. Refined meshes obtained in Example 2.

6. CONCLUSION

A mesh refinement method for the solution of three dimensional problems is described. The mesh refinement is acheived using the bisection and Rivara algorithms using an explicit mesh density function coupled with an automatic 3D mesh generator.

A couple of benchmark examples have been used to compare the performance of both refinement algorithms in terms of mesh and size qualities, number of generated elements and CPU time consumed.

It is found that the simple bisection algorithm generates elements of poor quality. When using such an algorithm, sorting has to be used to enhance the quality of the refined meshes.
Although the size quality is acceptable, the bisection algorithm generates meshes with large number of elements and nodes.

Rivara3D algorithm produces meshes of best quality. For both examples presented in this work and other benchmark tests we have carried out, the minimum values of the quality criteria are higher and the number of generated elements in the range $r / L_{Max} < 0.4\%$ and $L_{Min} / L_{Max} < 0.3\%$ is the smallest.
Moreover, in terms of size quality, when Rivara3D algorithm is used, the distribution of the ratio $L / \overline{L}$ is restrained and a minumum (optimum) number of elements and nodes is acheived.

Finally we note that the total cost (*i.e.* CPU time) for mesh refinement computations for all cases considered in this work is quite reasonable.

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List of figures and tables

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