

local data dependent vector  $h_i^j$ , i.e.,

$$\hat{x}_i^D = \sum_{j=1}^K [P_i(P_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j] \quad (13)$$

where

$$\hat{x}_i^{Dj} = \bar{x}_i^{Dj} + P_i(H^j)^T(V_i^j)^{-1}[\bar{z}_i^j - H^j \bar{x}_i^{Dj}];$$

$$\bar{x}_i^{Dj} = A \hat{x}_{i-1}^{Dj}$$

$$(P_i^j)^{-1} = (M_i^j)^{-1} + (H^j)^T(V_i^j)^{-1}H^j$$

$$h_{i+1}^j = F_{i+1}h_i^j + G_{i+1}^j \bar{x}_{i+1}^{Dj}; h_1^j = 0$$

$$F_i = \left[ I - \sum_{j=1}^K P_i(H^j)^T(V_i^j)^{-1}H^j \right] A = P_i M_i^{-1} A \quad (17)$$

$$G_{i+1}^j = P_{i+1} M_{i+1}^{-1} A P_i(P_i^j)^{-1} A^{-1} - P_{i+1}(M_{i+1}^j)^{-1}. \quad (18)$$

Now, define

$$\alpha_i^{Dj} = (B^j)^T S_{i+1} [P_i(P_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j]. \quad (19)$$

Equation (4) becomes

$$u_i^j = -[R^j + (B^j)^T S_{i+1} B^j]^{-1} \left\{ \sum_{m=1}^K \alpha_i^{Dm} + (B^j)^T S_{i+1} x_i^C \right\}. \quad (20)$$

Since  $x_i^C$  can be calculated from (10) by using the one-step delayed information  $u_{i-1}^j$ ; therefore, in order to compute (20), only  $(K-1)$  signals of the form (19) need to be transmitted immediately. Since  $\alpha_i^{Dj}$  needs only local information, the problem has been solved by a decentralized scheme with communication.

#### PROPOSED NEW SCHEME

In order to construct  $\alpha_i^{Dj}$  in (19), both  $\hat{x}_i^{Dj}$  and  $h_i^j$  must be generated locally. In this paragraph, we show that the required linear combination of  $\hat{x}_i^{Dj}$  and  $h_i^j$  in (19) can be always obtained directly by using a linear difference equation. To be precise, if we define a new quantity  $y_i^j$  according to (22), then we can show that (23) is correct. Therefore, we can obtain  $\alpha_i^{Dj}$  from  $y_i^j$  directly by (24). Since  $y_i^j$  also depends on the local information only, we then, without changing the connection structure in [1], obtain a new scheme just by using  $y_i^j$  to replace  $\hat{x}_i^{Dj}$  and  $h_i^j$ .

Let rewrite (11) as follows:

$$\begin{aligned} \hat{x}_{i+1}^D &= \left( I - \sum_{j=1}^K K_{i+1}^j H^j \right) \bar{x}_{i+1}^D + \sum_{j=1}^K K_{i+1}^j \bar{z}_{i+1}^j \\ &= P_{i+1} M_{i+1}^{-1} A \bar{x}_{i+1}^D + \sum_{j=1}^K K_{i+1}^j \bar{z}_{i+1}^j. \end{aligned} \quad (21)$$

Note that (21) is a linear equation with different driving signal  $\bar{z}_i^j, j=1, \dots, K$ . Therefore, we can define  $y_i^j$  to be the component of  $\hat{x}_i^D$  which is driven by  $\bar{z}_i^j$  only, i.e.,

$$y_{i+1}^j = P_{i+1} M_{i+1}^{-1} A \bar{y}_{i+1}^j + K_{i+1}^j \bar{z}_{i+1}^j; \bar{y}_{i+1}^j = A y_i^j \quad y_i^j = 0; j=1, \dots, K. \quad (22)$$

Now, we want to show that

$$y_i^j = P_i(P_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j. \quad (23)$$

Therefore, we can obtain  $\alpha_i^{Dj}$  from  $y_i^j$  according to

$$\alpha_i^{Dj} = (B^j)^T S_{i+1} y_i^j. \quad (24)$$

To show (23), consider the following identities:

$$\begin{aligned} &P_{i+1}(P_{i+1}^j)^{-1} \hat{x}_{i+1}^{Dj} + h_{i+1}^j \\ &\stackrel{(14)(16)}{=} P_{i+1}(P_{i+1}^j)^{-1} \{ \bar{x}_{i+1}^{Dj} + P_{i+1}^j(H^j)^T(V_{i+1}^j)^{-1}[\bar{z}_{i+1}^j - H^j \bar{x}_{i+1}^{Dj}] \} \\ &\quad + P_{i+1} M_{i+1}^{-1} A h_i^j + G_{i+1}^j \bar{x}_{i+1}^{Dj} \\ &\stackrel{(18)}{=} P_{i+1}(H^j)^T(V_{i+1}^j)^{-1} \bar{z}_{i+1}^j + P_{i+1} M_{i+1}^{-1} A h_i^j + P_{i+1} M_{i+1}^{-1} A P_i(P_i^j)^{-1} \hat{x}_i^{Dj} \\ &\quad + P_{i+1} [(P_{i+1}^j)^{-1} - (H^j)^T(V_{i+1}^j)^{-1} H^j - (M_{i+1}^j)^{-1}] A \hat{x}_i^{Dj} \\ &\stackrel{(15)(7)}{=} P_{i+1} M_{i+1}^{-1} A [P_i(P_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j] + K_{i+1}^j \bar{z}_{i+1}^j \end{aligned} \quad (25)$$

and

$$P_i(P_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j = 0 = y_i^j. \quad (26)$$

Since (22) and (25) have the same form and initial conditions, we can conclude that (23) is correct.

#### CONCLUSION

Although there is only a minor change in the proposed new scheme, we can actually achieve a lot of savings in computations. To see this, let us find the difference between the total number of equations of these two schemes for generating  $\alpha_i^{Dj}$ . Without counting the items useful to both schemes, we know from (19) and (24) that  $P_i^j, \hat{x}_i^{Dj}$  and  $h_i^j$  are required for the old scheme and  $y_i^j$  for the new one. Since it needs  $0.5n(n+1)$  equations for  $P_i^j$  from (16) and  $n$  equations for  $\hat{x}_i^{Dj}$  from (14),  $h_i^j$  from (16) and  $y_i^j$  from (22), respectively, the new scheme therefore saves a total of  $n + 0.5n(n+1)$  equations in computations.

#### Comments on "A Nonrecursive Algebraic Solution for the Discrete Riccati Equation"

G. J. GAALMAN

**Abstract**—This correspondence shows how a nonrecursive solution of the discrete matrix Riccati equation can be obtained when, in contrast with the assumption of Vaughan [1], the transition matrix is singular and discusses two numerical solution techniques.

#### I. INTRODUCTION

Vaughan [1] presents in his paper a nonrecursive algebraic solution for the discrete Riccati equation. The method involves finding the eigenvalues and eigenvectors of the canonical state-costate equations. Vaughan assumes that the transition matrix is nonsingular. In this correspondence the nonrecursive solution is derived (in Section III) for the optimal control problem with a singular transition matrix. In a recently published paper Michelsen [2] discusses the numerical problems encountered in the calculation of the eigenvalues and eigenvectors if the transition matrix has eigenvalues close to zero. He shows that this problem can be avoided by a simple reformulation. This reformulation, however, is based on the assumption of nonsingularity of the transition matrix. In Section IV two numerical methods to calculate the eigenvalues and eigenvectors in the singular case are presented. One of these methods gives the method of Michelsen as a special case.

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The author is with the Department of Mechanical Engineering, Twente University of Technology, Enschede, The Netherlands.

Note: A paper of Pappas, Laub, and Sandell, which also uses the generalized eigenvalue concept introduced in Section III, appeared in IEEE TRANSACTIONS ON AUTOMATIC CONTROL after the submission of this paper.

II. PROBLEM FORMULATION

Consider the discrete system described by

$$x_{t+1} = Ax_t + Bu_t, \quad x_0 = x(0) \tag{1}$$

and a quadratic performance criteria

$$x_T' P(T) x_T + \sum_{t=0}^{T-1} x_t' Q x_t + u_t' R u_t \tag{2}$$

where  $x_t \in R^n, u_t \in R^m$  are the state and the control vector, respectively, and  $A, B, Q, R$  and  $P(T)$  are real matrices of appropriate dimensions. In addition,  $Q$  and  $P(T)$  are positive semidefinite matrices and  $R$  is positive definite.

It is well known that the optimal control is given by

$$u_t = -R^{-1} B' y_{t+1} \tag{3}$$

with  $y_t \in R^n$  and

$$y_t = A' y_{t+1} + Q x_t, y_T = P(T) x_T. \tag{4}$$

Assuming a solution of the form

$$y_t = P(t) x_t \tag{5}$$

produces the matrix Riccati equation for  $P(t)$

$$P(t) = A' P(t+1) (I + B R^{-1} B' P(t+1))^{-1} A + Q. \tag{6}$$

This Riccati equation can be solved recursively. Vaughan develops a nonrecursive method for a control problem with a nonsingular transition matrix  $A$ . In Section III the singular case is discussed. Singular transition matrices are frequently encountered if artificial state variables are introduced in order to transform a control problem to the standard control problem (1)-(2).

III. NONRECURSIVE ALGEBRAIC SOLUTION

The state and costate equations (1) and (4) can be written as

$$C \begin{pmatrix} x_t \\ y_t \end{pmatrix} = D \begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} \tag{7}$$

with

$$C = \begin{pmatrix} A & 0 \\ -Q & I \end{pmatrix}, \quad D = \begin{pmatrix} I & B R^{-1} B' \\ 0 & A' \end{pmatrix}, \quad x_0 = x(0), \quad y_T = P(T) x_T.$$

$C$  and  $D$  are invertible if and only if  $A$  is invertible. The method of Vaughan solves the (standard) eigenvalue problem

$$\mathcal{H}_c v = \lambda v \tag{8}$$

with  $\mathcal{H}_c = C^{-1} D$ . From this solution the response of (7) and the nonrecursive solution of  $P(t)$  can be obtained. If  $A$  is singular the system matrix  $\mathcal{H}_c$  does not exist and the eigenvalue problem (8) cannot be solved.

It will now be shown that in this case a nonrecursive solution can be obtained from the eigenvalues and eigenvectors of the generalized matrix eigenvalue problem

$$Cv = \lambda Dv \tag{9}$$

and its reciprocal eigenvalue problem.

First, the eigenvalue problem (9) is considered, the reciprocal problem is defined and introduced later on. Since  $D$  is singular the characteristic polynomial

$$p(\lambda) = \det(C - \lambda D) \tag{10}$$

is of degree less than  $2n$ , so that there is not a complete set of eigenvalues and (generalized) eigenvectors.

For some generalized matrix eigenvalue problems  $p(\lambda)$  is identically zero regardless of the value of  $\lambda$ . The matrix  $C - \lambda D$  may have reduced rank for all  $\lambda$  and is then called degenerate. In the above case this pathological feature does not appear because there exist scalars which are not eigenvalues of (9). For example, the stabilizability and detectability conditions prevent that the scalar  $\rho$  with  $|\rho|=1$  is an eigenvalue. Therefore,  $p(\lambda)$  can be written as

$$p(\lambda) = \gamma_p \lambda^{p_0} (\lambda - \lambda_1)^{p_1} (\lambda - \lambda_2)^{p_2} \dots (\lambda - \lambda_k)^{p_k} \tag{11}$$

with  $p_0, p_1, \dots, p_k$  positive integers,  $\lambda_i \neq 0$ , and  $\sum_{i=0}^k p_i \leq 2n$ .

The eigenvectors and generalized eigenvectors are formed by  $Cv_j = \lambda_j Dv_j$  and  $Cv_j = \lambda_j Dv_j + Cv_{j-1}, j=1, \dots$ . So there are  $2n - \sum_{i=0}^k p_i$  (generalized) eigenvectors.

To complete the set of eigenvalues and (generalized) eigenvectors the reciprocal matrix eigenvalue problem

$$\mu Cw = Dw \tag{12}$$

is considered. Let  $q(\mu)$  be the characteristic polynomial of (12). Then, because of the relation  $q(\mu) = \mu^{2n} p(1/\mu)$  this polynomial becomes

$$q(\mu) = \gamma_q \mu^{q_0} (\mu - 1/\lambda_1)^{p_1} (\mu - 1/\lambda_2)^{p_2} \dots (\mu - 1/\lambda_k)^{p_k} \tag{13}$$

with  $\gamma_q = \gamma_p \prod_{i=1}^k (-\lambda_i)^{p_i}$  and  $q_0 = 2n - \sum_{i=0}^k p_i$ .

The  $q_0$  zero eigenvalues and the associated (generalized) eigenvectors can be used to complete the set of eigenvalues and eigenvectors of (9). (The zero eigenvalues of the reciprocal problem can be considered as infinite eigenvalues of the original problem.) A complete set of eigenvectors and eigenvectors is a set of eigenvalues of the original and reciprocal problem for which the (generalized) eigenvectors span  $\mathcal{C}^{2n}$ . There are several alternatives to form a complete set of eigenvalues and eigenvectors. Every  $\lambda_i \neq 0$  of the original problem can be exchanged for  $\mu_i = 1/\lambda_i$  of the reciprocal problem. The associated (generalized) eigenvectors are linearly related. A special combination will be selected. The matrices  $C$  and  $D$  are such that the relation

$$(C - \lambda D)' = \begin{pmatrix} 0 & I \\ -\lambda I & 0 \end{pmatrix} (C - 1/\lambda D) \begin{pmatrix} 0 & I \\ -\lambda I & 0 \end{pmatrix} \tag{14}$$

holds. From this relation immediately follows that

$$p(\lambda) = \lambda^{2n} p(1/\lambda) \tag{15}$$

and consequently

$$p(\lambda) = q(\lambda) = \gamma_p \lambda^{p_0} [(\lambda - \lambda_1)(\lambda - 1/\lambda_1)]^{s_1} \dots [(\lambda - \lambda_l)(\lambda - 1/\lambda_l)]^{s_l} \tag{16}$$

with  $k=2l, 2\sum_{i=1}^l s_i = \sum_{i=1}^k p_i$ , and  $p_0 = q_0 = n - \sum_{i=1}^l s_i$ . Now the following complete set is chosen

- $\lambda = 0$  (with multiplicity  $p_0$ ),
- $\lambda = \lambda_i$  (with multiplicity  $s_i$ ),  $|\lambda_i| < 1, i=1, \dots, l$ ,
- $\mu = 0$  (with multiplicity  $p_0$ ),
- $\mu = \lambda_i$  (with multiplicity  $s_i$ ),  $|\lambda_i| < 1, i=1, \dots, l$ .

The associated (generalized) eigenvectors are given by

$$CV = DVJ, \quad CWJ = DW \tag{17}$$

where the columns of  $V$  are  $n$  generalized eigenvectors of (9), the columns of  $W$  are  $n$  (generalized) eigenvectors of (12) and  $J$  is the Jordan matrix with diagonal elements  $\lambda = \lambda_i, |\lambda_i| < 1$  and  $\lambda = 0$ . The  $2n$  independent vectors  $Z = (V, W)$  constitute a basis for  $\mathcal{C}^{2n}$ . These eigenvalues can be used to derive a nonrecursive solution of  $P(t)$ . Define a new state vector of (7) by the transformation

$$\begin{pmatrix} x_t^0 \\ y_t^0 \end{pmatrix} = Z^{-1} \begin{pmatrix} x_t \\ y_t \end{pmatrix}. \tag{18}$$

Then (7) can be written as

$$CZ \begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} - DZ \begin{pmatrix} x_{i+1}^0 \\ y_{i+1}^0 \end{pmatrix} = 0$$

or by substituting (17)

$$(DV, CW) \left[ \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} x_{i+1}^0 \\ y_{i+1}^0 \end{pmatrix} \right] = 0. \quad (19)$$

The first term of (19) is invertible because  $\mathcal{R}(D, C)$  forms  $\mathcal{C}^{2n}$  and  $\mathcal{R} \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}$  forms a  $2n$ -dimensional space independent of  $\mathcal{U}(D, C)$ . Consequently, the second term is zero. This gives

$$\begin{pmatrix} x_T^0 \\ y_T^0 \end{pmatrix} = \begin{pmatrix} J^{T-t} & 0 \\ 0 & J^{T-t} \end{pmatrix} \begin{pmatrix} x_t^0 \\ y_t^0 \end{pmatrix}. \quad (20)$$

This relation corresponds to relation (17) of Vaughan and one can proceed along his lines in order to obtain the nonrecursive solution of  $P(t)$ . Thus,

$$P(t) = (V_2 + W_2 J^{T-t} S J^{T-t}) (V_1 + W_1 J^{T-t} S J^{T-t})^{-1} \quad (21)$$

where

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, S = (P(T)W_1 - W_2)^{-1}(V_2 - P(T)V_1).$$

This shows that the nonrecursive solution of  $P(t)$  is obtained from the eigenvalues and (generalized) eigenvectors of the generalized matrix eigenvalue problem and its reciprocal eigenvalue problem. Note that in the nonsingular case this approach can also be used.

The steady-state solution  $P$  of (6), which equals the unique positive semidefinite solution of the associated discrete algebraic matrix Riccati equation, can be found by letting  $T \rightarrow \infty$

$$P = V_2 V_1^{-1}. \quad (22)$$

So the steady-state solution can be derived from the stable eigenvalues and (generalized) eigenvectors of the generalized matrix eigenvalue problem (9). The remaining solutions of the algebraic matrix Riccati equation can be obtained by selecting  $n$  eigenvectors of (9) such that  $V_1^{-1}$  exists, all zero (generalized) eigenvectors are present and the lowest consecutive generalized eigenvectors of a selected chain of generalized eigenvectors are present.

#### NUMERICAL SOLUTION TECHNIQUES

The  $QZ$  algorithm developed by Moler and Stewart [3] can be used to compute the eigenvalues and eigenvectors of (9) and (12). The  $QZ$  algorithm is a generalization of the  $QR$  algorithm.

Another solution technique is to transform the generalized matrix eigenvalue problem (9) to a standard eigenvalue problem which, for example, can be solved using the  $QR$  algorithm. Consider the modified generalized eigenvalue problem

$$(C + \beta D)z = \nu(\alpha C + D)z \quad (23)$$

with  $\alpha \neq 0$ ,  $\alpha\beta \neq 1$  and  $\alpha$  such that  $\alpha C + D$  is nonsingular. Then this problem can be transformed to the standard problem

$$Ez = \nu z \quad (24)$$

with  $E = (\alpha C + D)^{-1}(C + \beta D)$ . Now, it is not difficult to show that the eigenvalues  $\lambda$ ,  $\mu$ , and  $\nu$  are related by

$$\lambda = (\nu - \beta)/(1 - \alpha\nu), \mu = (1 - \alpha\nu)/(\nu - \beta).$$

The (generalized) eigenvectors of (24) correspond with (generalized) eigenvectors of (9) and (12). Consequently, by solving the standard eigenvalue problem (24) the generalized eigenvalue problems (9) and (12) are solved. The condition  $\alpha\beta \neq 1$  is introduced in order to avoid that all eigenvalues are transformed to the value  $\nu = \beta$ .

Since the generalized eigenvalue problem has a finite number of eigenvalues it is always possible to find  $\alpha$  such that  $\alpha C + D$  is invertible. Difficulties arise when  $-\alpha$  equals  $\lambda_i$ , respectively  $1/\lambda_i$ ,  $i=1, \dots$ , or is near to these values. However, these values are not *a priori* known. Because of the stabilizability and detectability conditions a safe value seems to be  $\alpha$  with  $|\alpha|=1$ . For  $\alpha=\beta$  the matrix  $E$  is a symplectic or discrete-time Hamiltonian matrix, i.e., there exists a discrete linear optimal control problem for which the system matrix is precisely  $E$ . For  $\beta=-\alpha$  and  $\alpha=\pm 1$   $E$  is a Hamiltonian matrix, i.e., there exists a continuous linear optimal control problem with system matrix  $E$ . For  $\alpha=1, \beta=-1$ ,  $E$  is equal to the matrix  $\mathcal{H}_c^*$  which Michelsen [2] introduces in order to find the positive semidefinite solution of the algebraic matrix Riccati equation in the case of a nearly singular  $A$ . In contrast with the above approach the derivation of Michelsen cannot handle singular transition matrices.

#### ACKNOWLEDGMENT

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#### Correction to "Performance of State Regulator Systems with Floating-Point Computation"

R. E. RINK

In the above paper,<sup>1</sup> the example discussed on page 416 should have the spectral radius parameter

$$\gamma^2 = \sqrt{46.325}.$$

The square root was omitted. This correction has a negligible effect on the upper bound shown in Fig. 2 of the paper, but it has significant effect on that of Fig. 1. The correct ordinate values for Fig. 1 are

$l$	2	3	4	5	6	7
Bound	211.22	37.38	16.99	12.55	11.47	11.21

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The author is with the Department of Electrical Engineering, University of Alberta, Edmonton, Alta., Canada.

<sup>1</sup>R. E. Rink and H. Y. Chong, *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 411-421, June 1979.