Asymptotic Solutions of Singular Perturbation Problems for Linear Differential Equations of Elliptic Type

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Contents

1. Introduction

We consider a class of linear second order partial differential equations in two independent variables

(1.1) $$\varepsilon L_2[\Phi]+L_1[\Phi]=h(x, y)$$

where L_2 is a linear second order differential operator and L_1 is a linear first order differential operator, $h(x,y)$ is a given function, and ε is positive. The function Φ satisfies the differential equation in a bounded convex domain G and takes along the boundary Γ of G prescribed values $\varphi(x,y)$. When ε is very small, the boundary value problem described above is a so-called singular perturbation problem. Problems of this type frequently arise in mathematical physics, for example in oceanography (see ROBINSON [4]) and in magnetohydrodynamics (see SHERCLIFF [5]).

Confronted with a singular perturbation problem, one is usually interested in constructing an asymptotic approximation of the function Φ. Such an approximation contains so called "boundary layer" terms, which are asymptotically equivalent to zero everywhere in G except for a small neighborhood of a part Γ^* of the boundary Γ. Moreover, the derivatives of the boundary layer in a direction normal to Γ^* are of the order of magnitude of inverse powers of ε in the neighborhood of Γ^*. This behavior of the asymptotic approximation is indeed characteristic for the solution of singular perturbation problems.

It is usually not too difficult to construct a formal expansion of the function Φ which has the appearance of an asymptotic series. However, serious difficulties can arise when one attempts to *prove* that one really has constructed an asymptotic expansion. It is typical of the singular perturbation problems that the construction of the formal expansion is in most cases based on "intuition" and "past experience" which in the proof of the asymptotic properties must *a posteriori* be justified by rigorous mathematical argument. In many papers on singular perturbation problems in mathematical physics such proof is not attempted.

The development of the theory of singular perturbation problems of the type described above is mostly due to LEVINSON [3] and VIŠIK & LYUSTERNIK [6]. The latter authors have simplified the formal expansion of LEVINSON; their proofs of the asymptotic properties are mainly based on norm-estimates. LEVINSON'S original proof was based on a maximum principle that can be derived for the differential equations (1.1).

The work of LEVINSON and of VIŠIK & LYUSTERNIK has advanced considerably the theory of singular perturbation problems; however, a closer analysis of this work reveals that various aspects and phenomena have been left out of consideration. Moreover, the proofs of the asymptotic properties do not appear entirely satisfactory (see, for discussion, Section 3.1). It is the aim of this paper to present a more complete theory of singular perturbation problems for second order linear differential equations of elliptic type, while simplifying and clarifying the proofs of asymptotic behavior.

The starting point of our analysis is the maximum principle. Using the concept of barrier functions, we derive from this principle a number of asymptotic estimate theorems which provide the fundamental elements for all the proofs of asymptotic behavior that are given in the sequel. These theorems, presented in Chapter 2, are independent of the construction of the formal expansions.

Now in studying singular perturbation problems for the differential equation (1.1) in a bounded convex domain G two cases must be distinguished: first, the case in which the boundary Γ of G is a smooth curve of which no part coincides with a characteristic line of the operator L_1, and secondly, the case in which Γ contains parts which are characteristics of L_1.

We study the case of non-characteristic boundaries in Chapter 3. Reconsidering VIŠIK & LYUSTERNIK'S expansion procedure, we establish conditions for its validity, and give proofs of the asymptotic properties of the expansion. The proofs are very simple and are a direct consequence of the general theory of Chapter 2. Furthermore, in proving the asymptotic properties more general results are obtained than those contained in VIŠIK & LYUSTERNIK's work.

Next we study problems in which characteristic boundaries occur. These problems have been discussed only very briefly by VIŠIK & LYUSTERNIK [6] and have been reconsidered in some more detail by KNOWLES & MESSICK [2]. In Chapter 4 we investigate problems with characteristic boundaries for a particularly simple equation of the type (1.1). We study with special care the singularities that generally arise when use is made of the "parabolic boundary layers" introduced by VIŠIK & LYUSTERNIK. These singularities have not been investigated in the past. Restricting ourselves to first order asymptotic theory, we present the complete asymptotic solution and give proof of its uniform validity in the domain G. We remark that proofs presented by previous authors are concerned only with the special non-singular case and even for that case do not attain uniform validity in G. In Chapter 5 we develop an analogous asymptotic theory for the general equation (1.1) in a bounded convex region with characteristic boundaries. All proofs of asymptotic behavior that are presented in Chapters 4 and 5 are again a direct application of the theory of Chapter 2.

In the course of our analysis various questions are encountered, the elucidation of which would have diverted our attention from the main line of investigation presented in this paper. In a concluding chapter we summarize these as yet "open questions", some of which will be analysed in a subsequent paper.

2. Estimates of Solutions of Elliptic Boundary Value Problems with Applications to Singular Perturbation Problems

2.1. Introductory Remarks

In this paper we investigate asymptotic approximations of solutions of elliptic boundary value problems of the following type: The function $\Phi_\varepsilon(x,y)$ satisfies in a bounded convex domain G the elliptic differential equation

(2.1) $$L_\varepsilon[\Phi_\varepsilon] \equiv \varepsilon L_2[\Phi_\varepsilon] + L_1[\Phi_\varepsilon] = h(x, y),$$

with

(2.2) $$L_2 = a(x, y)\frac{\partial^2}{\partial x^2} + 2b(x, y)\frac{\partial^2}{\partial x\,\partial y} + c(x, y)\frac{\partial^2}{\partial y^2} + d(x, y)\frac{\partial}{\partial x} + e(x, y)\frac{\partial}{\partial y} + f(x, y)$$

and

(2.3) $$L_1 = -\frac{\partial}{\partial y} - g(x, y)$$

while

(2.4) $$\Phi_\varepsilon(x, y) = \varphi(x, y)$$

along the boundary Γ of the domain G. The constant ε is a small positive number, and we are interested in asymptotic approximations of $\Phi_\varepsilon(x,y)$, for $\varepsilon \to 0$.

The coefficients of the differential operator, the right hand side, the boundary Γ and the boundary values are called the parameters of the boundary value problem. Under very general conditions involving Hölder continuity of the coefficients and the right hand side of equation (2.1) the boundary value problem possesses a unique solution (*cf.* [*1*], p. 336). We shall assume throughout this paper that the problem (2.1)–(2.4) indeed has a unique solution.

We may assume without loss of generality that $a(x,y)>0$ in G; in order to guarantee the uniqueness of the solution, we must then take $g(x,y)-\varepsilon f(x,y)\geqq 0$ in G. Other conditions to be imposed on the parameters of the boundary value problem will be specified in the sequel.

Considering the solution of the boundary value problem, we shall show that it is possible to construct a function $\Phi_\nu(x,y,\varepsilon)$ such that

$$\Phi_\varepsilon(x,y)=\Phi_\nu(x,y,\varepsilon)+z_\nu(x,y,\varepsilon), \tag{2.5}$$

where z_ν is of the order $O(\varepsilon^\nu)$. The value of ν depends on the differentiability properties of the coefficients, the right hand side of (2.1) and the boundary values $\varphi(x,y)$ and, in addition, on the shape of the boundary Γ. One of the main objects of this research is the proof that indeed $z_\nu=O(\varepsilon^\nu)$. This proof involves an estimate of the remainder term z_ν which in the course of the development of the theory will appear to be the solution of an elliptic boundary value problem with $L_\varepsilon z_\nu=O(\varepsilon^\nu)$. In this way we are led to the problem of estimating solutions of elliptic boundary value problems with given right hand side of the differential equation and given boundary values.

Since we shall use such estimates later on in several different situations, we present in this chapter all the estimate theorems needed in the sequel. Moreover, the estimate theorems will be seen to be useful as a guide in the effective construction of the function $\Phi_\nu(x,y,\varepsilon)$. The main tool used in this paper for estimating solutions of elliptic boundary value problems is furnished by the so-called maximum principle and the concept of barrier function. For the sake of completeness we deal with these well known concepts in Section 2.2 and derive there an estimate for solutions of general elliptic boundary value problems.

Next in Section 2.3 we consider boundary value problems of the type (2.1)–(2.4) and prove several theorems which will appear to be fundamental in the development of the asymptotic theory.

2.2. The Maximum Principle and Some Applications

Let

$$\begin{aligned} L[\Phi]\equiv\, & a(x,y)\,\Phi_{xx}+2b(x,y)\,\Phi_{xy}+c(x,y)\,\Phi_{yy}+d(x,y)\,\Phi_x+e(x,y)\,\Phi_y+\\ & +f(x,y)\,\Phi=M[\Phi]+f(x,y)\,\Phi \end{aligned} \tag{2.6}$$

be a differential expression elliptic in a bounded domain G while the coefficients a, b, *etc.* are continuous within G with $a(x,y)>0$. The maximum principle may be formulated as follows: if a twice continuously differentiable function Φ attains a maximum in an interior point P of G, then $M[\Phi]\leqq 0$ in P. If this maximum is positive, and if $f(P)\leqq 0$, it follows that $L[\Phi]\leqq 0$ in P. For the proof of this well known principle the reader is referred to [*1*], p. 321.

The maximum principle has an interesting and useful corollary which is stated in the following lemma:

Lemma 1. *If the twice continuously differentiable functions $\Phi(x,y)$ and $\Psi(x,y)$ satisfy within a bounded domain G the relation*

$$|L[\Phi]| < L[-\Psi], \tag{2.7}$$

where L is the differential operator given by (2.6) with $f(x,y) \leqq 0$ in G, and if along the boundary Γ of G

$$|\Phi| \leqq \Psi, \tag{2.8}$$

then we also have the relation

$$|\Phi| \leqq \Psi \tag{2.9}$$

within $\bar{G}$.

Proof. Suppose $\Phi - \Psi$ possesses a positive maximum in an interior point P of G; then according to the maximum principle

$$L[\Phi - \Psi]|_P \leqq 0,$$

which contradicts the assumption (2.7). Hence $\Phi - \Psi$ does not have a positive maximum within G, and due to (2.8) $\Phi - \Psi \leqq 0$ along the boundary Γ of G; it follows that

$$\Phi - \Psi \leqq 0 \tag{2.10}$$

in $\bar{G}$. Now suppose that $-\Phi - \Psi$ possesses a positive maximum in an interior point P of G; then, again according to the maximum principle,

$$L[-\Phi - \Psi]|_P \leqq 0$$

which also contradicts the assumption (2.7). Hence $-\Phi - \Psi$ does not have a positive maximum within G, and due to (2.8) $-\Phi - \Psi \leqq 0$ along the boundary Γ of G; it now follows that

$$-\Phi - \Psi \leqq 0 \tag{2.11}$$

in $\bar{G}$. Combining (2.10) and (2.11) yields the result (2.9) which is valid in all of $\bar{G}$. Q.e.d.

The function $\Psi(x,y)$ is called a barrier function for the function $\Phi(x,y)$ since it gives an upper bound for the absolute value of $\Phi(x,y)$ in $\bar{G}$. Estimates of solutions of elliptic boundary value problems can be obtained by constructing suitable barrier functions satisfying the conditions of Lemma 1. This principle is illustrated by the following important theorem, where for simplicity and for future convenience we assume that the coefficient $e(x,y)$ is either positive or negative in $\bar{G}$.

Theorem I. *Let $\Phi(x,y)$ be the solution of the elliptic differential equation of the second order*

$$L[\Phi(x,y)] = h(x,y), \tag{2.12}$$

valid in G, while along the boundary Γ of G the relation

$$\Phi(x,y)|_\Gamma = (\varphi x, y)|_\Gamma \tag{2.13}$$

holds. L is the differential operator given in (2.6) *with* $e(x,y)$ *continuous and either positive or negative, and* $f(x,y) \leqq 0$ *in G.*

If there exists a constant m with the properties

$$|h(x,y)| \leqq m \quad in\ \bar{G} \tag{2.14}$$

and

$$|\varphi(x,y)| \leqq m \quad along\ \Gamma, \tag{2.15}$$

then there exists also a real number K independent of m such that

$$|\Phi(x,y)| \leqq K\,m \quad in\ \bar{G}. \tag{2.16}$$

Proof. We consider the case of $e(x,y)$ negative and continuous in $\bar{G}$; there exists a positive number $\alpha^2 < 1$ such that

$$e(x,y) < -\alpha^2 \quad \text{in } \bar{G}. \tag{2.17}$$

Putting $\operatorname{Min}_{\Gamma} y = y_0$, we can easily show that the function

$$\Psi(x,y) \equiv \frac{m}{\alpha^2}(y-y_0)+m$$

is a barrier function for the function $\Phi(x,y)$. Applying L to $-\Psi$, we obtain

$$L[-\Psi] = -e(x,y)\frac{m}{\alpha^2} - f(x,y)\left\{\frac{m}{\alpha^2}(y-y_0)+m\right\} > m,$$

$$m \geqq |h(x,y)| = |L[\Phi]|,$$

while along the boundary Γ the relation

$$\Psi|_\Gamma \geqq m \geqq |\varphi(x,y)|_\Gamma$$

holds. Using the lemma, we immediately obtain

$$|\Phi(x,y)| \leqq \frac{m}{\alpha^2}(y-y_0)+m \leqq K\,m \quad \text{in } \bar{G},$$

where K is a suitable constant depending on the size of G and α^2. In the case of $e(x,y)$ positive a similar analysis holds. Q.e.d.

2.3. Application of Barrier Function Technique to Asymptotic Solutions of Singular Elliptic Perturbation Problems

In this section we consider boundary value problems formulated in Section 1, equations (2.1)–(2.4), *viz*:

$$L_\varepsilon[\Phi_\varepsilon] \equiv \varepsilon\, L_2[\Phi_\varepsilon] + L_1[\Phi_\varepsilon] = h(x,y), \tag{2.1}$$

valid in a convex bounded domain G with

$$\begin{aligned} L_2 = {} & a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x\,\partial y} + c(x,y)\frac{\partial^2}{\partial y^2} + d(x,y)\frac{\partial}{\partial x} + \\ & + e(x,y)\frac{\partial}{\partial y} + f(x,y) \end{aligned} \tag{2.2}$$

and

$$L_1 = -\frac{\partial}{\partial y} - g(x, y), \tag{2.3}$$

while

$$\Phi_\varepsilon(x, y) = \varphi(x, y) \tag{2.4}$$

along the boundary Γ of the domain G. The operator L_2 is elliptic in G and ε is a small positive number. We assume that the coefficients and the right hand side of the differential equation are all Hölder-continuous in $\bar{G}$ and that the boundary Γ of G and the boundary values $\varphi(x,y)|_\Gamma$ are sufficiently smooth. Moreover we suppose $a(x,y)>0$ and $g(x,y)-\varepsilon f(x,y) \geqq 0$ in $\bar{G}$. In order that the latter inequality holds for all sufficiently small values of ε, say $\varepsilon<\varepsilon_0$, it is necessary to take $g(x,y)$ non-negative everywhere in $\bar{G}$. Having guaranteed by these conditions the existence and uniqueness of the solution $\Phi_\varepsilon(x,y)$ (*cf.* [*1*], p. 336), we can now derive almost immediately from Theorem I the following results.

Theorem II. *If Φ_ε is a solution of the boundary value problem*

$$L_\varepsilon[\Phi_\varepsilon] = h(x, y),$$

valid in a bounded domain G with

$$\Phi_\varepsilon(x, y)|_\Gamma = \varphi(x, y)|_\Gamma$$

along the boundary Γ of G, and if $h(x,y)$ and $\varphi(x,y)$ are bounded respectively in $\bar{G}$ and along Γ, then for sufficiently small values of ε the solutions Φ_ε are bounded uniformly with respect to ε in $\bar{G}$.

Proof. Since the coefficient $e(x,y)$ is continuous in $\bar{G}$, there exist numbers ε_0 and $\alpha^2<1$, independent of ε, such that

$$-1+\varepsilon\, e(x, y) < -\alpha^2 \qquad \text{in } \bar{G}, \tag{2.18}$$

valid for all ε with $\varepsilon<\varepsilon_0$. Due to the assumptions of the theorem there exists also a number m with the property

$$\begin{aligned} |h(x, y)| &\leqq m \qquad \text{in } \bar{G}, \\ |\varphi(x, y)| &\leqq m \qquad \text{along } \Gamma. \end{aligned} \tag{2.19}$$

The function

$$\Psi(x, y) \equiv \frac{m}{\alpha^2}(y-y_0)+m, \tag{2.20}$$

with $\operatorname{Min}_\Gamma y = y_0$, is now a barrier function for all solutions Φ_ε with $\varepsilon<\varepsilon_0$, and hence

$$|\Phi_\varepsilon(x, y)| \leqq K\, m \tag{2.21}$$

for $\varepsilon<\varepsilon_0$ and $(x,y)\in\bar{G}$. Q.e.d.

Theorem III. *If $\Phi_\varepsilon(x,y)$ is the solution of the boundary value problem*

$$L_\varepsilon[\Phi_\varepsilon] = h_\varepsilon(x, y),$$

valid in a bounded domain G with

$$\Phi_\varepsilon(x, y)|_\Gamma = \varphi_\varepsilon(x, y)|_\Gamma$$

along the boundary Γ of G, and if

$$h_\varepsilon(x,y)=O(\varepsilon^\mu) \qquad \text{in } \bar{G},\ \mu\geqq 0,$$

and

$$\Phi_\varepsilon(x,y)|_\Gamma=O(\varepsilon^\nu) \qquad \text{along } \Gamma,\ \nu\geqq 0,$$

then at most

(2.22) $$\Phi_\varepsilon(x,y)=O(\varepsilon^{\min(\mu,\nu)}) \qquad \text{in } \bar{G}.$$

Proof. Introducing the constant m, defined by

$$m=\operatorname{Max}\left\{\underset{\bar{G}}{\text{l.u.b.}}\left(\lim_{\varepsilon\to 0}\frac{h_\varepsilon(x,y)}{\varepsilon^\mu}\right),\ \underset{\Gamma}{\text{l.u.b.}}\left(\lim_{\varepsilon\to 0}\frac{\Phi_\varepsilon(x,y)}{\varepsilon^\nu}\right)\right\}+1,$$

we obtain for sufficiently small values of ε, say $\varepsilon<\varepsilon_1$, the following inequalities:

$$|h_\varepsilon(x,y)|<m\,\varepsilon^\mu\leqq m\,\varepsilon^{\min(\mu,\nu)} \qquad \text{in } \bar{G},$$

$$|\Phi_\varepsilon(x,y)|<m\,\varepsilon^\nu\leqq m\,\varepsilon^{\min(\mu,\nu)} \qquad \text{along } \Gamma.$$

The function

$$\psi_\varepsilon(x,y)\equiv\frac{m\,\varepsilon^{\min(\mu,\nu)}}{\alpha^2}(y-y_0)+m\,\varepsilon^{\min(\mu,\nu)}$$

is now a barrier function for all solutions $\Phi_\varepsilon(x,y)$ with $\varepsilon<\min(\varepsilon_0,\varepsilon_1)$. Hence

$$|\Phi_\varepsilon(x,y)|\leqq K\,m\,\varepsilon^{\min(\mu,\nu)} \qquad \text{in } \bar{G},$$

and thus $\Phi_\varepsilon(x,y)$ is at most of the order $O(\varepsilon^{\min(\mu,\nu)})$. Q.e.d.

In the next theorem we give another useful upper bound for the functions $\Phi_\varepsilon(x,y)$, which, as in Theorem II, is uniform with respect to ε. We consider again the boundary value problem (2.1)–(2.4) defined for a convex bounded domain G. Suppose $A(x_1,y_1)$ and $B(x_2,y_2)$ are the points of the boundary Γ with respectively

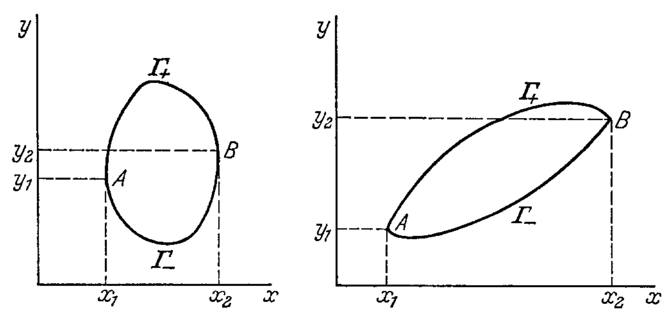

Fig. 1

a minimum and a maximum value of the abscissa x (see Fig. 1). The points of the lower part Γ_- of the boundary Γ satisfy the equation $y=\gamma_-(x)$, whereas the points of the upper part Γ_+ satisfy the equation $y=\gamma_+(x)$. Some typical configurations which will be allowed are illustrated by Fig. 1. The boundary condition for $\Phi_\varepsilon(x,y)$ along the boundary Γ can now be expressed as follows:

(2.23) $$\Phi_\varepsilon(x,y)|_{\Gamma_-}=\Phi_\varepsilon(x,\gamma_-(x))=\varphi_-(x),$$

(2.24) $$\Phi_\varepsilon(x,y)|_{\Gamma_+}=\Phi_\varepsilon(x,\gamma_+(x))=\varphi_+(x).$$

The extreme case of a rectangular boundary is also allowed. In this case we take the lower corner points as the points A and B (see Fig. 2). The boundary condition

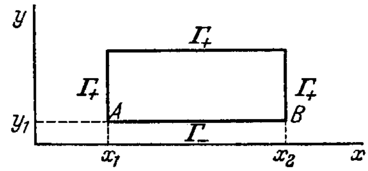

Fig. 2

(2.24) should now be split into three parts; however, in order to have a uniform notation for all possible cases, we write (2.24) in the form

(2.25) $$\Phi_\varepsilon(x,y)|_{\Gamma_+} = \varphi_+(x,y)|_{\Gamma_+}.$$

After these introductory remarks we now formulate our theorem.

Theorem IV. *Let $\Phi_\varepsilon(x,y)$ satisfy in a convex bounded domain G the differential equation*

(2.1) $$L_\varepsilon[\Phi_\varepsilon(x,y)] = h(x,y),$$

with $h(x,y)$ bounded in $\bar{G}$. The lower part Γ_- of the boundary Γ is given by the equation $y=\gamma_-(x)$ which is assumed twice continuously differentiable for $x_1 < x < x_2$. The function $\Phi_\varepsilon(x,y)$ is subjected along Γ_- to the boundary condition

(2.23) $$\Phi_\varepsilon(x,y)|_{\Gamma_-} = \Phi_\varepsilon(x,\gamma_-(x)) = \varphi_-(x),$$

with $\varphi_-(x)$ twice continuously differentiable for $x_1 \leqq x \leqq x_2$. $\Phi_\varepsilon(x,y)$ satisfies along Γ_+ the boundary condition

(2.25) $$\Phi_\varepsilon(x,y)|_{\Gamma_+} = \varphi_+(x,y)|_{\Gamma_+}.$$

$\varphi_+(x,y)$ is continuous along Γ_+, $\varphi_+(A)=\varphi_-(A)$, $\varphi_+(B)=\varphi_-(B)$ and $\varphi_+(x,y)$ has continuous directional derivatives along Γ_+ at the points A and B. Under these conditions there exists a number M independent of ε such that for sufficiently small values of ε the relation

(2.26) $$|\Phi_\varepsilon(x,y) - \varphi_-(x)| < M\{y-\gamma_-(x)\}$$

holds.

Proof. We consider the function $\Phi_\varepsilon^*(x,y) = \Phi_\varepsilon(x,y) - \varphi_-(x)$. Since $\varphi_-(x)$ is twice differentiable, we may apply the operator L_ε to $\Phi_\varepsilon^*(x,y)$, This function satisfies in G the differential equation

(2.27) $$L_\varepsilon[\Phi_\varepsilon^*(x,y)] = h(x,y) - L_\varepsilon[\varphi_-(x)] = h_\varepsilon^*(x,y)$$

with the boundary conditions

(2.28) $$\Phi_\varepsilon^*(x,y)|_{\Gamma_-} = 0 \quad \text{and}$$
$$\Phi_\varepsilon^*(x,y)|_{\Gamma_+} = \{\varphi_+(x,y) - \varphi_-(x)\}|_{\Gamma_+}.$$

We now introduce the function

(2.29) $$\Psi(x,y) = M_0\{y-\gamma_-(x)\}\,\chi(x),$$

where M_0 is some positive constant independent of ε, while $\chi(x)$ is some positive function also independent of ε and twice continuously differentiable in $[x_1, x_2]$; the function $\chi(x)$ depends only on x. It will be shown that M_0 and $\chi(x)$ can be chosen such that $\Psi(x,y)$ is a barrier function for all functions $\Phi_\varepsilon^*(x,y)$, if ε is taken sufficiently small. By taking M_0 and

$$\operatorname*{Min}_{x_1 \leqq x \leqq x_2} [\chi(x)]$$

sufficiently large it follows from the given conditions for the boundary values that the relation

$$\Psi(x,y)|_\Gamma \geqq |\Phi_\varepsilon^*(x,y)|_\Gamma \tag{2.30}$$

can be satisfied for all values of ε. Applying L_ε to $-\Psi(x,y)$, we obtain

$$\begin{aligned} L_\varepsilon[-\Psi] = \varepsilon M_0 \Big[& -a(x,y)\{y-\gamma_-(x)\}\frac{d^2\chi}{dx^2} + 2a(x,y)\frac{d\gamma_-}{dx}\frac{d\chi}{dx} + \\ & + a(x,y)\frac{d^2\gamma_-}{dx^2}\chi(x) - 2b(x,y)\frac{d\chi}{dx} - d(x,y)\{y-\gamma_-(x)\}\frac{d\chi}{dx} + \\ & + d(x,y)\frac{d\gamma_-}{dx}\chi(x) - e(x,y)\chi(x) - f(x,y)\{y-\gamma_-(x)\}\chi(x)\Big] + \\ & + M_0\chi(x) + M_0 g(x,y)\{y-\gamma_-(x)\}\chi(x). \end{aligned}$$

Apart from the terms

$$2a(x,y)\frac{d\gamma_-}{dx}\frac{d\chi}{dx}, \quad a(x,y)\frac{d^2\gamma_-}{dx^2}\chi(x) \quad \text{and} \quad d(x,y)\frac{d\gamma_-}{dx}\chi(x),$$

the factor between the brackets is continuous in $\bar{G}$; moreover, since M_0 and $\chi(x)$ are positive and $g(x,y)$ and $\{y-\gamma_-(x)\}$ non-negative in $\bar{G}$, it follows that in G for sufficiently small values of ε, say $\varepsilon<\varepsilon_0$, we have the estimate

$$\begin{aligned} L_\varepsilon[-\Psi] > \tfrac{1}{2}M_0\chi(x) + \varepsilon M_0 \left\{2a(x,y)\frac{d\chi}{dx} + d(x,y)\chi(x)\right\}\frac{d\gamma_-}{dx} + \\ + \varepsilon M_0 a(x,y)\frac{d^2\gamma_-}{dx^2}\cdot\chi(x). \end{aligned} \tag{2.31}$$

Since the curve $y=\gamma_-(x)$ is concave upwards,

$$\frac{d^2\gamma_-}{dx^2}>0 \qquad \text{for } x_1<x<x_2,$$

and hence we obtain from (2.31)

$$L_\varepsilon[-\Psi] > \tfrac{1}{2}M_0\chi(x) + \varepsilon M_0 \left\{2a(x,y)\frac{d\chi}{dx} + d(x,y)\chi(x)\right\}\frac{d\gamma_-}{dx}, \tag{2.32}$$

valid in G for $\varepsilon<\varepsilon_0$.

We consider a fixed right neighborhood Δ_1 of (x_1, y_1) lying in G with $x_1<x<x_1+\delta_1$ and a fixed left neighborhood Δ_2 of (x_2, y_2) lying in G with $x_2-\delta_2<x<x_2$ such that $d\gamma_-/dx$ does not change sign for $x_1<x<x_1+\delta_1$ and

for $x_2-\delta_2<x<x_2$; further, $x_1+\delta_1<x_2-\delta_2$. Hence

$$\frac{d\gamma_-}{dx}<0 \qquad \text{for } x_1<x<x_1+\delta_1$$

and

$$\frac{d\gamma_-}{dx}>0 \qquad \text{for } x_2-\delta_2<x<x_2.$$

We now choose $\chi(x)$ such that

$$2a(x,y)\frac{d\chi}{dx}+d(x,y)\chi(x)<0 \qquad \text{in } \Delta_1 \tag{2.33a}$$

and

$$2a(x,y)\frac{d\chi}{dx}+d(x,y)\chi(x)>0 \qquad \text{in } \Delta_2. \tag{2.33b}$$

Because $a(x,y)$, $d(x,y)$, $\chi(x)$ and $d\chi/dx$ are continuous and $a(x,y)$ is positive in $\bar{\Delta}_1$ and $\bar{\Delta}_2$, we can always satisfy the requirements (2.33) by taking $d\chi/dx$ sufficiently large negative in the segment $[x_1, x_1+\delta_1]$ and sufficiently large positive in the segment $[x_2-\delta_2, x_2]$. ($d\chi/dx \ll 0$ for $x_1 \leqq x \leqq x_1+\delta_1$ and $d\chi/dx \gg 0$ for $x_2-\delta_2 \leqq x \leqq x_2$.) With this choice of the function $\chi(x)$ we get from (2.32) the estimate

$$L_\varepsilon[-\Psi]>\tfrac{1}{2}M_0\chi(x) \qquad \text{for } (x,y)\in\Delta_i \ (i=1,2). \tag{2.34}$$

In $G-\Delta_1-\Delta_2$ the expression

$$M_0\left\{2a(x,y)\frac{d\chi}{dx}+d(x,y)\chi(x)\right\}\frac{d\gamma_-}{dx}$$

is bounded, and thus for sufficiently small values of ε, say $\varepsilon<\varepsilon_1<\varepsilon_0$,

$$L_\varepsilon[-\Psi]>\tfrac{1}{3}M_0\chi(x) \qquad \text{for } (x,y)\in G-\Delta_1-\Delta_2. \tag{2.34*}$$

Combining (2.34) and (2.34*), we obtain the relation

$$L_\varepsilon[-\Psi]>\tfrac{1}{3}M_0\chi(x) \qquad \text{for } (x,y)\in G.$$

Since $\chi(x)$ is positive and continuous for $x_1 \leqq x \leqq x_2$, there exists a positive constant N (independent of ε) such that

$$L_\varepsilon[-\Psi]>\tfrac{1}{3}M_0\chi(x)>\tfrac{1}{3}M_0 N \qquad \text{for } (x,y)\in G.$$

Finally, taking M_0 such that

$$M_0>\frac{3}{N}\operatorname{Sup}|h_\varepsilon^*(x,y)|, \quad \varepsilon<\varepsilon_1, \quad (x,y)\in\bar{G},$$

we get the result

$$L_\varepsilon[-\Psi]>|L_\varepsilon[\Phi_\varepsilon^*]|, \tag{2.35}$$

valid in G for all values of ε with $\varepsilon<\varepsilon_1$.

Finally, if we combine the results (2.30) and (2.35), it follows immediately from Lemma 1 that we have in $\bar{G}$, for $\varepsilon<\varepsilon_1$, the uniform estimate

$$|\Phi_\varepsilon^*(x, y)|=|\Phi_\varepsilon(x, y)-\varphi_-(x)|\leqq\Psi(x, y)=M_0\{y-\gamma_-(x)\}\chi(x).$$

Putting

$$M=M_0 \underset{x_1\leqq x\leqq x_2}{\mathrm{Max}} [\chi(x)],$$

we get the desired result:

$$|\Phi_\varepsilon(x, y)-\varphi_-(x)|\leqq M\{y-\gamma_-(x)\}, \tag{2.26}$$

uniformly valid in $\bar{G}$ for ε sufficiently small. Q.e.d.

This theorem has two important consequences:

1. The normal derivative of the functions $\Phi_\varepsilon(x,y)$ along the lower part Γ_- of the boundary Γ is uniformly bounded with respect to ε. This means that the solution $\Phi_\varepsilon(x,y)$ does not exhibit the character of a boundary layer along the lower part Γ_- of the boundary. (For the concept of boundary layer, see Chapters 3, 4 and 5.)

2. If we consider an upper neighborhood of the lower part Γ_- of the boundary Γ which has a width δ, then in this neighborhood we have the estimate

$$\Phi_\varepsilon(x, y)=\varphi_-(x)+O(\delta) \tag{2.36}$$

valid uniformly for sufficiently small values of ε.

We conclude this chapter by proving still another theorem which is an extension of Theorem III to certain cases in which the function $h_\varepsilon(x,y)$ has a singular behavior. This extension will appear very useful in the development of the theory (*cf.* Chapter 3, 4 and 5). As to the boundary of the convex bounded region G, we shall now distinguish explicitly the case when there exist two unique points A and B where the abscissa x takes on minimum and maximum values (see Fig. 1) from the case when there do not exist such points as *e.g.* for a rectangular boundary (Fig. 2). The first case is dealt with in Theorem V and the second in Theorem $\mathrm{V}^{\mathrm{bis}}$.

Theorem V. *Let $\Phi_\varepsilon(x,y)$ satisfy in a convex domain G the differential equation*

$$L_\varepsilon[\Phi_\varepsilon(x, y)]=h_\varepsilon(x, y)$$

with

$$\Phi_\varepsilon(x, y)|_\Gamma=\varphi_\varepsilon(x, y)|_\Gamma$$

valid along the boundary Γ, on which are two unique points $A(x_1, y_1)$ and $B(x_2, y_2)$ where the abscissa x takes on minimal and maximal values, respectively. It is further assumed that the functions $\Phi_\varepsilon(x,y)$ are uniformly bounded in G for sufficiently small values of ε. If $\Phi_\varepsilon(x,y)=O(\varepsilon^\nu)$ $(\nu\geqq 0)$ along Γ, $h_\varepsilon(x,y)=O(\varepsilon^\mu)$ $(\mu\geqq 0)$ in $\bar{G}$ with the exception of arbitrarily small neighborhoods $V(A)$ and $V(B)$ of the points A and B, where $h_\varepsilon(x,y)$ is singular, and if $\min(\mu, \nu)\leqq 1$, then $\Phi_\varepsilon(x,y)$ is at most of the order $O(\varepsilon^{\min(\mu,\nu)})$ uniformly in $\overline{G-V(A)-V(B)}$.

Proof. By supposition there exists a positive number M independent of ε such that

$$|\Phi_\varepsilon(x, y)|<M \tag{2.37}$$

for sufficiently small values of ε. We consider first the region G^* containing all points of G with $x_1+\frac{1}{2}\delta<x<x_2-\frac{1}{2}\delta$, where δ is an arbitrarily small fixed number. According to the assumptions of the theorem there exists also a constant m independent of ε such that the functions $\Phi_\varepsilon(x,y)$ in G^* for sufficiently small values of ε satisfy the relation

$$|L_\varepsilon[\Phi_\varepsilon]|=|h_\varepsilon(x,y)|<m\,\varepsilon^\mu, \tag{2.38}$$

while along the boundary Γ^* of G^* the following conditions hold:

$$|\Phi_\varepsilon(x,y)|\,|_\Gamma<m\,\varepsilon^\nu \tag{2.39}$$

and

$$|\Phi_\varepsilon(x,y)|\,\Big|_{\substack{x=x_1+\frac{1}{2}\delta\\ x=x_2-\frac{1}{2}\delta}}<M. \tag{2.40}$$

We construct now the infinitely differentiable function $\Psi_\varepsilon(x,y)$ which is subjected to the following requirements:

$$\begin{aligned}
&\Psi_\varepsilon(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}(y-y_0)+m\right\} && \text{for } x_1+\delta\leqq x\leqq x_2-\delta,\\
&\Psi_\varepsilon(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}(y-y_0)+m\right\}+M && \text{for } x\leqq x_1+\tfrac{1}{2}\delta \text{ and}\\
& && \quad x\geqq x_2-\tfrac{1}{2}\delta,\\
&\Psi_\varepsilon(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}(y-y_0)+m\right\}+\rho_1(x) && \text{for } x_1+\tfrac{1}{2}\delta\leqq x\leqq x_1+\delta,\\
&\Psi_\varepsilon(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}(y-y_0)+m\right\}+\rho_2(x) && \text{for } x_2-\delta\leqq x\leqq x_2-\tfrac{1}{2}\delta,
\end{aligned} \tag{2.41}$$

where α^2 is a positive number smaller than 1 such that

$$-1+\varepsilon\, e(x,y)<-\alpha^2 \quad \text{in } \bar{G} \tag{2.18}$$

for sufficiently small values of ε;

$$y_0=\min_\Gamma y,$$

$\rho_1(x)$ is a decreasing C^∞-function with $\rho_1(x_1+\frac{1}{2}\delta)=M$ and $\rho_1(x_1+\delta)=0$, and similarly $\rho_2(x)$ is an increasing C^∞-function with $\rho_2(x_2-\delta)=0$ and $\rho_2(x_2-\frac{1}{2}\delta)=M$. We shall show now that the function $\Psi_\varepsilon(x,y)$ is a barrier function for the function $\Phi_\varepsilon(x,y)$ in the region $\overline{G^*}$, if ε is sufficiently small. It is clear from (2.39) and (2.40) that along the boundary Γ^* of G^*

$$|\Phi_\varepsilon(x,y)|\,|_{\Gamma^*}<\Psi_\varepsilon(x,y)\,|_{\Gamma^*}. \tag{2.42}$$

In G^* we investigate the expression $L_\varepsilon[-\Psi_\varepsilon]$, which may be written as

$$L_\varepsilon[-\Psi_\varepsilon]=-\varepsilon\left[a(x,y)\frac{\partial^2\Psi_\varepsilon}{\partial x^2}+d(x,y)\frac{\partial\Psi_\varepsilon}{\partial x}\right]+\{1-\varepsilon\, e(x,y)\}\frac{\partial\Psi_\varepsilon}{\partial y}+$$
$$+\{g(x,y)-\varepsilon f(x,y)\}\,\Psi_\varepsilon(x,y).$$

For sufficiently small values of ε it follows now from (2.18) and from the fact that $g(x,y)-\varepsilon f(x,y)\geqq 0$ in $\bar{G}$ that

$$(2.43)\quad L_\varepsilon[-\Psi_\varepsilon]>2m\,\varepsilon^{\min(\mu,\nu)}-\varepsilon\left[a(x,y)\frac{\partial^2\Psi_\varepsilon}{\partial x^2}+d(x,y)\frac{\partial\Psi_\varepsilon}{\partial x}\right]\quad \text{in } G^*.$$

Since

$$a\frac{\partial^2\Psi_\varepsilon}{\partial x^2}+d\frac{\partial\Psi_\varepsilon}{\partial x}$$

is independent of ε (*cf.* (2.41)) and uniformly bounded in $\overline{G^*}$, we can take m so large that, apart from the requirements (2.38) and (2.39), we also have the estimate

$$\left|a\frac{\partial^2\Psi_\varepsilon}{\partial x^2}+d\frac{\partial\Psi_\varepsilon}{\partial x}\right|<m\qquad \text{in } G^*.$$

Substituting this result into (2.43) and remembering that we have assumed $\min(\mu,\nu)\leqq 1$, we obtain

$$(2.44)\qquad L_\varepsilon[-\Psi_\varepsilon]>m\,\varepsilon^{\min(\mu,\nu)}>|h_\varepsilon(x,y)|=|L_\varepsilon[\Phi_\varepsilon(x,y)]|,$$

valid in G^*. If we combine the results (2.42) and (2.44), it is clear that $\Psi_\varepsilon(x,y)$ is indeed a barrier function in $\overline{G^*}$ for the function $\Phi_\varepsilon(x,y)$ when ε is taken sufficiently small.

Again using the fundamental Lemma 1, we obtain the result

$$|\Phi_\varepsilon(x,y)|\leqq\overline{\Psi}_\varepsilon(x,y)$$

valid in $\overline{G^*}$ for sufficiently small values of ε.

This result is *a fortiori* also valid in the subregion of $\overline{G^*}$ with $x_1+\delta\leqq x\leqq x_2-\delta$. In the latter region $\overline{\Psi}_\varepsilon(x,y)$ is the function

$$\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}(y-y_0)+m\right\},$$

and hence

$$(2.45)\qquad |\Phi_\varepsilon(x,y)|\leqq K\,m\,\varepsilon^{\min(\mu,\nu)}$$

for $x_1+\delta\leqq x\leqq x_2-\delta$ and for sufficiently small values of ε. K is an appropriate constant depending on α^2 and on the size of the region G. Since δ may be taken arbitrarily small, it is clear that the estimate (2.45) is valid in the region $\overline{G-V(A)-V(B)}$, where $V(A)$ and $V(B)$ are arbitrarily small fixed neighborhoods of the points A and B. Q.e.d.

We shall use this theorem in the next chapter in the following slightly modified form.

Let $V(A)$ and $V(B)$ again be arbitrarily small neighborhoods of the points A and B. Suppose that $\Phi_\varepsilon(x,y)$ in $G-V(A)-V(B)$ for any fixed $V(A)$ and $V(B)$ satisfies the differential equation

$$L_\varepsilon[\Phi_\varepsilon(x,y)]=h_\varepsilon(x,y)$$

with the boundary condition

$$\Phi_\varepsilon(x, y)|_{\Gamma^*} = \varphi_\varepsilon(x, y)$$

where Γ^* is the part of Γ which partly bounds $G - V(A) - V(B)$.

Further, it is assumed that $\Phi_\varepsilon(x,y)$ is uniformly bounded in $\overline{G - V(A) - V(B)}$ for sufficiently small values of ε.

If $\varphi_\varepsilon(x,y) = O(\varepsilon^\nu)$ $(\nu \geqq 0)$ along Γ^*, $h_\varepsilon(x,y) = O(\varepsilon^\mu)$ $(\mu \geqq 0)$ in $\overline{G - V(A) - V(B)}$, and if $\min(\mu, \nu) \leqq 1$, then $\Phi_\varepsilon(x,y)$ is at most of the order $O(\varepsilon^{\min(\mu,\nu)})$ uniformly in $\overline{G - V(A) - V(B)}$.

The validity of this modification can be proved along exactly the same lines as that given in the proof of the theorem above.

We shall now treat a variant of Theorem V which applies to the case of a rectangular boundary and which will be used in the theory of asymptotic solutions of singular perturbation problems of the type dealt with in the Chapter 4 and 5.

Theorem V$^{\text{bis}}$. *Let $\Phi_\varepsilon(x,y)$ in the region G defined by $0 < x < l_1$, $0 < y < l_2$, satisfy the differential equation*

$$L_\varepsilon[\Phi_\varepsilon(x, y)] = h_\varepsilon(x, y),$$

while along the boundary Γ of G the relation

$$\Phi_\varepsilon(x, y)|_\Gamma = \varphi_\varepsilon(x, y)|_\Gamma$$

holds. If $h_\varepsilon(x,y) = O(e^\mu)$ $(0 \leqq \mu)$ in $\bar{G}$ and $\varphi_\varepsilon(x,y) = O(\varepsilon^\nu)$ $(0 \leqq \nu)$ along Γ with the exception of arbitrarily small neighborhoods of the corner points $(0, l_2)$ and (l_1, l_2) where $h_\varepsilon(x,y)$ or (and) $\varphi_\varepsilon(x,y)$ is (are) $O(1)$, and if $\min(\mu, \nu) \leqq 1$, then $\Phi_\varepsilon(x,y)$ is at most of the order $O(\varepsilon^{\min(\mu,\nu)})$ uniformly in $\bar{G}$ with the exception of arbitrarily small neighborhoods of the points $(0, l_2)$ and (l_1, l_2).

If we assume that $\Phi_\varepsilon(x,y)$ is uniformly bounded in $\bar{G}$, then it is even permissible for the function $h_\varepsilon(x,y)$ to have singularities in the corner points $(0, l_2)$ and (l_1, l_2).

Proof. Either by Theorem II or by supposition there exists a fixed number M independent of ε such that for sufficiently small values of ε the absolute value of the function $\Phi_\varepsilon(x,y)$ is bounded in $\bar{G}$ by M. We consider first the region G^* defined by $\frac{1}{2}\delta < x < l_1 - \frac{1}{2}\delta$, $0 < y < l_2$, where δ is an arbitrarily small fixed number. According to the assumption of the theorem there exists another constant m independent of ε such that the functions $\Phi_\varepsilon(x,y)$ in G^* for sufficiently small values of ε satisfy the relation

$$|L_\varepsilon[\Phi_\varepsilon(x, y)]| = |h_\varepsilon(x, y)| < m\,\varepsilon^\mu; \tag{2.46}$$

along the boundary Γ^* of G^* the following conditions hold:

$$\begin{aligned}
&|\Phi_\varepsilon(x, 0)| < m\,\varepsilon^\nu, && \tfrac{1}{2}\delta \leqq x \leqq l_1 - \tfrac{1}{2}\delta,\\
&|\Phi_\varepsilon(x, l_2)| < m\,\varepsilon^\nu, && \tfrac{1}{2}\delta \leqq x \leqq l_1 - \tfrac{1}{2}\delta,\\
&|\Phi_\varepsilon(\tfrac{1}{2}\delta, y)| < M, && 0 \leqq y \leqq l_2,\\
&|\Phi_\varepsilon(l_1 - \tfrac{1}{2}\delta, y)| < M, && 0 \leqq y \leqq l_2.
\end{aligned} \tag{2.47}$$

We now construct the C^∞-function $\Psi_\varepsilon^*(x,y)$ which has the following properties:

$$
(2.48)\qquad
\begin{aligned}
&\Psi_\varepsilon^*(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}\,y+m\right\} && \text{for } +\delta\leqq x\leqq l_1-\delta,\\
&\Psi_\varepsilon^*(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}\,y+m\right\}+M && \text{for } x\leqq +\tfrac{1}{2}\delta \text{ and } x\geqq l_1-\tfrac{1}{2}\delta,\\
&\Psi_\varepsilon^*(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}\,y+m\right\}+\rho_1(x) && \text{for } +\tfrac{1}{2}\delta\leqq x\leqq +\delta,\\
&\Psi_\varepsilon^*(x,y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}\,y+m\right\}+\rho_2(x) && \text{for } l_1-\delta\leqq x\leqq l_1-\tfrac{1}{2}\delta,
\end{aligned}
$$

where the constant α^2 and the functions $\rho_1(x)$ and $\rho_2(x)$ are defined similarly as in the equations (2.41). The function $\Psi_\varepsilon^*(x,y)$ is now a barrier function for the function $\Phi_\varepsilon(x,y)$ in the region $\overline{G^*}$ if ε is taken sufficiently small. The proof is quite similar to the one given in Theorem V and is therefore omitted here.

The result is that we have in $\overline{G^*}$ for sufficiently small values of ε the estimate

$$|\Phi_\varepsilon(x,y)|\leqq \Psi_\varepsilon^*(x,y).$$

This result is *a fortiori* also valid in the subregion $+\delta\leqq x\leqq l_1-\delta$, $0\leqq y\leqq l_2$, and hence

$$(2.49)\qquad |\Phi_\varepsilon(x,y)|\leqq \varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}\,y+m\right\}\leqq K\,m\,\varepsilon^{\min(\mu,\nu)}$$

for $+\delta\leqq x\leqq l_1-\delta$, $0\leqq y\leqq l_2$ and ε sufficiently small. K may be taken as

$$K=\frac{2\,l_2}{\alpha^2}+1.$$

In order to extend this result to the regions $0\leqq x\leqq \delta$, $l_1-\delta\leqq x\leqq l_1$ with the exception of neighborhoods of the points $(0,l_2)$ and (l_1,l_2), we consider now the region G^{**} defined by $0<x<l_1$, $0<y<l_2-\frac{1}{2}\delta$. According to the assumptions of the theorem there exists a constant m independent of ε such that the function $\Phi_\varepsilon(x,y)$ in the region G^{**} for sufficiently small values of ε satisfies the relation

$$(2.50)\qquad |L_\varepsilon[\Phi_\varepsilon(x,y)]|=|h_\varepsilon(x,y)|<m\,\varepsilon^\mu;$$

along the boundary Γ^{**} of G^{**} the following conditions hold:

$$
(2.51)\qquad
\begin{aligned}
&|\Phi_\varepsilon(x,0)|<m\,\varepsilon^\nu, && 0\leqq x\leqq l_1,\\
&|\Phi_\varepsilon(0,y)|<m\,\varepsilon^\nu, && 0\leqq y\leqq l_2-\tfrac{1}{2}\delta,\\
&|\Phi_\varepsilon(l_1,y)|<m\,\varepsilon^\nu, && 0\leqq y\leqq l_2-\tfrac{1}{2}\delta,\\
&|\Phi_\varepsilon(x,l_2-\tfrac{1}{2}\delta)|<M, && 0\leqq x\leqq l_1.
\end{aligned}
$$

By taking m sufficiently large we can use the same value for m in the formulae (2.46), (2.47), (2.50) and (2.51).

We construct the C^∞-function $\Psi_\varepsilon^{**}(y)$ with the following properties:

$$\Psi_\varepsilon^{**}(y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}y+m\right\} \qquad \text{for } 0\leqq y\leqq l_2-\delta,$$

(2.52) $$\Psi_\varepsilon^{**}(y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}y+m\right\}+\rho_3(y) \qquad \text{for } l_2-\delta\leqq y\leqq l_2-\tfrac{1}{2}\delta,$$

$$\Psi_\varepsilon^{**}(y)=\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}y+m\right\}+M \qquad \text{for } l_2-\tfrac{1}{2}\delta\leqq y,$$

where α^2 is again the same constant used in (2.41) and (2.48); it has the property that

(2.18) $$-1+\varepsilon e(x,y)<-\alpha^2 \qquad \text{in } \bar{G},$$

for sufficiently small values of ε. $\rho_3(y)$ is an increasing C^∞-function, depending only on y, with $\rho_3(l_2-\delta)=0$ and $\rho_3(l_2-\frac{1}{2}\delta)=M$. It is not difficult to show that $\Psi_\varepsilon^{**}(y)$ is a barrier function in $\overline{G^{**}}$ for the function $\Phi_\varepsilon(x,y)$ if ε is taken sufficiently small. It is clear that along the boundary we have

(2.53) $$|\Phi_\varepsilon(x,y)|_{\Gamma^{**}}<\Psi_\varepsilon^{**}(y)|_{\Gamma^{**}}.$$

Applying the operator L_ε to $-\Psi_\varepsilon^{**}(y)$, we obtain

(2.54) $$L_\varepsilon[-\Psi^{**}(y)]=-\varepsilon\left\{c(x,y)\frac{d^2\Psi_\varepsilon^{**}}{dy^2}\right\}+\{1-\varepsilon e(x,y)\}\frac{d\Psi_\varepsilon^{**}}{dy}+ \\ +\{g(x,y)-\varepsilon f(x,y)\}\Psi_\varepsilon^{**}.$$

Using the definition of $\Psi_\varepsilon^{**}(y)$, the relation (2.18) and the inequality $g-\varepsilon f\geqq 0$ in $\bar{G}$, and taking into account that $\rho_3(y)$ is an increasing function of y and that its derivatives are independent of ε, we may estimate the right hand side of (2.54) as

$$L_\varepsilon[-\Psi_\varepsilon^{**}]>2m\,\varepsilon^{\min(\mu,\nu)}+O(\varepsilon), \qquad \text{valid in } G^{**}.$$

Since $\min(\mu,\nu)\leqq 1$, we obtain for sufficiently large values of m the result

(2.55) $$L_\varepsilon[-\Psi_\varepsilon^{**}]>m\,\varepsilon^{\min(\mu,\nu)}>|h_\varepsilon(x,y)|=|L_\varepsilon[\Phi_\varepsilon(x,y)]|.$$

When we combine formulae (2.53) and (2.55), it follows that $\Psi_\varepsilon^{**}(y)$ is a barrier function in $\overline{G^{**}}$ for the function $\Phi_\varepsilon(x,y)$ if ε is sufficiently small, and hence

$$|\Phi_\varepsilon(x,y)|\leqq\Psi_\varepsilon^{**}(y), \qquad \text{valid in } \overline{G^{**}}.$$

Restricting this result to the subregion $0\leqq x\leqq l_1$, $0\leqq y\leqq l_2-\delta$, we obtain from (2.52) the estimate

(2.56) $$|\Phi_\varepsilon(x,y)|<\varepsilon^{\min(\mu,\nu)}\left\{\frac{2m}{\alpha^2}l_2+m\right\}=K\,m\,\varepsilon^{\min(\mu,\nu)},$$

valid for $0\leqq x\leqq l_1$, $0\leqq y\leqq l_2-\delta$ and ε sufficiently small. It follows finally from (2.49) and (2.56) that $\Phi_\varepsilon(x,y)$ is at most of the order $O(\varepsilon^{\min(\mu,\nu)})$ in $\bar{G}$ with the exception of arbitrarily small δ-neighborhoods of the corner points $(0,l_2)$ and (l_1,l_2). Q.e.d.

Remark. The proof of an analogous theorem in which there may also occur irregularities of the functions $h_\varepsilon(x,y)$ and $\varphi_\varepsilon(x,y)$ in the lower corner points (0,0) and (0, l_1) is much more difficult. This arises from the fact that in order to exclude neighborhoods of the points (0,0) and (0, l_1) we are led to a barrier function $\Psi^{**}(y)$ which is strongly decreasing in a neighborhood of $y=0$, and hence $L_\varepsilon[-\Psi_\varepsilon^{**}(y)]$ may become negative due to the presence of the term $d\Psi^{**}/dy$ (see (2.54)). In some special cases the difficulties may be circumvented by using Theorem IV which gives an estimate of $\Phi_\varepsilon(x,y)$ in the neighborhood of $y=0$. Examples of this situation will be found in Chapters 4 and 5.

3. Asymptotic Solutions of Singular Perturbation Problems for Elliptic Differential Equations in Bounded Convex Domains with Smooth Boundary

3.1. Introductory Remarks

We recall the formulation of the singular perturbation problem, that will be investigated in this chapter. Within a bounded convex domain G of the (x,y) plane the function $\Phi_\varepsilon(x,y)$ satisfies the elliptic differential equation

$$L_\varepsilon[\Phi_\varepsilon(x,y)] \equiv \varepsilon L_2[\Phi_\varepsilon(x,y)] + L_1[\Phi_\varepsilon(x,y)] = h(x,y), \tag{2.1}$$

where the operators L_2 and L_1 are given by

$$\begin{aligned} L_2 = a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x\,\partial y} + c(x,y)\frac{\partial^2}{\partial y^2} + d(x,y)\frac{\partial}{\partial x} + \\ + e(x,y)\frac{\partial}{\partial y} + f(x,y) \end{aligned} \tag{2.2}$$

and

$$L_1 = -\frac{\partial}{\partial y} - g(x,y) \tag{2.3}$$

and ε is a small positive number.

The coefficients and the right-hand side of the differential equation are assumed continuously differentiable in the closure $\bar{G}$ of the domain G up to a certain order which will be specified later.

We suppose $a(x,y)>0$ in $\bar{G}$; in order to guarantee the uniqueness of the solutions of boundary value problems associated with (2.1), we suppose also $g(x,y)-\varepsilon f(x,y)\geqq 0$ in $\bar{G}$. The boundary Γ of the convex domain G is assumed to have a continuously changing tangent. Moreover there exist on Γ precisely two points $A(x_1, y_1)$ and $B(x_2, y_2)$, which are the points of tangency of the characteristics $x=x_1$ and $x=x_2$ of L_1 to the boundary Γ. These points divide Γ into a lower part Γ_- and an upper part Γ_+ having respectively the equations $y=\gamma_-(x)$ and $y=\gamma_+(x)$ (see Fig. 3). Along the boundary Γ the function $\Phi_\varepsilon(x,y)$ is subject to the conditions

$$\Phi_\varepsilon(x,y)|_{\Gamma_-} = \Phi_\varepsilon(x,\gamma_-(x)) = \varphi_-(x), \tag{2.23}$$

$$\Phi_\varepsilon(x,y)|_{\Gamma_+} = \Phi_\varepsilon(x,\gamma_+(x)) = \varphi_+(x). \tag{2.24}$$

The boundaries $y=\gamma_\pm(x)$ and the boundary values $\varphi_\pm(x)$ are assumed to be continuously differentiable up to a certain order which will also be specified later on. The differentiability of $y=\gamma_\pm(x)$ is of course restricted to the interval $x_1<x<x_2$.

This chapter will be devoted to the effective construction of an asymptotic expansion into powers of the small parameter ε of the solution $\Phi_\varepsilon(x,y)$ of the boundary value problem formulated in (2.1), (2.23), and (2.24). The problem has

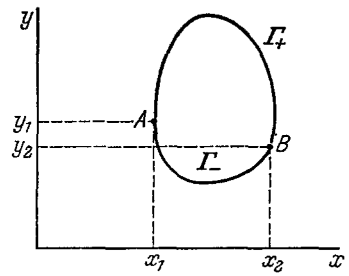

Fig. 3

already been investigated by WASOW [7], LEVINSON [3] and VIŠIK & LYUSTERNIK [6]. WASOW considered the special case in which L_2 is the Laplacian operator, while LEVINSON and VIŠIK & LYUSTERNIK have studied the more general case of (2.1). LEVINSON has shown that the solution $\Phi_\varepsilon(x,y)$ can be written in the form

(3.1) $$\Phi_\varepsilon(x,y)=w_0(x,y)+v_0(x,y,\varepsilon)+z_0(x,y,\varepsilon),$$

where the functions w_0 and v_0 are of the order $O(1)$, while the remainder term z_0 is of the order $O(\varepsilon^{\frac{1}{2}})$ in every subregion of G bounded by Γ and by two characteristics of the operator L_1.

LEVINSON's analysis is of considerable complexity, due partially to an unnecessarily complicated form of the function $v_0(x,y,\varepsilon)$. Moreover, as will be seen shortly, the estimate of the remainder term can be considerably improved.

The problem has been taken up by VIŠIK & LYUSTERNIK in [6]. These authors construct a formal asymptotic expansion of Φ_ε given by

(3.2) $$\Phi_\varepsilon(x,y)=\sum_{i=0}^{m}\varepsilon^i w_i(x,y)+\sum_{i=0}^{m+1}\varepsilon^i \tilde{v}_i(x,y,\varepsilon)+z_m(x,y,\varepsilon)$$

where w_i and $\tilde{v}_i$ are functions that will be defined explicitly in this chapter. VIŠIK & LYUSTERNIK have shown that under certain conditions concerning the differentiability of the parameters of the boundary value problem the remainder term satisfies the differential equation

(3.3) $$L_\varepsilon(z_m)=\varepsilon^{m+1}g_m(x,y,\varepsilon)$$

with boundary condition

(3.4) $$z_m|_\Gamma=0,$$

where $g_m(x,y,\varepsilon)$ is of the order one uniformly in $\bar{G}$ with the exception of arbitrarily small fixed neighborhoods $V(A)$ and $V(B)$ of the points A and B.

Proceeding from (3.3) and (3.4), VIŠIK & LYUSTERNIK derive norm-estimates of the remainder z_m and its derivatives, such as

(3.5) $$\| z_m \| = O(\varepsilon^{m+1})$$

where

$$\| z_m \|^2 = \iint |z_m|^2 \, dx \, dy,$$

the integral being taken over the region $\overline{G - V(A) - V(B)}$.

Moreover, for the case $m=0$ a pointwise estimate is given, valid in all of $\bar{G}$, which reads

(3.6) $$|z_0| \leqq C \min\{(x-x_1)^{\frac{1}{p}}, (x_2-x)^{\frac{1}{q}}, \varepsilon(x-x_1)^{\frac{1}{p}-1}, \varepsilon(x_2-x)^{\frac{1}{q}-1}\}$$

where C is a constant while $(p-1)$ and $(q-1)$ denote the respective orders of tangency of the boundary Γ with the characteristics $x=x_1$ and $x=x_2$ at the points A and B respectively. In particular, it follows from the result (3.6) that z_0 is of order ε in $\overline{G-V(A)-V(B)}$.

VIŠIK & LYUSTERNIK'S work represents considerable progress; nevertheless, a closer analysis reveals a number of shortcomings. First, if we reconsider VIŠIK & LYUSTERNIK'S construction of the formal asymptotic expansion (3.2), we shall find that the differentiability conditions given by these authors are not entirely correct.

Second, if we consider the problem of estimating the remainder term z_m, our opinion is that norm-estimates of the type (3.5), stressed in VIŠIK & LYUSTERNIK'S work, do not have much meaning as proof of asymptotic properties of the expansion (3.2). The relation (3.5) remains true, if, for instance, z_m is locally of the order of magnitude of ε^{-1} in a subdomain which is of the order of magnitude of ε^{m+4}. In this case, however, the asymptotic expansion (3.1) is locally meaningless. Although norm-estimates can be useful in demonstrating in an elegant way some global properties of asymptotic expansions, the proof of asymptotic properties of the solution can be obtained only by means of a pointwise estimate of the remainder term[1]. In this respect the result (3.6) is certainly of great importance. However, the analysis leading to this result is of great complexity, since VIŠIK & LYUSTERNIK'S aim was to obtain an estimate valid in all of $\bar{G}$. Moreover, only the case $m=0$ was studied.

It thus appears that simple proofs of the asymptotic properties of expansion (3.2), valid for all m, have not yet been obtained. Restricting our attention to the domain $\overline{G-V(A)-V(B)}$, we shall show in Section 3.5 of this chapter that z_m is indeed of the order ε^{m+1} uniformly in $\overline{G-V(A)-V(B)}$. Our proof will be seen to be very simple and will be a direct consequence of Theorem V of the preceding chapter.

In the pages that follow we first reconsider carefully VIŠIK & LYUSTERNIK'S construction of the formal asymptotic series in order to establish the correct conditions for its validity. We then proceed to the proofs of the asymptotic properties.

[1] It should be remarked that norm-estimates of the derivatives of z_m given by VIŠIK & LYUSTERNIK are not sufficiently strong for embedding the pointwise-estimate $z_m = O(\varepsilon^{m+1})$.

3.2. The Reduced Equation

In order to obtain a first rough approximation of the function $\Phi_\varepsilon(x,y)$ for small values of the parameter ε, we consider a function $w_0(x,y)$ which satisfies the differential equation obtained from the original one (2.1) by putting $\varepsilon=0$. This equation is called the reduced equation and reads as follows:

$$(3.7) \qquad L_1[w_0] \equiv -\frac{\partial w_0}{\partial y} - g(x,y)\, w_0 = h(x,y).$$

The characteristics of this equation are the lines x=constant. The function $w_0(x,y)$ can satisfy only one of the boundary conditions (2.23) and (2.24), *viz*

$$(3.8) \qquad w_0(x,\gamma_-(x)) = \varphi_-(x)$$

or

$$(3.9) \qquad w_0(x,\gamma_+(x)) = \varphi_+(x).$$

According to Theorem IV of the preceeding chapter we know that under certain conditions there exists a constant M independent of ε, such that for sufficiently small values of ε

$$(2.26) \qquad |\Phi_\varepsilon(x,y) - \varphi_-(x)| \leqq M[y-\gamma_-(x)].$$

This suggests the choice of the equation (3.8) as the proper boundary condition for the approximation $w_0(x,y)$. The function w_0 is now easily determined, and the result is

$$(3.10) \qquad w_0(x,y) = \varphi_-(x) - \int_{\gamma_-(x)}^{y} \exp\left[-\int_{\eta}^{y} g(x,\zeta)\, d\zeta\right] h^*(x,\eta)\, d\eta$$

with

$$h^*(x,y) = h(x,y) + g(x,y)\, \varphi_-(x).$$

It is quite evident that this approximation for $\Phi_\varepsilon(x,y)$ is not valid in the neighborhood of the upper part Γ_+ of the boundary Γ, since the boundary condition (2.24) is not satisfied.

A better approximation of the solution $\Phi_\varepsilon(x,y)$ is to be obtained by adding to $w_0(x,y)$ a term $v_0(x,y,\varepsilon)$ which equals zero along Γ_- and yields the boundary value $\varphi_+(x) - w_0(x,\gamma_+(x))$ along Γ_+, and which has, moreover, the property that $w_0(x,y)+v_0(x,y,\varepsilon)$ satisfies the equation (2.1) up to the order $O(\varepsilon)$.

The function v_0 will be called a boundary layer function. Such a function has the property of being asymptotically equal to zero everywhere in G except for a small neighborhood of Γ_+. The first and the second derivatives of v_0 are in this neighborhood of the order $O(1/\varepsilon)$ and $O(1/\varepsilon^2)$, respectively.

Before constructing a formal asymptotic expansion for Φ_ε, we draw attention to the fact that the function $\gamma_-(x)$, and hence also $w_0(x,y)$, has singular derivatives with respect to x at the points A and B. If $g(x,y)$, $h(x,y)$ and $\varphi_-(x)$ are sufficiently regular, it follows that in the neighborhood of the points A and B

$$(3.11) \qquad \frac{\partial w_0}{\partial x} = O\left(\frac{\partial \gamma_-}{\partial x}\right)$$

and

$$\frac{\partial^2 w_0}{\partial x^2} = O\left(\frac{\partial^2 y_-}{\partial x^2}\right). \tag{3.12}$$

Therefore in constructing a formal asymptotic expansion for the function $\Phi_\varepsilon(x, y)$ it will be necessary to exclude small neighborhoods of the points A and B.

3.3. The Local Coordinate System

For the construction of the boundary layer terms it is necessary to introduce a new coordinate system in the neighborhood of the upper boundary Γ_+ (see Fig. 4).

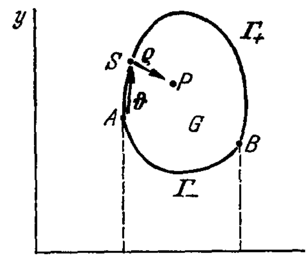

Fig. 4

Let ρ be the distance PS from the point $P \in G$ to the boundary Γ_+ measured along the normal on Γ_+ and ϑ the distance from A to S measured along Γ_+. The parametric representation of Γ_+ is given by

$$\begin{aligned} x &= x_b(\vartheta), \\ y &= y_b(\vartheta) \end{aligned} \tag{3.13}$$

with $0 \leqq \vartheta \leqq \vartheta_0$; the point A has the coordinates $(x_b(0), y_b(0))$, the point $B, (x_b(\vartheta_0), y_b(\vartheta_0))$. We suppose henceforth that the equations (3.13) are $(n+3)$ times continuously differentiable with respect to ϑ for $0 \leqq \vartheta \leqq \vartheta_0$, $n = 0, 1, 2, \ldots$.

We consider a lower neighborhood $\bar{\Omega}(0 \leqq \rho \leqq \rho_0, 0 \leqq \vartheta \leqq \vartheta_0)$ of the curve Γ_+ which is sufficiently small such that the normals from points of Γ_+ do not meet within $\bar{\Omega}$.

Hence there exists in $\bar{\Omega}$ a $1-1$ correspondence between the coordinates (x, y) and the coordinates (ρ, ϑ) and vice versa; this correspondence is described by the relations

$$\begin{aligned} x &= x_b(\vartheta) + \rho\, \frac{\dot{y}_b(\vartheta)}{\dot{x}_b(\vartheta)} \left[1 + \left\{\frac{\dot{y}_b(\vartheta)}{\dot{x}_b(\vartheta)}\right\}^2\right]^{-\frac{1}{2}}, \\ y &= y_b(\vartheta) - \rho \left[1 + \left\{\frac{\dot{y}_b(\vartheta)}{\dot{x}_b(\vartheta)}\right\}^2\right]^{-\frac{1}{2}}, \end{aligned} \tag{3.14}$$

where the dots denote differentiation with respect to ϑ. Substituting this transformation into the differential equation (2.1), we obtain the equation

$$\begin{aligned} &\varepsilon \left\{\alpha(\rho,\vartheta) \frac{\partial^2 \Phi_\varepsilon}{\partial \rho^2} + 2\beta(\rho,\vartheta) \frac{\partial^2 \Phi_\varepsilon}{\partial \rho\, \partial \vartheta} + \gamma(\rho,\vartheta) \frac{\partial^2 \Phi_\varepsilon}{\partial \vartheta^2} + \zeta(\rho,\vartheta) \frac{\partial \Phi_\varepsilon}{\partial \rho} + \right. \\ &\quad \left. + \eta(\rho,\vartheta) \frac{\partial \Phi_\varepsilon}{\partial \vartheta} + f(\rho,\vartheta)\, \Phi_\varepsilon \right\} - \mu(\rho,\vartheta) \frac{\partial \Phi_\varepsilon}{\partial \rho} - \nu(\rho,\vartheta) \frac{\partial \Phi_\varepsilon}{\partial \vartheta} - \\ &\quad - g(\rho,\vartheta)\, \Phi_\varepsilon = h(\rho,\vartheta), \end{aligned} \tag{3.15}$$

valid for $0<\rho\leqq\rho_0$ and $0\leqq\vartheta\leqq\vartheta_0$. The coefficients α, β, γ *etc.* of this differential equation can easily be calculated from (2.1)–(2.3), and one finds

$$\alpha(\rho,\vartheta)=a(\rho,\vartheta)\left(\frac{\partial\rho}{\partial x}\right)^2+2b(\rho,\vartheta)\frac{\partial\rho}{\partial x}\frac{\partial\rho}{\partial y}+c(\rho,\vartheta)\left(\frac{\partial\rho}{\partial y}\right)^2,$$

$$\beta(\rho,\vartheta)=a(\rho,\vartheta)\frac{\partial\rho}{\partial x}\frac{\partial\vartheta}{\partial x}+b(\rho,\vartheta)\left(\frac{\partial\rho}{\partial x}\frac{\partial\vartheta}{\partial y}+\frac{\partial\rho}{\partial y}\frac{\partial\vartheta}{\partial x}\right)+c(\rho,\vartheta)\frac{\partial\rho}{\partial y}\frac{\partial\vartheta}{\partial y},$$

$$\gamma(\rho,\vartheta)=a(\rho,\vartheta)\left(\frac{\partial\vartheta}{\partial x}\right)^2+2b(\rho,\vartheta)\frac{\partial\vartheta}{\partial x}\frac{\partial\vartheta}{\partial y}+c(\rho,\vartheta)\left(\frac{\partial\vartheta}{\partial y}\right)^2,$$

$$\zeta(\rho,\vartheta)=a(\rho,\vartheta)\frac{\partial^2\rho}{\partial x^2}+2b(\rho,\vartheta)\frac{\partial^2\rho}{\partial x\partial y}+c(\rho,\vartheta)\frac{\partial^2\rho}{\partial y^2}+d(\rho,\vartheta)\frac{\partial\rho}{\partial x}+$$

$$(3.16)\qquad +e(\rho,\vartheta)\frac{\partial\rho}{\partial y},$$

$$\eta(\rho,\vartheta)=a(\rho,\vartheta)\frac{\partial^2\vartheta}{\partial x^2}+2b(\rho,\vartheta)\frac{\partial^2\vartheta}{\partial x\partial y}+c(\rho,\vartheta)\frac{\partial^2\vartheta}{\partial y^2}+d(\rho,\vartheta)\frac{\partial\vartheta}{\partial x}+$$

$$+e(\rho,\vartheta)\frac{\partial\vartheta}{\partial y},$$

$$\mu(\rho,\vartheta)=\frac{\partial\rho}{\partial y},$$

$$\nu(\rho,\vartheta)=\frac{\partial\vartheta}{\partial y},$$

with $a(\rho,\vartheta)=a(x,y)$, $b(\rho,\vartheta)=b(x,y)$, *etc.* Since the differential operator L_2 (2.2) is elliptic with $a(x,y)>0$ in $\bar{G}$, it follows that also

$$(3.17)\qquad \alpha(\rho,\vartheta)>0 \quad \text{in } \bar{\Omega}.$$

Furthermore, we have

$$(3.18)\qquad \mu(0,\vartheta)=\cos(\rho,y)|_{\rho=0}<0 \quad \text{for } 0<\vartheta<\vartheta_0.$$

Finally we assume that the coefficients of the original differential equation (2.1) possess continuous partial derivatives of order n in $\bar{G}$. Because we have supposed that the parametric representation (3.13) of Γ_+ is $(n+3)$ times continuously differentiable, it follows now that the coefficients (3.16) of the equation (3.15) possess also continuous partial derivatives of order n with respect to ρ and ϑ in $\bar{\Omega}$. Expanding these coefficients in a Taylor series, for $N+1\leqq n$ we may write

$$\alpha(\rho,\vartheta)=\alpha_0(\vartheta)+\alpha_1(\vartheta)\rho+\alpha_2(\vartheta)\rho^2\cdots+\alpha_N(\vartheta)\rho^N+\alpha_{N+1}(\rho,\vartheta)\rho^{N+1},$$

$$\beta(\rho,\vartheta)=\beta_0(\vartheta)+\beta_1(\vartheta)\rho+\beta_2(\vartheta)\rho^2\cdots+\beta_N(\vartheta)\rho^N+\beta_{N+1}(\rho,\vartheta)\rho^{N+1},$$

$$(3.19)\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mu(\rho,\vartheta)=\mu_0(\vartheta)+\mu_1(\vartheta)\rho+\mu_2(\vartheta)\rho^2\cdots+\mu_N(\vartheta)\rho^N+\mu_{N+1}(\rho,\vartheta)\rho^{N+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

with $\alpha_0(\vartheta)>0$ and $\mu_0(\vartheta)<0$ for $0<\vartheta<\vartheta_0$.

We introduce now the local variable t defined by $\rho=\varepsilon t$. The differential expression $\varepsilon L_\varepsilon[\Phi_\varepsilon]$ can then be expressed in Ω as follows:

(3.20) $$\varepsilon L_\varepsilon[\Phi_\varepsilon]=M_0[\Phi_\varepsilon]+\varepsilon M_1[\Phi_\varepsilon]+\cdots+\varepsilon^N M_N[\Phi_\varepsilon]+\varepsilon^{N+1}M_{N+1}[\Phi_\varepsilon]$$

with

$$M_0=\alpha_0(\vartheta)\frac{\partial^2}{\partial t^2}-\mu_0(\vartheta)\frac{\partial}{\partial t},$$

$$M_1=\alpha_1(\vartheta)\,t\frac{\partial^2}{\partial t^2}+2\beta_0(\vartheta)\frac{\partial^2}{\partial t\,\partial\vartheta}+\{\zeta_0(\vartheta)-\mu_1(\vartheta)\,t\}\frac{\partial}{\partial t}-$$

(3.21) $$-\nu_0(\vartheta)\frac{\partial}{\partial\vartheta}-g_0(\vartheta),$$

$$M_2=\alpha_2(\vartheta)\,t^2\frac{\partial^2}{\partial t^2}+2\beta_1(\vartheta)\,t\frac{\partial^2}{\partial t\,\partial\vartheta}+\gamma_0(\vartheta)\frac{\partial^2}{\partial\vartheta^2}+$$

$$+\{\zeta_1(\vartheta)\,t-\mu_2(\vartheta)\,t^2\}\frac{\partial}{\partial t}+\{\eta_0(\vartheta)-\nu_1(\vartheta)\,t\}\frac{\partial}{\partial\vartheta}+\{f_0(\vartheta)-g_1(\vartheta)\,t\},$$

and similarly for M_i, $i>2$. M_i $(0\leqq i\leqq N)$ is a differential operator of the second order whose coefficients are polynomials in t of degree $\leqq i$; the coefficients of the polynomials are bounded functions of ϑ for $0\leqq\vartheta\leqq\vartheta_0$. The operator M_{N+1} is likewise a differential operator of the second order with bounded coefficients which are, however, no longer simple polynomials in t.

3.4. The Fundamental Iteration Process

In this section we describe the procedure of VIŠIK & LYUSTERNIK for obtaining a formal asymptotic solution of the boundary value problem given by equations (2.1), (2.23), and (2.24). This procedure leads to the following theorem:

Theorem VI. *Let there be a given bounded convex domain G with a smooth boundary Γ which has the property that its parametric representation, with arc length as parameter, is continuously differentiable up to the order* $2m+6$ ($m=0, 1, 2, \ldots$). *If the coefficients and the right hand side of the elliptic differential equation*

(2.1) $$L_\varepsilon[\Phi_\varepsilon(x,y)]\equiv\varepsilon L_2[\Phi_\varepsilon(x,y)]+L_1[\Phi_\varepsilon(x,y)]=h(x,y)$$

are continuously differentiable up to the order $(2m+3)$ *in* $\bar{G}$, *and if the boundary values*

(2.23), (2.24) $$\Phi_\varepsilon(x,y)|_{\Gamma_\pm}=\varphi_\pm(x)$$

are also continuously differentiable up to the order $(2m+3)$ *for all points on the boundary* Γ, *then it is possible to express the solution* $\Phi_\varepsilon(x,y)$ *of the boundary value problem* (2.1), (2.23), (2.24) *in the form*

(3.2) $$\Phi_\varepsilon(x,y)=\sum_{i=0}^{m}\varepsilon^i w_i(x,y)+\sum_{i=0}^{m+1}\varepsilon^i\tilde{v}_i(x,y,\varepsilon)+z_m(x,y,\varepsilon).$$

The functions $w_i(x,y)$ and $\tilde{v}_i(x,y,\varepsilon)$ are uniformly bounded for $\varepsilon>0$ in $\overline{G-V(A)-V(B)}$ where $V(A)$ and $V(B)$ are fixed arbitrarily small neighborhoods of the points

A and B which are defined as the points of tangency to the boundary Γ of the characteristics $x=x_1$ and $x=x_2$ of L_1. In particular, for small values of ε the functions $\tilde{v}_i(x, y, \varepsilon)$ are zero everywhere in $G-V(A)-V(B)$ with the exception of a small neighborhood of the upper part Γ_+ of the boundary Γ (boundary layer functions). The remainder term $z_m(x, y, \varepsilon)$ satisfies in $G-V(A)-V(B)$ the differential equation

$$(3.3)\qquad L_\varepsilon[z_m(x,y,\varepsilon)]=\varepsilon^{m+1}\, g_m(x,y,\varepsilon),$$

where $g_m(x, y, \varepsilon)$ is of the order $O(1)$ in $\overline{G-V(A)-V(B)}$, while $z_m=0$ along the part Γ^* of Γ which bounds the subregion $G-V(A)-V(B)$.

Proof. We divide the proof into three parts; in the first part the functions $w_i(x, y)$ are constructed, in the second part the boundary layer terms $\tilde{v}_i(x, y, \varepsilon)$ are constructed; finally in the last part we prove the properties of the remainder term $z_m(x, y, \varepsilon)$ stated in the theorem.

a. The Construction of the Functions $w_i(x, y)$. The function $w_0(x, y)$ is a solution of the reduced equation

$$(3.7)\qquad L_1[w_0(x,y)]=-\left(\frac{\partial}{\partial y}+g(x,y)\right) w_0(x,y)=h(x,y)$$

with the boundary condition

$$(3.8)\qquad w_0(x,y)|_{\Gamma_-}=w_0(x,\gamma_-(x))=\varphi_-(x).$$

The functions $w_i(x, y)$ are determined by iteration from the equations

$$(3.22)\qquad L_1[w_i(x,y)]=-L_2[w_{i-1}(x,y)]$$

with the boundary conditions

$$(3.23)\qquad w_i(x,y)|_{\Gamma_-}=0, \qquad i=1,2,\dots,m.$$

Hence it follows that the functions $w_i(x, y)$ are given by the expressions

$$(3.10)\quad w_0(x,y)=\varphi_-(x)-\int_{\gamma_-(x)}^{y} \exp\left[-\int_{\eta}^{y} g(x,\zeta)\,d\zeta\right]\{h(x,\eta)+g(x,\eta)\,\varphi_-(x)\}\,d\eta,$$

and

$$(3.24)\quad w_i(x,y)=+\int_{\gamma_-(x)}^{y} \exp\left[-\int_{\eta}^{y} g(x,\zeta)\,d\zeta\right]\{L_2[w_{i-1}(x,\eta)]\}\,d\eta, \qquad i=1,2,\dots,m.$$

From the data in the theorem it follows that $w_0(x, y)$ has continuous partial derivatives up to the order $(2m+3)$ in G, $w_1(x, y)$ up to the order $(2m+1)$, $w_2(x, y)$ up to the order $(2m-1)$, *etc.* and $w_m(x, y)$ up to the order 3. Hence we may apply in G the operator L_ε to the first sum of the right hand side of (3.2).

Moreover it follows from (3.10) and (3.24) that the functions $w_i(x, y)$ ($i=0, 1, \dots, m$) are uniformly bounded in $\overline{G-V(A)-V(B)}$. It is necessary to exclude the neighborhoods $V(A)$ and $V(B)$ since $L_2[w_{i-1}(x, y)]$ has a singularity in the points A and B.

b. The Construction of the Functions $\tilde{v}_i(x, y, \varepsilon)$. The function $w_0(x, y)$ satisfies the differential equation (2.1) up to the order $O(\varepsilon)$ and the boundary condition along the lower part Γ_- of the boundary. In order to satisfy the boundary condition along the upper part Γ_+, we add to the function $w_0(x, y)$ a function $v_0(x, y, \varepsilon)$ which has the following properties:

1. $v_0(x, y, \varepsilon)$ is defined only in the neighborhood Ω of Γ_+ and satisfies there the homogeneous equation

$$L_\varepsilon[v_0(x, y, \varepsilon)] = 0 \tag{3.25}$$

up to the order ε. Hence $v_0(x, y, \varepsilon)$ is a solution of the differential equation

$$M_0[v_0] = \alpha_0(\vartheta)\frac{\partial^2 v_0}{\partial t^2} - \mu_0(\vartheta)\frac{\partial v_0}{\partial t} = 0 \tag{3.26}$$

(see (3.20) and (3.21)).

2. $w_0(x, y) + v_0(x, y, \varepsilon)$ satisfies the boundary condition along Γ_+, *i.e.*

$$[w_0(x, y) + v_0(x, y, \varepsilon)]\,|_{\Gamma_+} = [w_0(\rho, \vartheta) + v_0(\rho, \vartheta, \varepsilon)]\,|_{\rho=0} = \varphi_+(x_b(\vartheta))$$

or

$$v_0\,|_{t=0} = \varphi_+(x_b(\vartheta)) - w_0(0, \vartheta). \tag{3.27}$$

3. $w_0(x, y) + v_0(x, y, \varepsilon)$ approaches the function $w_0(x, y)$ for $\rho \neq 0$ and $\varepsilon \to 0$. This means

$$\lim_{t\to\infty} v_0(t, \vartheta) = 0. \tag{3.28}$$

The general solution of equation (3.26) is

$$\begin{aligned} v_0(t, \vartheta) = v_0\left(\frac{\rho}{\varepsilon}, \vartheta\right) &= C_1(\vartheta)\exp\{+\lambda(\vartheta)\, t\} + C_2(\vartheta) \\ &= C_1(\vartheta)\exp\left\{+\lambda(\vartheta)\frac{\rho}{\varepsilon}\right\} + C_2(\vartheta) \end{aligned} \tag{3.29}$$

with

$$\lambda(\vartheta) = \frac{\mu_0(\vartheta)}{\alpha_0(\vartheta)}. \tag{3.30}$$

Excluding arbitrarily small fixed neighborhoods $V(A)$ and $V(B)$ and taking ρ_0 sufficiently small, we have $\lambda(\vartheta) < 0$ in $\Omega - V(A) - V(B)$ (*cf.* (3.17) and (3.18)). Thus it follows from (3.27) and (3.28) that

$$C_1(\vartheta) = \varphi_+(x_b(\vartheta)) - w_0(0, \vartheta) \tag{3.31a}$$

and

$$C_2(\vartheta) = 0. \tag{3.31b}$$

Hence

$$v_0(x, y, \varepsilon) = v_0(\rho, \vartheta, \varepsilon) = v_0\left(\frac{\rho}{\varepsilon}, \vartheta\right) = \{\varphi_+(x_b(\vartheta)) - w_0(0, \vartheta)\}\exp\left(+\lambda(\vartheta)\frac{\rho}{\varepsilon}\right). \tag{3.32}$$

The function $|v_0(\rho/\varepsilon, \vartheta)|$ decreases rapidly for increasing values of ρ and for decreasing values of ε ($\varepsilon > 0$); $v_0(\rho/\varepsilon, \vartheta)$ is for $\rho \neq 0$ even asymptotically equal to zero in $\Omega - V(A) - V(B)$. Due to these properties $v_0(\rho/\varepsilon, \vartheta)$ is called a boundary layer function. It is now clear that the function $w_0(x, y) + v_0(x, y, \varepsilon)$ is a much better approximation to the solution $\Phi_\varepsilon(x, y)$ than the function $w_0(x, y)$ alone, since outside $V(A)$ and $V(B)$ $w_0(x, y)+v_0(x, y, \varepsilon)$ satisfies the boundary conditions and the differential equation up to the order ε. Similarly the functions $v_i(x, y, \varepsilon)$ are determined in $\Omega - V(A) - V(B)$ by means of the differential equations

$$M_0[v_i] = -\sum_{s=1}^{i} M_s[v_{i-s}], \qquad i=1,2,\dots,m \tag{3.33}$$

with the boundary conditions

$$v_i|_{\Gamma_+} = v_i|_{t=\rho=0} = -w_i(0, \vartheta), \tag{3.34}$$

where the differential operators M_s are defined by (3.21). The result is

$$\begin{aligned} v_i(t,\vartheta) = v_i\left(\frac{\rho}{\varepsilon}, \vartheta\right) &= P_i(t,\vartheta)\exp\{+\lambda(\vartheta)\,t\} \\ &= P_i\left(\frac{\rho}{\varepsilon}, \vartheta\right)\exp\left\{+\lambda(\vartheta)\frac{\rho}{\varepsilon}\right\}, \qquad i=1,2,\dots,m. \end{aligned} \tag{3.35}$$

$P_i(t, \vartheta)$ is a polynomial in t with $-w_i(0, \vartheta)$ as the term independent of t. From (3.35) it again follows that $v_i(\rho/\varepsilon, \vartheta)$ is asymptotically zero for $\rho \neq 0$; for $\varepsilon \neq 0$, $v_i(\rho/\varepsilon, \vartheta)$ is infinitely differentiable with respect to ρ, but, due to the occurrence of the term $w_i(0, \vartheta)$, only $(2m+3-2i)$ times differentiable with respect to Δ. Hence the function $v_m(\rho/\varepsilon, \vartheta)$ is just 3 times differentiable with respect to Δ in $\Omega - V(A) - V(B)$.

Finally the function $v_{m+1}(x, y, \varepsilon)$ is determined by the equation

$$M_0[v_{m+1}] = -\sum_{s=1}^{m+1} M_s[v_{m+1-s}] \tag{3.36}$$

with the boundary condition

$$v_{m+1}|_{\Gamma_+} = v_{m+1}|_{t=\rho=0} = 0. \tag{3.37}$$

The result is

$$\begin{aligned} v_{m+1}(t,\vartheta) &= P_{m+1}(t,\vartheta)\exp\{+\lambda(\vartheta)\,t\} \\ &= P_{m+1}\left(\frac{\rho}{\varepsilon}, \vartheta\right)\exp\left\{+\lambda(\vartheta)\frac{\rho}{\varepsilon}\right\}. \end{aligned} \tag{3.38}$$

$P_{m+1}(t, \vartheta)$ is a polynomial in t without a "constant" term. Hence $v_{m+1}(\rho/\varepsilon, \vartheta)$ is again asymptotically equal to zero for $\rho \neq 0$ and infinitely differentiable with respect to ρ if ε is not zero. Because of the above mentioned differentiability properties of the functions v_i ($i=1, 2, \dots, m$) and the definitions of the operators M_s (*cf.* (3.21)), in particular, of v_m and M_1, the right hand side of (3.36) is only twice continuously differentiable with respect to ϑ, and thus the term $v_{m+1}(\rho/\varepsilon, \vartheta)$ is also only twice continuously differentiable with respect to ϑ.

Hence in $\Omega - V(A) - V(B)$ we may apply the operator L_ε to the expression

$$\sum_{i=0}^{m+1} \varepsilon^i\, v_i(x, y, \varepsilon).$$

In order to express the boundary layer terms as functions defined in all of $\overline{G - V(A) - V(B)}$ and not only in $\overline{\Omega - V(A) - V(B)}$, we multiply them by an infinitely differentiable smoothing factor $\psi(\rho/\rho_0)$ which is identically equal to 1 for $\rho \leqq \frac{1}{3}\rho_0$ and equal to zero for $\rho \geqq \frac{2}{3}\rho_0$; in this way we obtain functions $\tilde{v}_i(x, y, \varepsilon)$:

(3.39) $$\tilde{v}_i(x, y, \varepsilon) = \tilde{v}_i\left(\frac{\rho}{\varepsilon}, \vartheta\right) = \psi\left(\frac{\rho}{\rho_0}\right) v_i\left(\frac{\rho}{\varepsilon}, \vartheta\right).$$

Having defined the function $w_i(x, y)$ and $\tilde{v}_i(x, y, \varepsilon)$, we proceed now to the proof of the properties of the remainder term z_m.

c. The Remainder Term z_m. The remainder term z_m is defined by equation (3.2). Applying the operator L_ε to z_m in $G - V(A) - V(B)$, we obtain

(3.40)
$$\begin{aligned} L_\varepsilon[z_m] &= L_\varepsilon[\Phi_\varepsilon] - L_\varepsilon\left[w_0 + \sum_{i=1}^{m} \varepsilon^i w_i\right] - L_\varepsilon\left[\sum_{j=0}^{m+1} \varepsilon^j \tilde{v}_j\right] \\ &= h - \left\{L_1[w_0] + L_1\left[\sum_{i=1}^{m} \varepsilon^i w_i\right] + \varepsilon L_2\left[\sum_{i=0}^{m} \varepsilon^i w_i\right]\right\} - \\ &\quad - \varepsilon^{-1}\left\{\left(M_0 + \sum_{i=1}^{m+2} \varepsilon^i M_i\right)\left[\sum_{j=0}^{m+1} \varepsilon^j \tilde{v}_j\right]\right\} \\ &= -\varepsilon^{m+1} L_2[w_m] - \varepsilon^{-1}\left\{\left(M_0 + \sum_{i=1}^{m+2} \varepsilon^i M_i\right)\left[\sum_{j=0}^{m+1} \varepsilon^j \tilde{v}_j\right]\right\} \\ &= -\varepsilon^{m+1} L_2[w_m] - \varepsilon^{-1}\left\{\left(M_0 + \sum_{i=1}^{m+2} \varepsilon^i M_i\right)\left[\psi\left(\frac{\rho}{\rho_0}\right)\sum_{j=0}^{m+1} \varepsilon^j v_j\right]\right\}. \end{aligned}$$

The factor $\psi(\rho/\rho_0)$ may be omitted in the second term on the right hand side of equation (3.40) if $\rho \leqq \frac{1}{3}\rho_0$; the second term vanishes identically for $\rho \geqq \frac{2}{3}\rho_0$.

For $\frac{1}{3}\rho_0 \leqq \rho \leqq \frac{2}{3}\rho_0$ we may write

$$\frac{\partial^p}{\partial t^p}\left[\psi\left(\frac{\varepsilon t}{\rho_0}\right) v_j(t, \vartheta)\right] \sim \psi\left(\frac{\varepsilon t}{\rho_0}\right)\frac{\partial^p}{\partial t^p}[v_j(t, \vartheta)],$$

because v_j contains a factor

$$\exp\left[\lambda(\vartheta)\frac{\rho}{\varepsilon}\right]$$

which equals zero asymptotically for $\rho \geqq \frac{1}{3}\rho_0$. Hence we may write

$$L_\varepsilon[z_m] = -\varepsilon^{m+1} L_2[w_m] - \psi\left(\frac{\rho}{\rho_0}\right)\varepsilon^{-1}\left\{\left(M_0 + \sum_{i=1}^{m+2} \varepsilon^i M_i\right)\left[\sum_{j=0}^{m+1} \varepsilon^j v_j\right]\right\}.$$

Using the relations (3.26), (3.33), and (3.36), we find

$$\begin{aligned} L_\varepsilon[z_m] &= -\varepsilon^{m+1}\left\{L_2[w_m] + \psi\left(\frac{\rho}{\rho_0}\right)\sum_{i=1}^{m+2} M_i\left[\sum_{j=0}^{i-1} \varepsilon^j v_{m+2-i+j}\right]\right\} \\ &\sim -\varepsilon^{m+1}\left\{L_2[w_m] + \sum_{i=1}^{m+2} M_i[\tilde{v}_{m+2-i}]\right\}, \end{aligned}$$

and it follows that

(3.41) $$L_\varepsilon[z_m] = \varepsilon^{m+1} g_m(x, y, \varepsilon),$$

with $g_m(x, y, \varepsilon)$ uniformly bounded in $G-V(A)-V(B)$. Finally it follows easily from the boundary conditions imposed on the functions $\Phi_\varepsilon(x, y)$, $w_i(x, y)$ and $v_i(x, y, \varepsilon)$ that $z_m = 0$ along the part Γ^* of Γ that bounds the region $G-V(A)-V(B)$. Q.e.d.

Remark. For the validity of the theorem we have required that the coefficients of the differential operator, the right hand side, and the boundary values be continuously differentiable up to the order $2m+3$, while the parametric representation of the boundary should be differentiable up to the order $2m+6$. The latter condition, for $m=0$, is in agreement with the requirement of LEVINSON, that the boundary Γ should belong to the class C^{VI} (*cf. lit.* 3).

VIŠIK & LYUSTERNIK require that all parameters of the boundary value problem be continuously differentiable up to the order $2(m+1)+p$, where $p-1$ is the order of tangency of the boundary Γ with the characteristics $x=x_1$ and $x=x_2$ at the points A and B (*cf.* [6]).

Reviewing the proof of Theorem VI, we believe, as we have mentioned above, that the conditions of VIŠIK & LYUSTERNIK are not entirely correct, since the order of tangency of the boundary Γ with the characteristics $x=x_1$ and $x=x_2$ is irrelevant for the differentiability conditions of the theorem.

3.5. The Asymptotic Expansion of the Function $\Phi_\varepsilon(x, y)$

In the preceeding chapter it has been shown that for the remainder term z_m of the expansion

$$\Phi_\varepsilon(x,y) = \sum_{i=0}^{m} \varepsilon^i w_i(x,y) + \sum_{i=0}^{m+1} \varepsilon^i \tilde{v}_i(x,y,\varepsilon) + z_m(x,y,\varepsilon) \tag{3.42}$$

the following relation holds in $G-V(A)-V(B)$:

$$L_\varepsilon[z_m] = \varepsilon^{m+1} g_m(x,y,\varepsilon) \tag{3.43}$$

with $g_m(x, y, \varepsilon) = O(1)$ in $G-V(A)-V(B)$ and $z_m = 0$ along the part Γ^* of Γ that bounds the region $G-V(A)-V(B)$. From these relations it is not immediately evident that (3.42) yields an asymptotic expansion for the function $\Phi_\varepsilon(x, y)$; in order to prove the asymptotic properties we have still to make an estimate for the function z_m. This can now easily be performed by using the general theorems of the preceeding chapter.

The following important theorem will now be established:

Theorem VII. *If the conditions of Theorem VI are satisfied, then*

$$\Phi_\varepsilon(x,y) = \sum_{i=0}^{m} \varepsilon^i w_i(x,y) + \sum_{i=0}^{m} \varepsilon^i \tilde{v}_i(x,y,\varepsilon) + O(\varepsilon^{m+1}) \tag{3.44}$$

in every part $\overline{G-V(A)-V(B)}$ of $\bar{G}$,where $V(A)$ and $V(B)$ are arbitrarily small neighborhoods of the points A and B.

Proof. First we consider the case $m=0$. The function $z_0(x, y, \varepsilon)$ is defined in the region $\overline{G-V(A)-V(B)}$, and according to (3.42) it can be written as

$$z_0(x,y,\varepsilon) = \Phi_\varepsilon(x,y) - w_0(x,y) - \tilde{v}_0(x,y,\varepsilon) - \varepsilon\,\tilde{v}_1(x,y,\varepsilon). \tag{3.45}$$

From Theorem VI it is clear that in $G-V(A)-V(B)$ $z_0(x, y, \varepsilon)$ satisfies the differential equation

$$L_\varepsilon[z_0]=\varepsilon\, g_0(x, y, \varepsilon) \tag{3.46}$$

with $g_0(x, y, \varepsilon)=O(1)$ in $G-V(A)-V(B)$.

Along the part Γ^* of the boundary of $G-V(A)-V(B)$ we have the condition

$$z_0|_{\Gamma^*}=0. \tag{3.47}$$

According to Theorem II of Chapter 2 the function $\Phi_\varepsilon(x, y)$ is uniformly bounded in $\overline{G-V(A)-V(B)}$ for sufficiently small values of ε.

Hence it follows that $z_0(x, y, \varepsilon)$ is also uniformly bounded in $\overline{G-V(A)-V(B)}$ for sufficiently small values of ε, because this is also the case for the functions $w_0(x, y)$ and $\tilde{v}_i(x, y, \varepsilon)$ $(i=0, 1)$.

If we now apply the modified form of Theorem V of Chapter 2, it is evident that

$$z_0=O(\varepsilon), \tag{3.48}$$

valied in $\overline{G-V(A)-V(B)}$. Because the function $\tilde{v}_1(x, y, \varepsilon)$ is uniformly bounded in $\overline{G-V(A)-V(B)}$, it follows finally that

$$\Phi_\varepsilon(x, y)=w_0(x, y)+\tilde{v}_0(x, y, \varepsilon)+O(\varepsilon), \tag{3.49}$$

a result valid in $\overline{G-V(A)-V(B)}$. We have thus proved (3.42) for the case of $m=0$.

The general case will now be proved by induction. Consider the remainder term

$$z_k(x, y, \varepsilon)=\Phi_\varepsilon(x, y)-\sum_{i=0}^{k}\varepsilon^i w_i(x, y)-\sum_{i=0}^{k+1}\varepsilon^i \tilde{v}_i(x, y, \varepsilon), \qquad 1\leqq k\leqq m. \tag{3.50}$$

Suppose now that

$$z_{k-1}=O(\varepsilon^k), \qquad 1\leqq k\leqq m. \tag{3.51}$$

Hence it follows that

$$z_k(x, y, \varepsilon)=O(\varepsilon^k)-\varepsilon^k w_k(x, y)-\varepsilon^{k+1}\tilde{v}_{k+1}(x, y, \varepsilon).$$

Putting

$$z_k=\varepsilon^k z_k^*,$$

we obtain

$$z_k^*=O(1)-w_k(x, y)-\varepsilon\,\tilde{v}_{k+1}(x, y, \varepsilon).$$

Since w_k and $\tilde{v}_{k+1}$ are uniformly bounded in $\overline{G-V(A)-V(B)}$, the function z_k^* is also uniformly bounded in $\overline{G-V(A)-V(B)}$. Moreover, according to Theorem VI z_k^* satisfies the differential equation

$$L_\varepsilon[z_k^*]=\varepsilon\, g_k(x. y, \varepsilon)$$

with the boundary condition

$$z_k^*|_{\Gamma^*}=0.$$

Again applying Theorem V in the modified form, we obtain the result

$$z_k^*=O(\varepsilon)$$

in $\overline{G-V(A)-V(B)}$, and hence

$$z_k = O(\varepsilon^{k+1}) \tag{3.52}$$

in $\overline{G-V(A)-V(B)}$. Thus if $z_{k-1}=O(\varepsilon^k)$ in $\overline{G-V(A)-V(B)}$, then $z_k=O(\varepsilon^{k+1})$ in $\overline{G-V(A)-V(B)}$. According to (3.48) we have $z_0=O(\varepsilon)$ in $\overline{G-V(A)-V(B)}$, and hence

$$z_m = O(\varepsilon^{m+1}) \tag{3.53}$$

in $\overline{G-V(A)-V(B)}$. It follows now from (3.2) that

$$\Phi_\varepsilon(x,y) = \sum_{i=0}^{m} \varepsilon^i w_i(x,y) + \sum_{i=0}^{m+1} \varepsilon^i \tilde{v}_i(x,y,\varepsilon) + O(\varepsilon^{m+1}).$$

Because $\tilde{v}_{m+1}(x,y,\varepsilon)$ is uniformly bounded in $\overline{G-V(A)-V(B)}$, we obtain finally the result

$$\Phi_\varepsilon(x,y) = \sum_{i=0}^{m} \varepsilon^i w_i(x,y) + \sum_{i=0}^{m} \varepsilon^i \tilde{v}_i(x,y,\varepsilon) + O(\varepsilon^{m+1}),$$

valid in $\overline{G-V(A)-V(B)}$. Q.e.d.

We conclude this chapter by giving an interesting result which follows immediately from Theorems VI and VII.

Theorem VIII. *If $\Phi_\varepsilon(x,y)$ is the solution of the boundary value problem*

$$L_\varepsilon[\Phi_\varepsilon(x,y)] = h(x,y), \quad \textit{valid in a convex bounded domain } G,$$

$$\Phi_\varepsilon(x,y)|_{\Gamma_\pm} = \varphi_\pm(x), \quad \textit{valid along the smooth boundary of } G,$$

and if the parameters of this boundary value problem, such as the coefficients of the differential operator, the right hand side, the boundary values, and the parametric representation of the boundary, are infinitely continuously differentiable, then the solution $\Phi_\varepsilon(x,y)$ has the asymptotic expansion

$$\Phi_\varepsilon(x,y) \sim \sum_{i=0}^{\infty} \varepsilon^i w_i(x,y) + \sum_{i=0}^{\infty} \varepsilon^i \tilde{v}_i(x,y,\varepsilon). \tag{3.54}$$

This expansion is valid in every part $\overline{G-V(A)-V(B)}$ of $\overline{G}$, where $V(A)$ and $V(B)$ are arbitrarily small fixed neighborhoods of the points A and B.

Remarks. 1. The term $\varepsilon^{m+1}\,\tilde{v}_{m+1}(x,y,\varepsilon)$ does not appear in the final asymptotic expansion (3.44). This term was needed only in order to prove that the remainder term z_m satisfies in $G-V(A)-V(B)$ the differential equation

$$L_\varepsilon[z_m] = O(\varepsilon^{m+1}),$$

from which we could easily derive the wanted estimate of z_m.

2. We have not obtained a suitable asymptotic expansion for $\Phi_\varepsilon(x,y)$ in the neighborhoods $V(A)$ and $V(B)$. However, using Theorem IV of Chapter 2, we can obtain in $V(A)$ and $V(B)$ an estimate for the function $\Phi_\varepsilon(x,y)$ which depends on the size of these neighborhoods, but not on ε. In $V(A)$ and $V(B)$ we have,

according to formula (2.26), the estimate

$$|\Phi_\varepsilon(x, y) - \varphi_-(A)| < M(y - \gamma_-(x)), \tag{3.55}$$

where M does not depend on ε.

The analysis of an asymptotic expansion for $\Phi_\varepsilon(x, y)$, valid in the vicinity of points A and B, requires a modification of the method followed so far. We shall return to this question in a subsequent paper.

4. Regions with Characteristic Boundaries

4.1. Introductory Remarks

We shall presently consider singular perturbation problems associated with the partial differential equation

$$L_\varepsilon(\Phi) = \varepsilon L_2(\Phi) + L_1(\Phi) = h(x, y) \tag{4.1}$$

in cases in which the boundaries of the region contain curves that are characteristics of the reduced equation

$$L_1(\Phi) = 0. \tag{4.2}$$

Such problems have been discussed briefly by Višik & Lyusternik [*6*] and by Knowles & Messick [*2*]. In our study we investigate various aspects that have not been considered by these authors and present asymptotic proofs of a more general scope.

In this chapter we study a particularly simple partial differential equation of the class (4.1); the general theory will be developed in Chapter 5. Throughout we restrict ourselves to the study of the first asymptotic approximation.

4.2. The Parabolic Boundary Layer

Consider the equation

$$\varepsilon\left[\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right] - \frac{\partial \Phi}{\partial y} = 0, \tag{4.3}$$

for $x \geqq 0$ and $y \geqq 0$, with boundary conditions

$$\Phi(x, 0) = 0, \tag{4.4}$$

$$\Phi(0, y) = \varphi(y) \tag{4.5}$$

where $\varphi(y)$ has bounded derivatives up to the second order. We suppose continuity of the assigned boundary conditions, that is

$$\varphi(0) = 0. \tag{4.6}$$

We remark that the boundary $x = 0$ is a characteristic of the reduced equation

$$\frac{\partial \Phi}{\partial y} = 0.$$

Solutions of the reduced equation do not satisfy the given boundary conditions. We therefore introduce the local coordinate

(4.7) $$\xi=\frac{x}{\sqrt{\varepsilon}}.$$

Equation (4.1) transforms into

(4.8) $$\frac{\partial^2 \Phi}{\partial \xi^2}-\frac{\partial \Phi}{\partial y}=-\varepsilon\frac{\partial^2 \Phi}{\partial y^2}.$$

We define the function $u_0(\xi, y)$ as the solution of the reduced equation in local coordinates, that is

(4.9) $$\frac{\partial^2 u_0}{\partial \xi^2}-\frac{\partial u_0}{\partial y}=0,$$

satisfying the boundary conditions (4.4), (4.5). An explicit form of this function is easily obtained as follows:

(4.10) $$u_0(\xi, y)=\sqrt{\frac{2}{\pi}}\int\limits_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2}\varphi\left[y-\frac{\xi^2}{2t^2}\right]dt.$$

If $|\varphi|\leqq M$, then

$$|u_0|\leqq M\frac{2}{\sqrt{u}}\operatorname{erfc}\left[\frac{\xi}{2\sqrt{y}}\right].$$

The function $u_0(\xi, y)$ is therefore uniformly bounded.

Moreover, analysing expression (4.10) for large values of ξ, one easily finds that for $x>\delta$, δ being a fixed arbitrarily small number,

(4.11) $$u_0\left(\frac{x}{\sqrt{\varepsilon}}, y\right)=o(\varepsilon^N)$$

where N is an arbitrarily large positive number. The function $u_0(x/\sqrt{\varepsilon}, y)$ is therefore a boundary layer function. We shall call this function a *parabolic boundary layer,* since it arises as a solution of a parabolic differential equation. One easily establishes that derivatives of the parabolic boundary layer also are boundary layer functions. In particular, we have

(4.12) $$\frac{\partial^2 u_0}{\partial y^2}=\sqrt{\frac{2}{\pi}}\left\{\frac{\xi}{(2y)^{\frac{3}{2}}}e^{-\frac{1}{2}\frac{\xi^2}{2y}}\varphi'(0)+\int\limits_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2}\varphi''\left[y-\frac{\xi^2}{2t^2}\right]dt\right\}$$

where the primes indicate derivatives with respect to the argument.

Returning now to the problem defined by equations (4.3), (4.4), (4.5), we attempt to construct its asymptotic solution by introducing

(4.13) $$\Phi(x, y)=u_0\left(\frac{x}{\sqrt{\varepsilon}}, y\right)+z_0(x, y).$$

For the function z_0 we then have the following problem:

(4.14) $$\varepsilon\left(\frac{\partial^2 z_0}{\partial x^2}+\frac{\partial^2 z_0}{\partial y^2}\right)-\frac{\partial z_0}{\partial y}=-\varepsilon\frac{\partial^2 u_0}{\partial y^2},$$

(4.15) $$z_0(x,0)=0,$$

(4.16) $$z_0(0,y)=0.$$

The parabolic boundary layer u_0 is the first asymptotic approximation of the function Φ, if the function z_0 is uniformly small in the region under consideration.

Investigating the right hand side of equation (4.14), given explicitely in equation (4.12), we see that $\partial^2 u_0/\partial y^2$ is uniformly bounded in the region $x\geqq 0$, $y\geqq 0$ if and only if $\varphi'(0)=0$. In the case of general boundary conditions, when $\varphi'(0)\neq 0$, the right hand side of equation (4.14) has a singularity at the origin $x=0$, $y=0$. The nature of the singularity is most clearly revealed if in equation (4.12) the origin is approached along any curve $\xi=m\,y^\alpha$, where m and α are constants. The presence of this "corner singularity" indicates that in attempting a proof of the asymptotic properties of the parabolic boundary layer difficulties should be expected. It turns out that these difficulties can be circumvented by two different but equivalent methods, which lead to identical results and which will both be presented in the sequel. Firstly, the corner singularity can be removed by a regularization procedure. We shall follow this method in the present chapter. Secondly, proofs of asymptotic behavior can also be based directly on the Theorem IV of Chapter 2. This method will be exposed in Chapter 5.

4.3. Regularization of the Parabolic Boundary Layer

The aim of the present section is to construct a regularized parabolic boundary layer function $\bar{u}_0$ which would satisfy equations (4.3), (4.4), (4.5) up to a certain order of approximation and which moreover would be uniformly bounded and have uniformly bounded first and second derivatives in $y\geqq 0$, $x\geqq 0$. For this purpose we write the boundary condition (4.5) as follows:

(4.17) $$\Phi(0,y)=\varphi(y)=\varphi_0(y)+y\,\varphi'(0)\cdot\exp\left[-\frac{y}{\varepsilon^\alpha}\right]$$

where α is a positive constant that will be determined later on. Equation (4.17) implies that

(4.18) $$\varphi_0(0)=0,\qquad \varphi_0'(0)=0.$$

We presently define the regularized parabolic boundary layer function $\bar{u}_0(\xi,y)$ as the solution of the reduced equation in local coordinates (4.9) that satisfies the boundary conditions

(4.19) $$\bar{u}_0(\xi,0)=0,$$

(4.20) $$\bar{u}_0(0,y)=\varphi_0(y).$$

Explicitly we have

$$\bar{u}_0(\xi, y) = \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi_0 \left[y - \frac{\xi^2}{2t^2} \right] dt. \tag{4.21}$$

Evidently $\bar{u}_0$ is an uniformly bounded boundary layer function; in particular, if $|\varphi_0| \leq M_0$, then

$$|\bar{u}_0(\xi, y)| \leq M_0 \frac{2}{\sqrt{\pi}} \operatorname{erfc}\left[\frac{\xi}{2\sqrt{y}} \right].$$

It is easily shown that the first derivatives of $\bar{u}_0$ also are boundary layer functions, uniformly bounded in $y \geq 0$, $x \geq 0$.

We consider now the second derivative with respect to y, that is,

$$\begin{aligned}
\frac{\partial^2 \bar{u}_0}{\partial y^2} &= \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi_0'' \left[y - \frac{\xi^2}{2t^2} \right] dt \\
&= \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi'' \left[y - \frac{\xi^2}{2t^2} \right] dt - \\
&\quad - \varepsilon^{-2\alpha} \varphi'(0) \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \left(y - \frac{\xi^2}{2t^2} \right) \exp\left[-\frac{1}{\varepsilon^\alpha} \left(y - \frac{\xi^2}{2t^2} \right) \right] dt + \\
&\quad + 2\varepsilon^{-\alpha} \varphi'(0) \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \exp\left[-\frac{1}{\varepsilon^\alpha} \left(y - \frac{\xi^2}{2t^2} \right) \right] dt.
\end{aligned} \tag{4.22}$$

The following estimates can be established without difficulty:

$$\sqrt{\frac{2}{\pi}} \left| \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi'' \left[y - \frac{\xi^2}{2t^2} \right] dt \right| \leq M'' \frac{2}{\sqrt{\pi}} \operatorname{erfc}\left[\frac{\xi}{2\sqrt{y}} \right] \tag{4.23}$$

where $M'' = \operatorname{Max} |\varphi''|$, and

$$\sqrt{\frac{2}{\pi}} \left| \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \left[y - \frac{\xi^2}{2t^2} \right] \exp\left[-\frac{1}{\varepsilon^\alpha} \left(y - \frac{\xi^2}{2t^2} \right) \right] dt \right| \leq \varepsilon^\alpha \frac{2}{\sqrt{\pi}} \operatorname{erfc}\left[\frac{\xi}{2\sqrt{y}} \right] \tag{4.24}$$

$$\sqrt{\frac{2}{\pi}} \left| \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \exp\left[-\frac{1}{\varepsilon^\alpha} \left(y - \frac{\xi^2}{2t^2} \right) \right] dt \right| \leq \frac{2}{\sqrt{\pi}} \operatorname{erfc}\left[\frac{\xi}{2\sqrt{y}} \right]. \tag{4.25}$$

Therefore $\partial^2 \bar{u}_0 / \partial y^2$ is bounded in $x \geq 0$, $y \geq 0$ for each $\varepsilon \neq 0$, and moreover

$$\frac{\partial^2 \bar{u}_0}{\partial y^2} = O(1) + O(\varepsilon^{-\alpha}) \tag{4.26}$$

uniformly in $x \geq 0$, $y \geq 0$.

We return to the problem defined in equations (4.3), (4.4), (4.5) and attempt to approximate its solution by the regularized parabolic boundary layer, that is, we write

$$\Phi(x, y, \varepsilon)=\bar{u}_0\left(\frac{x}{\sqrt{\varepsilon}}, y\right)+\bar{z}_0(x, y, \varepsilon). \tag{4.27}$$

For the function $\bar{z}_0$ we then have the following problem:

$$\varepsilon\left(\frac{\partial^2 \bar{z}_0}{\partial x^2}+\frac{\partial^2 \bar{z}_0}{\partial y^2}\right)-\frac{\partial \bar{z}_0}{\partial y}=-\varepsilon\frac{\partial^2 \bar{u}_0}{\partial y^2}, \tag{4.28}$$

$$\bar{z}_0(x, 0)=0, \tag{4.29}$$

$$\bar{z}_0(0, y)=+\varphi'(0)\, y \exp\left[-\frac{y}{\varepsilon^\alpha}\right]. \tag{4.30}$$

The function $\bar{u}_0$ is indeed an asymptotic approximation for the function Φ if $\bar{z}_0$ is asymptotically small for $\varepsilon \to 0$.

On the basis of equation (4.26) we have, uniformly in $x \geqq 0$, $y \geqq 0$,

$$\varepsilon\frac{\partial^2 \bar{u}_0}{\partial y^2}=O(\varepsilon)+O(\varepsilon^{1-\alpha}). \tag{4.31}$$

Moreover, we shall use the estimate

$$\frac{y}{\varepsilon^\alpha}\exp\left[-\frac{y}{\varepsilon^\alpha}\right] \leqq C \qquad \text{for } y \geqq 0 \tag{4.32}$$

where C is a constant independent of ε.

The problem for $\bar{z}_0$ can then be formulated as follows:

$$\varepsilon\left(\frac{\partial^2 \bar{z}_0}{\partial x^2}+\frac{\partial^2 \bar{z}_0}{\partial y^2}\right)-\frac{\partial \bar{z}_0}{\partial y}=O(\varepsilon)+O(\varepsilon^{1-\alpha}), \tag{4.33}$$

$$\bar{z}_0(x, 0)=0, \tag{4.34}$$

$$\bar{z}_0(0, y)=O(\varepsilon^\alpha). \tag{4.35}$$

Obviously, the optimal choice of the constant α is

$$1-\alpha=\alpha, \qquad \alpha=\tfrac{1}{2}. \tag{4.36}$$

The problem for the remainder term $\bar{z}_0$, unlike that for the remainder term z_0, is free of singularities.

Finally, let us analyse the regularized boundary layer function $\bar{u}_0$. From the definition it follows that

$$\begin{aligned}\bar{u}_0(\xi, y)&=u_0(\xi, y)-\\ &-\sqrt{\frac{2}{\pi}}\,\varphi'(0)\int_{\frac{\xi}{\sqrt{2}}}^{\infty} e^{-\frac{1}{2}t^2}\left(y-\frac{\xi^2}{2t^2}\right)\exp\left[-\frac{1}{\sqrt{\varepsilon}}\left(y-\frac{\xi^2}{2t^2}\right)\right]dt\end{aligned} \tag{4.37}$$

where $u_0(\xi, y)$ is the parabolic boundary layer function introduced in Section 4.2, equation (4.10). Using estimate (4.24), we find that

$$\bar{u}_0(\xi, y) = u_0(\xi, y) + O(\sqrt{\varepsilon}) \tag{4.38}$$

uniformly in $x \geqq 0$, $y \geqq 0$.

Thus the regularization of the parabolic boundary layer that removes the corner singularity is obtained by an asymptotically small modification of the original parabolic boundary layer. Using the regularized parabolic boundary layer instead of the original one, we shall be able to give proofs of asymptotic properties without difficulties. However, before demonstrating that the parabolic boundary layer is indeed the asymptotic solution of the problem (4.3), (4.4), (4.5), we shall study a somewhat different boundary value problem.

4.4. A Boundary Value Problem for a Finite Region

We consider the equation

$$\varepsilon\left[\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right] - \frac{\partial \Phi}{\partial y} = 0 \tag{4.39}$$

in the region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$, l_1 and l_2 being arbitrary positive constants. The boundary conditions are

$$\Phi(x, 0) = f_1(x), \tag{4.40}$$

$$\Phi(x, l_2) = f_2(x), \tag{4.41}$$

$$\Phi(0, y) = g_1(y), \tag{4.42}$$

$$\Phi(l_1, y) = g_2(y). \tag{4.43}$$

We require the boundary data to be four times continuously differentiable in the intervals $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$. Again we suppose continuity of the assigned boundary condition, that is

$$\begin{aligned} f_1(0) &= g_1(0), \qquad g_1(l_2) = f_2(0), \\ f_2(l_1) &= g_2(l_2), \qquad f_1(l_1) = g_2(0). \end{aligned} \tag{4.44}$$

We introduce the solution of the reduced equation satisfying the boundary condition (4.40), *i.e.*

$$w_0 = f_1(x). \tag{4.45}$$

Writing

$$\Phi = f_1(x) + \bar{\Phi}(x, y), \tag{4.46}$$

we have the following problem:

$$\varepsilon\left[\frac{\partial^2 \bar{\Phi}}{\partial x^2} + \frac{\partial^2 \bar{\Phi}}{\partial y^2}\right] - \frac{\partial \bar{\Phi}}{\partial y} = -\varepsilon f_1'', \tag{4.47}$$

$$\bar{\Phi}(x, 0) = 0, \tag{4.48}$$

(4.49) $$\bar{\Phi}(x, l_2) = f_2(x) - f_1(x),$$

(4.50) $$\bar{\Phi}(0, y) = g_1(y) - f_1(0),$$

(4.51) $$\bar{\Phi}(l_1, y) = g_2(y) - f_1(l_1).$$

Presently by following Section 4.3, we introduce a regularized parabolic boundary layer $\bar{u}_0^{(1)}$ along $x=0$ and another regularized parabolic boundary layer $\bar{u}_0^{(2)}$ along $x=l_1$.

Explicitly

(4.52) $$\bar{u}_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, y\right) = \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{2\varepsilon y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi_0^{(1)}\left[y - \frac{x^2}{2\varepsilon t^2}\right] dt,$$

(4.53) $$\bar{u}_0^{(2)}\left(\frac{l_1 - x}{\sqrt{\varepsilon}}, y\right) = \sqrt{\frac{2}{\pi}} \int_{\frac{l_1 - x}{\sqrt{2\varepsilon y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi_0^{(2)}\left[y - \frac{(l_1 - x)^2}{2\varepsilon t^2}\right] dt$$

where

(4.54) $$\varphi_0^{(1)}(y) = g_1(y) - f_1(0) - g_1'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}},$$
$$\varphi_0^{(2)}(y) = g_2(y) - f_1(l_1) - g_2'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}}.$$

We write

(4.55) $$\bar{\Phi} = \bar{u}_0^{(1)} + \bar{u}_0^{(2)} + \bar{\bar{\Phi}}.$$

There remains then to be solved the following problem:

(4.56) $$\varepsilon\left[\frac{\partial^2 \bar{\bar{\Phi}}}{\partial x^2} + \frac{\partial^2 \bar{\bar{\Phi}}}{\partial y^2}\right] - \frac{\partial \bar{\bar{\Phi}}}{\partial y} = -\varepsilon\left\{f_1'' + \frac{\partial^2 \bar{u}_0^{(1)}}{\partial y^2} + \frac{\partial^2 \bar{u}_0^{(2)}}{\partial y^2}\right\},$$

(4.57) $$\bar{\bar{\Phi}}(x, 0) = 0,$$

(4.58) $$\bar{\bar{\Phi}}(x, l_2) = f_2 - f_1 - \bar{u}_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, l_2\right) - \bar{u}_0^{(2)}\left(\frac{l_1 - x}{\sqrt{\varepsilon}}, l_2\right),$$

(4.59) $$\bar{\bar{\Phi}}(0, y) = -\bar{u}_0^{(2)}\left(\frac{l_1}{\sqrt{\varepsilon}}, y\right) + g_1'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}},$$

(4.60) $$\bar{\bar{\Phi}}(l_1, y) = -\bar{u}_0^{(1)}\left(\frac{l_1}{\sqrt{\varepsilon}}, y\right) + g_2'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}}.$$

It should be remarked that the boundary values assigned by conditions (4.59) (4.60) are asymptotically $O(\sqrt{\varepsilon})$. Only the boundary values assigned by conditions (4.58) are of the order unity. This suggests that we may complete the asymptotic solution by constructing along the boundary $y=l_2$ an ordinary boundary layer function of the type studied in the preceding chapter. We introduce for this purpose the local coordinate

(4.61) $$\eta = \frac{l_2 - y}{\varepsilon}.$$

Transforming equation (4.56), and subsequently taking the limit $\varepsilon \to 0$, we find

$$\frac{\partial^2 \bar{\bar{\Phi}}}{\partial \eta^2} + \frac{\partial \bar{\bar{\Phi}}}{\partial \eta} = 0. \tag{4.62}$$

We therefore define the boundary layer function V_0 as follows:

$$V_0\left(x, \frac{l_2 - y}{\varepsilon}\right) = \psi(x) \exp\left[-\frac{l_2 - y}{\varepsilon}\right], \tag{4.63}$$

$$\psi(x) = f_2(x) - f_1(x) - \bar{u}_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, l_2\right) - \bar{u}_0^{(2)}\left(\frac{l_1 - x}{\sqrt{\varepsilon}}, l_2\right). \tag{4.64}$$

We shall presently show that the asymptotic solution of the boundary value problem (4.39), with boundary conditions (4.40) to (4.43), is given by

$$\begin{aligned} \Phi(x, y, \varepsilon) = f_1(x) + \bar{u}_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, y\right) + \bar{u}_0^{(2)}\left(\frac{l_1 - x}{\sqrt{\varepsilon}}, y\right) + \\ + V_0\left(x, \frac{l_2 - y}{\varepsilon}\right) + O(\sqrt{\varepsilon}), \end{aligned} \tag{4.65}$$

the approximation being uniformly valid in the subregion obtained by excluding from the rectangular region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$ arbitrarily small neighborhoods of the two corner points C and D (see Fig. 5). To proof our assertion, we define

$$z_0 = \Phi - [f_1 + \bar{u}_0^{(1)} + \bar{u}_0^{(2)} + V_0]. \tag{4.66}$$

The function z_0 is uniformly bounded in the rectangular region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$ and is the solution of the following problem:

$$\varepsilon\left(\frac{\partial^2 z_0}{\partial x^2} + \frac{\partial^2 z_0}{\partial y^2}\right) - \frac{\partial z_0}{\partial y} = R_0, \tag{4.67}$$

$$R_0 = -\varepsilon\left\{f_1'' + \frac{\partial^2 \bar{u}_0^{(1)}}{\partial y^2} + \frac{\partial^2 \bar{u}_0^{(2)}}{\partial y^2} + \frac{d^2 \psi}{d x^2} \exp\left[-\frac{l_2 - y}{\varepsilon}\right]\right\}, \tag{4.68}$$

$$z_0(x, 0) = -\psi(x)\, e^{-\frac{l_2}{\varepsilon}}, \tag{4.69}$$

$$z_0(x, l_2) = 0, \tag{4.70}$$

$$z_0(0, y) = -\left[\bar{u}_0^{(2)}\left(\frac{l_1}{\sqrt{\varepsilon}}, y\right) + V_0\left(0, \frac{l_2 - y}{\varepsilon}\right) - g_1'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}}\right], \tag{4.71}$$

$$z_0(l_1, y) = -\left[\bar{u}_0^{(1)}\left(\frac{l_1}{\sqrt{\varepsilon}}, y\right) + V_0\left(l_1, \frac{l_2 - y}{\varepsilon}\right) - g_2'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}}\right]. \tag{4.72}$$

Analysis of the above expressions shows that along the boundaries of the rectangular region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$

$$z_0 = O(\sqrt{\varepsilon}). \tag{4.73}$$

We now investigate the term R_0 defined in equation (4.68). Since we have been using regularized parabolic boundary layers, we know from the discussion of

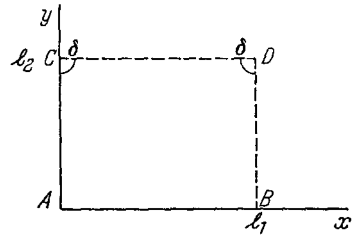

Fig. 5

Section 4.3 that corner singularities at the points A and B (see Fig. 5) do not occur. However, the function $d^2\psi/dx^2$ is locally of the order of magnitude of ε^{-1} in points C and D. This follows from the definition (4.64):

$$\begin{aligned}\frac{d^2\psi}{dx^2}&=f_2''(x)-f_1''(x)-\\ &-\frac{1}{\varepsilon}\left\{\left[\frac{\partial}{\partial y}\bar{u}_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}},y\right)\right]_{y=l_2}+\left[\frac{\partial}{\partial y}\bar{u}_0^{(2)}\left(\frac{l_1-x}{\sqrt{\varepsilon}},y\right)\right]_{y=l_2}\right\}.\end{aligned} \tag{4.74}$$

We therefore have the following asymptotic estimates:

$$\frac{d^2\psi}{dx^2}=f_2''(x)-f_1''(x)+o(\varepsilon^N) \qquad \text{for } \delta<x\leqq l_1-\delta \tag{4.75}$$

where δ is a fixed arbitrarily small positive number. However,

$$\begin{aligned}\left(\frac{d^2\psi}{dx^2}\right)_{x=0}&=f_2''(0)-f_1''(0)-\frac{1}{\varepsilon}g_1'(l_2)+o(\varepsilon^N),\\ \left(\frac{d^2\psi}{dx^2}\right)_{x=l_1}&=f_2''(l_1)-f_1''(l_1)-\frac{1}{\varepsilon}g_2'(l_2)+o(\varepsilon^N).\end{aligned} \tag{4.76}$$

It follows from these considerations and from the results of Section 4.3 that R_0 is uniformly bounded and of the order of magnitude $\sqrt{\varepsilon}$ in the region $0\leqq x\leqq l_1$, $0\leqq y\leqq l_2$ with the exception of the two upper corner points C and D, at which points R_0 is bounded but of the order of magnitude of unity. We exclude from the region under consideration the vicinities of the two corner points C and D by segments of circles with radius δ, as indicated in Fig. 5. In the subregion so defined we have

$$|R_0|<\sqrt{\varepsilon}\,M \tag{4.77}$$

where M is a constant independent of ε. Moreover, z_0 is uniformly bounded in the rectangular region and has along its boundaries the values estimated by equation (4.73). We can therefore immediately apply the Theorem $\mathrm{V}^{\mathrm{bis}}$ of Chapter 2 and conclude that uniformly in the subregion

$$z_0=O(\sqrt{\varepsilon}). \tag{4.78}$$

This proves our assertion and completes the asymptotic solution of the problem under consideration.

Thus our result states that the asymptotic solution (4.65) is valid uniformly in the rectangular region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$ with the exception of the vicinity of the two upper corner points. The asymptotic error is of the order of magnitude of $\sqrt{\varepsilon}$. It should be remarked that it is the corner singularity described in Sections 4.2 and 4.3 that determines the order of magnitude of the error term. Thus if one studies the exceptional case $g_1'(0)=0$, $g_2'(0)=0$, in which no corner singularities occur and the regularization procedure can therefore be omitted, one finds without difficulty that the error term in expansion (4.65) is of the order ε. The corner singularities should therefore be considered as important elements in the theory of parabolic boundary layers.

On the other hand, the difficulty that we have encountered at the upper corner points C and D are of a different nature. These difficulties arise from overlapping of two boundary layers and can be removed by an improved construction of the asymptotic solution. In fact we shall be able to show by a more refined analysis that the exclusion of the corner points C and D in the asymptotic solution (4.65) is unnecessary and that this solution is in fact uniformly valid in the region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$ including all corner points.

4.5. Improved Asymptotic Results

We shall presently investigate more carefully the construction of the boundary layer function along the boundary $y=l_2$. As a preliminary we consider the following problem:

(4.79) $$\varepsilon\left[\frac{\partial^2\Phi^*}{\partial x^2}+\frac{\partial^2\Phi^*}{\partial y^2}\right]-\frac{\partial\Phi^*}{\partial y}=0$$

in the region $y \leqq l_2$,

(4.80) $$\Phi^*(x,l_2)=\psi(x,\varepsilon),$$

(4.81) $$\lim_{y\to-\infty}\Phi^*(x,y)=0.$$

We wish to consider the case in which there exist points $x=x_i$ such that

(4.82) $$\frac{d^n\psi}{dx^n}=O(1) \qquad \text{if } x \neq x_i,$$

(4.83) $$\frac{d^n\psi}{dx^n}=O(\varepsilon^{-\frac{1}{2}n}) \qquad \text{if } x=x_i.$$

This character is exhibited by the function ψ defined in equation (4.64) of the preceding section.

We introduce the local coordinate

(4.84) $$\eta=\frac{l_2-y}{\varepsilon}.$$

Equation (4.79) transforms into

(4.85) $$\frac{\partial^2\Phi^*}{\partial\eta^2}+\frac{\partial\Phi^*}{\partial\eta}=-\varepsilon^2\frac{\partial^2\Phi^*}{\partial x^2}.$$

Equation (4.85) suggests the following expansion:

(4.86) $$\Phi^*(x,y,\varepsilon)=\sum_{n=0}^{N}\varepsilon^{2n}V_n(x,\eta,\varepsilon)+Z_N^*(x,y,\varepsilon)$$

where

(4.87) $$V_0(x,\eta,\varepsilon)=\psi(x,\varepsilon)\exp(-\eta),$$

while for $n>0$

(4.88) $$\frac{\partial^2 V_n}{\partial\eta^2}+\frac{\partial V_n}{\partial\eta}=-\frac{\partial^2 V_{n-1}}{\partial x^2},$$

(4.89) $$[V_n]_{\eta=0}=0,\qquad \lim_{\eta\to+\infty}V_n=0.$$

For Z_N^* we then have

(4.90) $$\varepsilon\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]Z_N^*-\frac{\partial Z_N^*}{\partial y}=R_N^*(x,y,\varepsilon),$$

(4.91) $$R_N^*=-\varepsilon^{2N+1}\frac{\partial^2 V_N}{\partial x^2}.$$

We first consider the case $N=0$:

(4.92) $$R_0^*=-\varepsilon\frac{d^2\psi}{dx^2}\exp\left[-\frac{l_2-y}{\varepsilon}\right].$$

Obviously R_0^* is at most of the order ε if $x\neq x_i$. However

(4.93) $$R_0^*=O(1)\qquad\text{for } y=l_2 \text{ and } x=x_i.$$

This was the case encountered in Section 4.4.

We next consider $N=1$. From equations (4.88), (4.89) we obtain

(4.94) $$V_1(x,\eta)=\frac{d^2\psi}{dx^2}\eta\exp(-\eta).$$

Therefore

(4.95) $$R_1^*=-\varepsilon^2(l_2-y)\frac{d^4\psi}{dx^4}\exp\left[-\frac{l_2-y}{\varepsilon}\right].$$

Now R_1^* is at most of the order ε^3 if $x\neq x_i$. However, if $x=x_i$, R_1^* is of the order ε. This fact follows from the estimate

(4.96) $$\left|\frac{l_2-y}{\varepsilon}\exp\left[-\frac{l_2-y}{\varepsilon}\right]\right|\leqq c_1$$

where c_1 is a constant independent of ε, the estimate being uniformly valid for all $y\leqq l_2$. We therefore arive at the conclusion that

(4.97) $$R_1^*=O(\varepsilon)$$

uniformly in x and $y\leqq l_2$.

It thus appears that the improved boundary layer construction along the boundary $y=l_2$ might remove the difficulties encountered at points $x=x_i$.

Returning to the problem studied in Section 4.4, we introduce

$$\Phi=f_1+\bar{u}_0^{(1)}+\bar{u}_0^{(2)}+\bar{V}_0+\bar{Z}_0, \tag{4.98}$$

$$\bar{V}_0\left(x,\frac{l_2-y}{\varepsilon}\right)=\left[\psi(x)+\varepsilon(l_2-y)\frac{d^2\psi}{dx^2}\right]\exp\left[-\frac{l_2-y}{\varepsilon}\right], \tag{4.99}$$

ψ being defined in equation (4.64). The function $\bar{Z}_0$ must satisfy the following equation:

$$\varepsilon\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\bar{Z}_0-\frac{\partial\bar{Z}_0}{\partial y}=\bar{R}_0, \tag{4.100}$$

$$\bar{R}_0=-\varepsilon\left\{f_1''+\frac{\partial^2\bar{u}_0^{(1)}}{\partial y^2}+\frac{\partial^2\bar{u}_0^{(2)}}{\partial y^2}\right\}+R_1^*. \tag{4.101}$$

Uniformly in the rectangular region $0\leqq x\leqq l_1$, $0\leqq y\leqq l_2$ we now have

$$\bar{R}_0=O(\sqrt{\varepsilon}). \tag{4.102}$$

There remain to be investigated the boundary conditions for $\bar{Z}_0$. These are

$$\begin{aligned}
\bar{Z}_0(x,0)&=-\bar{V}_0\left(x,\frac{l_2}{\varepsilon}\right),\\
\bar{Z}_0(x,l_2)&=0,\\
\bar{Z}_0(0,y)&=-\bar{u}_0^{(2)}\left(\frac{l_1}{\sqrt{\varepsilon}},y\right)-\bar{V}_0\left(0,\frac{l_2-y}{\varepsilon}\right)+g_1'(0)\,y\,e^{-\frac{y}{\sqrt{\varepsilon}}},\\
\bar{Z}_0(l_1,y)&=-\bar{u}_0^{(1)}\left(\frac{l_1}{\sqrt{\varepsilon}},y\right)-\bar{V}_0\left(l_1,\frac{l_2-y}{\varepsilon}\right)+g_2'(0)\,y\,e^{-\frac{y}{\sqrt{\varepsilon}}}.
\end{aligned} \tag{4.103}$$

Using equation (4.99), it follows that

$$\bar{Z}_0(x,0)=o(\varepsilon^N). \tag{4.104}$$

Moreover, we have already seen that

$$\bar{u}_0^{(1,2)}\left(\frac{l_1}{\sqrt{\varepsilon}},y\right)=o(\varepsilon^N). \tag{4.105}$$

We now investigate

$$\bar{V}_0\left(0,\frac{l_2-y}{\varepsilon}\right)=\left[\psi(0)+\varepsilon(l_2-y)\left(\frac{d^2\psi}{dx^2}\right)_{x=0}\right]\exp\left[-\frac{l_2-y}{\varepsilon}\right]. \tag{4.106}$$

From definition (4.64) we have

$$\psi(0)=-\bar{u}_0^{(2)}\left(\frac{l_1}{\sqrt{\varepsilon}},l_2\right)+g_1'(0)\,l_2\,e^{-\frac{l_2}{\sqrt{\varepsilon}}}=o(\varepsilon^N), \tag{4.107}$$

while

$$\left(\frac{d^2\psi}{dx^2}\right)_{x=0}=O(\varepsilon^{-1}). \tag{4.108}$$

Using the estimate (4.96), we arrive at the conclusion that

$$\bar{V}_0\left(0, \frac{l_2-y}{\varepsilon}\right)=O(\varepsilon) \tag{4.109}$$

uniformly for $y \leqq l_2$. In exactly the same way we can prove that

$$\bar{V}_0\left(l_1, \frac{l_2-y}{\varepsilon}\right)=O(\varepsilon) \tag{4.110}$$

uniformly for $y \leqq l_2$.

Combining these results with the estimate (4.32), we arrive at the conclusion that along the boundaries of the rectangular region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$ the function $\bar{Z}_0$ is at most of the order $\sqrt{\varepsilon}$. Moreover, throughout the rectangular region $\bar{Z}_0$ is uniformly bounded while $\bar{R}_0$ is uniformly of the order $\sqrt{\varepsilon}$. Applying Theorem III of Chapter 2, we have

$$\bar{Z}_0=O(\sqrt{\varepsilon}) \tag{4.111}$$

uniformly in $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$.

Finally let us inspect more closely the improved boundary layer function

$$\bar{V}_0\left(x, \frac{l_2-y}{\varepsilon}\right),$$

defined in equation (4 99). Using the estimate (4.96), we see that

$$\varepsilon(l_2-y)\frac{d^2\psi}{dx^2}\exp\left[-\frac{l_2-y}{\varepsilon}\right]=O(\varepsilon) \tag{4.112}$$

uniformly in the rectangular region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$. Therefore we have

$$\bar{V}_0\left(x, \frac{l_2-y}{\varepsilon}\right)=V_0\left(x, \frac{l_2-y}{\varepsilon}\right)+O(\varepsilon). \tag{4.113}$$

Similarly we have already shown in Section 4.3 that the regularized parabolic boundary layer is asymptotically equivalent to the ordinary parabolic boundary layer, that is

$$\bar{u}_0^{(1,2)}=u_0^{(1,2)}+O(\sqrt{\varepsilon}). \tag{4.114}$$

Summarizing our results, we have established the following theorem:

Theorem IX. *If the function $\Phi(x, y)$ is the solution of the boundary value problem*

$$\varepsilon\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]-\frac{\partial\Phi}{\partial y}=0$$

in the domain $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$ *with boundary conditions*

$$\Phi(x,0)=f_1(x), \quad \Phi(x,l_2)=f_2(x),$$
$$\Phi(0,y)=g_1(y), \quad \Phi(l_1,y)=g_2(y),$$

and if the boundary data are continuous at the corner points and possesses along the segments $0 \leqq x \leqq l_1$ *and* $0 \leqq y \leqq l_2$ *continuous derivatives up to the fourth order, then*

$$\Phi = f_1(x) + u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, y\right) + u_0^{(2)}\left(\frac{l_1-x}{\sqrt{\varepsilon}}, y\right) + V_0\left(x, \frac{l_2-y}{\varepsilon}\right) + O(\sqrt{\varepsilon})$$

uniformly in $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$, *including the four corner points.*

In the theorem above the functions $u_0^{(1,2)}$ are parabolic layers defined by

$$u_0^{(1)} = \sqrt{\frac{2}{\pi}} \int\limits_{\frac{x}{\sqrt{2\varepsilon y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi^{(1)}\left[y - \frac{x^2}{2\varepsilon t^2}\right] dt, \tag{4.115}$$

$$u_0^{(2)} = \sqrt{\frac{2}{\pi}} \int\limits_{\frac{l_1-x}{\sqrt{2\varepsilon y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi^{(2)}\left[y - \frac{(l_1-x)^2}{2\varepsilon t^2}\right] dt, \tag{4.116}$$

$$\begin{aligned} \varphi^{(1)} &= g_1(y) - f_1(0), \\ \varphi^{(2)} &= g_2(y) - f_1(l_1). \end{aligned} \tag{4.117}$$

The function V_0 is the boundary layer function

$$V_0\left(x, \frac{l_2-y}{\varepsilon}\right) = \psi(x) \exp\left[-\frac{l_2-y}{\varepsilon}\right], \tag{4.118}$$

$$\psi(x) = f_2(x) - f_1(x) - u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, l_2\right) - u_0^{(2)}\left(\frac{l_1-x}{\sqrt{\varepsilon}}, l_2\right). \tag{4.119}$$

Again we remark that the order of magnitude of the asymptotic error in Theorem IX is determined by the presence of corner singularities (see Section 4.2). If one studies the exceptional case $g_1'(0)=0$, $g_2'(0)=0$ in which the corner singularities are absent, one finds along the lines of the preceding analysis that the asymptotic error in Theorem IX is no longer $O(\sqrt{\varepsilon})$ but $O(\varepsilon)$.

4.6. The Boundary Value Problem for Quarter-Infinite Region

We presently return to the problem studied in Section 4.2. Using the results of Sections 4.3, 4.4 and 4.5, we shall investigate the asymptotic properties of the parabolic boundary layer in an unbounded region.

We thus study the function $\Phi(x, y, \varepsilon)$, defined by

$$\varepsilon\left[\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right] - \frac{\partial \Phi}{\partial y} = 0, \tag{4.120}$$

in the unbounded region $x \geqq 0, y \geqq 0$ and subject to boundary conditions

$$\begin{aligned} \Phi &= 0 \qquad \text{for } y=0, \\ \Phi &= \varphi(y) \qquad \text{for } x=0, \end{aligned} \tag{4.121}$$

the function $\varphi(y)$ having continuous derivatives up to the fourth order. We recall the definition of the parabolic boundary layer:

$$u_0\left(\frac{x}{\sqrt{\varepsilon}}, y\right) = \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi\left[y - \frac{\xi^2}{2t^2}\right] dt. \tag{4.122}$$

The following assertion will now be proved:

Theorem X. *If the function Φ satisfying equations* (4.120), (4.121) *is uniformly bounded in a region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$, where l_1 and l_2 are arbitrary constants, and possesses uniformly bounded derivatives up to the fourth order, then the asymptotic approximation*

$$\Phi(x, y, \varepsilon) = u_0\left(\frac{x}{\sqrt{\varepsilon}}, y\right) + O(\sqrt{\varepsilon}) \tag{4.123}$$

is uniformly valid in the subregion $0 \leqq x \leqq l_1 - \delta$, $0 \leqq y \leqq l_2 - \delta$, where δ is an arbitrarily small positive number.

To prove the assertion, we analyse

$$z_0 = \Phi - \bar{u}_0 \tag{4.124}$$

where $\bar{u}_0$ is the regularized parabolic boundary layer studied in Section 4.3; that is

$$\begin{aligned} \bar{u}_0 &= \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} e^{-\frac{1}{2}t^2} \varphi_0\left[y - \frac{\xi^2}{2t^2}\right] dt, \\ \varphi_0 &= \varphi(y) - \varphi'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}}. \end{aligned} \tag{4.125}$$

The function z_0 is then defined by

$$\varepsilon\left\{\frac{\partial^2 z_0}{\partial x^2} + \frac{\partial^2 z_0}{\partial y^2}\right\} - \frac{\partial z_0}{\partial y} = -\varepsilon \frac{\partial^2 \bar{u}_0}{\partial y^2}, \tag{4.126}$$

$$z_0(x, 0) = 0, \tag{4.127}$$

$$z_0(0, y) = \varphi'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}}. \tag{4.128}$$

The values taken by the function z_0 along $y = l_2$ and $x = l_1$ are as follows:

$$z_0(x, l_2) = \Phi(x, l_2) - \bar{u}_0\left(\frac{x}{\sqrt{\varepsilon}}, l_2\right) = f_2(x), \tag{4.129}$$

$$z_0(l_1, y) = \Phi(l_1, y) - \bar{u}_0\left(\frac{l_1}{\sqrt{\varepsilon}}, y\right) = g_2(y). \tag{4.130}$$

Although the functions $f_2(x)$ and $g_2(y)$ are unknown, they are uniformly bounded in the intervals that are considered, and so are the derivatives of f_2 and g_2 up to the fourth order.

We decompose the function z_0 into two parts:

$$z_0 = z_0^{(1)} + z_0^{(2)}, \tag{4.131}$$

$z_0^{(1)}$ being defined by

$$\begin{gathered} \varepsilon\left[\frac{\partial^2 z_0^{(1)}}{\partial x^2} + \frac{\partial^2 z_0^{(1)}}{\partial y^2}\right] - \frac{\partial z_0^{(1)}}{\partial y} = -\varepsilon \frac{\partial^2 \bar{u}_0}{\partial y^2}, \\ z_0^{(1)} = 0 \qquad \text{along } \Gamma \end{gathered} \tag{4.132}$$

where Γ is the boundary of the region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$. $z_0^{(2)}$ is defined by

$$\begin{aligned} &\varepsilon\left[\frac{\partial^2 z_0^{(2)}}{\partial x^2} + \frac{\partial^2 z_0^{(2)}}{\partial y^2}\right] - \frac{\partial z_0^{(2)}}{\partial y} = 0, \\ &z_0^{(2)}(x, 0) = 0, \\ &z_0^{(2)}(0, y) = \varphi'(0)\, y\, e^{-\frac{y}{\sqrt{\varepsilon}}} = g_1(y), \\ &z_0^{(2)}(x, l_2) = f_2(x), \\ &z_0^{(2)}(l_1, y) = g_2(y). \end{aligned} \tag{4.133}$$

We first consider the problem for $z_0^{(1)}$. In Section 4.3 it has been shown that

$$\varepsilon \frac{\partial^2 \bar{u}_0}{\partial y^2} = O(\sqrt{\varepsilon}) \qquad \text{for } x \geqq 0,\ y \geqq 0. \tag{4.134}$$

Applying Theorem III of Chapter 2, we thus find

$$z_0^{(1)} = O(\sqrt{\varepsilon}) \qquad \text{for } 0 \leqq x \leqq l_1,\ 0 \leqq y \leqq l_2. \tag{4.135}$$

We next consider the problem for $z_0^{(2)}$. Formally this problem has been solved in Section 4.4 and 4.5, and using Theorem IX, we have

$$z_0^{(2)} = u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, y\right) + u_0^{(2)}\left(\frac{l_1 - x}{\sqrt{\varepsilon}}, y\right) + V_0\left(x, \frac{l_2 - y}{\varepsilon}\right) + O(\sqrt{\varepsilon}). \tag{4.136}$$

The solution is valid uniformly in the region $0 \leqq x \leqq l_1$, $0 \leqq y \leqq l_2$.

Analysing the solution (4.136), we first write the explicit expression for the parabolic boundary layer $u_0^{(1)}$:

$$u_0^{(1)} = \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{2\varepsilon y}}}^{\infty} e^{-\frac{1}{2}t^2} g_1\left(y - \frac{x^2}{2\varepsilon t^2}\right) dt. \tag{4.137}$$

The function g_1 being defined in equation (4.133), we find from estimate (4.24) that

$$u_0^{(1)} = O(\sqrt{\varepsilon}). \tag{4.138}$$

Next we remark that for any bounded functions $f_2(x)$ and $g_2(y)$

$$u_0^{(2)}\left(\frac{l_1-x}{\sqrt{\varepsilon}}, y\right)=o(\varepsilon^N) \qquad \text{for } x\leqq l_1-\delta, \tag{4.139}$$

$$V_0\left(x, \frac{l_2-y}{\varepsilon}\right)=o(\varepsilon^N) \qquad \text{for } y\leqq l_2-\delta \tag{4.140}$$

where δ is a fixed positive arbitrarily small number. Therefore, in the subregion $0\leqq x\leqq l_1-\delta$, $0\leqq y\leqq l_2-\delta$, we have the uniform estimate

$$z_0=z_0^{(1)}+z_0^{(2)}=O(\sqrt{\varepsilon}). \tag{4.141}$$

We recall finally that according to Section 4.3

$$\bar{u}_0=u_0+O(\sqrt{\varepsilon}) \qquad \text{for } x\geqq 0,\ y\geqq 0. \tag{4.142}$$

Hence, uniformly in the subregion $0\leqq x\leqq l_1-\delta$, $0\leqq y\leqq l_2-\delta$, we have

$$\Phi-u_0=O(\sqrt{\varepsilon}) \tag{4.143}$$

which completes the proof of Theorem X.

5. General Asymptotic Theory for Regions with Characteristic Boundaries

5.1. Generalized Parabolic Boundary Layer

Presently we consider the general linear elliptic partial differential equation of second order

$$L_\varepsilon(\Phi)=h(x,y), \tag{5.1}$$

$$\begin{aligned} L_\varepsilon=\varepsilon\left\{a(x,y)\frac{\partial^2}{\partial x^2}+2b(x,y)\frac{\partial^2}{\partial x\,\partial y}+c(x,y)\frac{\partial^2}{\partial y^2}+\right.\\ \left.+d(x,y)\frac{\partial}{\partial x}+e(x,y)\frac{\partial}{\partial y}+f(x,y)\right\}-\left\{\frac{\partial}{\partial y}+g(x,y)\right\}. \end{aligned} \tag{5.2}$$

The conditions that are satisfied by the coefficients of the operator L_ε have been formulated in Chapter 2.

We again study problems in which the boundaries of the region under consideration contain characteristic lines. In analogy with Section 4.2 we first investigate a simple problem for the unbounded region $x\geqq 0$, $y\geqq 0$ and impose the boundary conditions

$$\begin{aligned} &\Phi(x,0)=0,\\ &\Phi(0,y)=\varphi(y), \qquad \varphi(0)=0. \end{aligned} \tag{5.3}$$

Furthermore, for simplicity, we consider here the case

$$h(x,y)=0. \tag{5.4}$$

Again we introduce the local coordinate

$$\xi=\frac{x}{\sqrt{\varepsilon}}. \tag{5.5}$$

Equation (5.1) transforms into

$$a\frac{\partial^2\Phi}{\partial\xi^2}-\frac{\partial\Phi}{\partial y}-g\,\Phi=-\sqrt{\varepsilon}\left\{2b\frac{\partial^2\Phi}{\partial\xi\,\partial y}+d\frac{\partial\Phi}{\partial\xi}\right\}-\varepsilon\left\{c\frac{\partial^2\Phi}{\partial y^2}+e\frac{\partial\Phi}{\partial y}+f\Phi\right\}. \tag{5.6}$$

Now we define the function $u_0(\xi, y)$ as the solution of the reduced equation in local coordinates, satisfying the boundary conditions (5.3). Explicitly

$$a_0(y)\frac{\partial^2 u_0}{\partial\xi^2}-\frac{\partial u_0}{\partial y}-g_0(y)\,u_0=0, \tag{5.7}$$

$$\begin{aligned} u_0(\xi,0)&=0\\ u_0(0,y)&=\varphi(y)\end{aligned} \tag{5.8}$$

where

$$a_0(y)=a(0,y), \qquad g_0(y)=g(0,y). \tag{5.9}$$

The function u_0 can easily be expressed in terms of the parabolic boundary layer function analysed in Section 4.2. This is accomplished by the transformation

$$u_0(\xi,y)=\tilde{u}_0(\xi,\eta_1)\exp\left[-\int_0^y g_0(y')\,dy'\right], \tag{5.10}$$

$$\eta_1=\int_0^y a_0(y')\,dy'. \tag{5.11}$$

One then obtains

$$\tilde{u}_0(\xi,\eta_1)=\sqrt{\frac{2}{\pi}}\int_{\frac{\xi}{\sqrt{2\eta_1}}}^{\infty} e^{-\frac{1}{2}t^2}\,\tilde{\varphi}\left[\eta_1-\frac{\xi^2}{2t^2}\right]dt \tag{5.12}$$

where

$$\tilde{\varphi}(\eta_1)=\varphi(y)\cdot\exp\left[\int_0^y g_0(y')\,dy'\right]. \tag{5.13}$$

It follows that $u_0(\xi, y)$ is a boundary layer function, and we shall call this function again a parabolic boundary layer. Moreover, the derivatives of the function $u_0(\xi, y)$ also are boundary layer functions.

We now investigate the appearance of singularities in the derivatives of u_0. Straightforward analysis shows that the derivatives

$$\frac{\partial u_0}{\partial\xi},\quad \frac{\partial u_0}{\partial y},\quad \frac{\partial^2 u_0}{\partial\xi^2}$$

are all uniformly bounded in every closed subregion of the region $x\geqq 0$, $y\geqq 0$. However,

$$\frac{\partial^2 u_0}{\partial y^2}=\sqrt{\frac{2}{\pi}}\left\{\frac{\xi}{(2\eta_1)^{\frac{3}{2}}}e^{-\frac{1}{2}\frac{\xi^2}{2\eta_1}}\tilde{\varphi}'(0)\,a_0^2+I_1\right\}\cdot\exp\left[-\int_0^y g_0(y')\,dy'\right], \tag{5.14}$$

$$\frac{\partial^2 u_0}{\partial\xi\,\partial y}=\sqrt{\frac{2}{\pi}}\left\{-\frac{1}{(2\eta_1)^{\frac{1}{2}}}e^{-\frac{1}{2}\frac{\xi^2}{2\eta_1}}\tilde{\varphi}'(0)\,a_0+I_2\right\}\cdot\exp\left[-\int_0^y g_0(y')\,dy'\right] \tag{5.15}$$

where I_1 and I_2 are functions that are uniformly bounded in the region under consideration. These derivatives thus possess singularities at the origin. Returning now to the problem defined in equations (5.1), (5.2), (5.3), we write

$$\Phi(x,y,\varepsilon)=u_0\left(\frac{x}{\sqrt{\varepsilon}},y\right)+Z_0(x,y,\varepsilon). \tag{5.16}$$

We obtain

$$L_\varepsilon(Z_0)=-R_0, \tag{5.17}$$

$$Z_0(0,y)=0, \quad Z_0(x,0)=0, \tag{5.18}$$

$$\begin{aligned} R_0=\sqrt{\varepsilon}\left\{2b\frac{\partial^2 u_0}{\partial\xi\,\partial y}+d\frac{\partial u_0}{\partial\xi}+\frac{a-a_0}{\sqrt{\varepsilon}}\frac{\partial^2 u_0}{\partial\xi^2}-\frac{g-g_0}{\sqrt{\varepsilon}}u_0\right\}+ \\ +\varepsilon\left\{c\frac{\partial^2 u_0}{\partial y^2}+e\frac{\partial u_0}{\partial y}+f\,u_0\right\}. \end{aligned} \tag{5.19}$$

As in Section 4.2, R_0 is unbounded at the origin, due to the "corner singularity". If a small neighborhood of the origin is excluded, and if the coefficients a and g are differentiable functions of x in the vicinity of $x=0$, then R_0 is uniformly of the order $\varepsilon^{\frac{1}{2}}$. In the special case that a and g are functions of y only, while b and d both equal zero, R_0 is uniformly of the order ε, the small neighborhood of the origin being again excluded.

The difficulties arising from the presence of the corner singularities can be circumvented by introducing regularized parabolic boundary layers, analogous to the theory of Section 4.3. However, as has already been mentioned, the asymptotic theory can also be developed without the regularization procedure when proper use is made of Theorem IV, Chapter 2. The two approaches are equivalent and lead to identical results. This will be illustrated in the present chapter, where the general theory will be developed without making use of the concept of regularized parabolic boundary layers.

5.2. Boundary Value Problem for a Finite Region

We now consider the equation

$$L_\varepsilon(\Phi)=h(x,y) \tag{5.20}$$

in the region $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, with boundary conditions

$$\begin{aligned} \Phi(x,0)&=f_1(x), \\ \Phi(x,1)&=f_2(x), \\ \Phi(0,y)&=g_1(y), \\ \Phi(1,y)&=g_2(y), \end{aligned} \tag{5.21}$$

the assigned boundary conditions being continuous.

The region that we consider is a particularly simple one, but this is not a serious restriction. As we shall see later on, the solutions of problems defined in a more general finite region can be obtained by a simple transformation from the solution of the problem considered here.

As in Chapter 4, we shall construct the asymptotic solution of the problem (5.20), (5.21) by superposition of the solution of the reduced equation, two parabolic boundary layers along $x=0$ and $x=1$, and an ordinary boundary layer along $y=1$. We shall not introduce regularized parabolic boundary layers; for the boundary layer along $y=1$ we shall directly use the improved construction, analogous to Section 4.5.

In the development that follows we shall omit the explicit analysis of the differentiability conditions that must be satisfied by the parameters of the boundary value problem. These conditions are easily established from the requirement that all derivatives of the coefficients of the operator L_ε as well as all derivatives of the boundary data (5.21) that appear in the theory should be continuous and uniformly bounded in $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

We define the solution of the reduced equation satisfying the boundary condition at $y=0$:

$$w_0(x,y)=\left\{f_1(x)-\int_0^y h(x,y'')\exp\left[\int_0^{y''} g(x,y')\,dy'\right]dy''\right\}\times \times\exp\left[-\int_0^y g(x,y')\,dy'\right]. \tag{5.22}$$

We also define the function $\rho_0(x,y,\varepsilon)$ by the relation:

$$\rho_0=-L_\varepsilon(w_0)+h(x,y). \tag{5.23}$$

Obviously we have

$$\rho_0=O(\varepsilon) \tag{5.24}$$

uniformly in $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Next, following Section 5.1, we introduce the parabolic boundary layer

$$u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}},y\right)=\sqrt{\frac{2}{\pi}}\int_{\frac{x}{\sqrt{2\varepsilon\eta_1}}}^{\infty} e^{-\frac{1}{2}t^2}\,\tilde{\varphi}^{(1)}\left[\eta_1-\frac{x^2}{2\varepsilon t^2}\right]dt\cdot\exp\left[-\int_0^y g_0(y')\,dy'\right] \tag{5.25}$$

where

$$\eta_1=\int_0^y a_0(y')\,dy', \tag{5.26}$$

$$\tilde{\varphi}^{(1)}(\eta_1)=[g_1(y)-w_0(0,y)]\exp\left[\int_0^y g_0(y')\,dy'\right]. \tag{5.27}$$

Analogously we define the second parabolic boundary layer

$$u_0^{(2)}\left(\frac{1-x}{\sqrt{\varepsilon}},y\right)=\sqrt{\frac{2}{\pi}}\int_{\frac{1-x}{\sqrt{2\varepsilon\eta_2}}}^{\infty} e^{-\frac{1}{2}t^2}\,\tilde{\varphi}^{(2)}\left[\eta_2-\frac{(1-x)^2}{2\varepsilon t^2}\right]dt\times \times\exp\left[-\int_0^y \tilde{g}_0(y')\,dy'\right] \tag{5.28}$$

where

$$\eta_2 = \int_0^y a_1(y')\,dy', \tag{5.29}$$

$$a_1(y) = a(1,y), \quad \tilde{g}_0(y) = g(1,y), \tag{5.30}$$

$$\tilde{\varphi}^{(2)}(\eta_2) = [g_2(y) - w_0(1,y)] \exp\left[\int_0^y \tilde{g}_0(y')\,dy'\right]. \tag{5.31}$$

We also define the functions $R_0^{(1)}$ and $R_0^{(2)}$ by the relation

$$R_0^{(1,2)} = -L_\varepsilon[u_0^{(1,2)}]. \tag{5.32}$$

$R_0^{(1)}$ is given explicitly by equation (5.19) where the substitution $u_0^{(1)}$ has to be made for u_0. $R_0^{(2)}$ is also given by (5.19), if in that equation a_0 is replaced by a_1 and g_0 is replaced by $\tilde{g}_0$, while for u_0 the function $u_0^{(2)}$ is substituted. Following the discussion in Section 5.1, we obtain the estimates

$$R_0^{(1,2)} = O(\sqrt{\varepsilon}) \tag{5.33}$$

valid uniformly in $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, with the exception of small fixed neighborhoods of the corner points $x=0$, $y=0$ and $x=1$, $y=0$. The estimate (5.33) should be replaced by

$$R_0^{(1,2)} = O(\varepsilon) \tag{5.34}$$

if the following conditions are satisfied:

$$\begin{aligned} a &= a(y), \quad & b &= 0, \\ d &= 0, \quad & g &= g(y). \end{aligned} \tag{5.35}$$

Now estimates (5.33), (5.34) are valid in, say, $0 \leqq x \leqq 1$, $\delta \leqq y \leqq 1$, where δ is a small *fixed* positive number. However, we shall need estimates valid in a more extended region $\overline{G^*}$, defined by $0 \leqq x \leqq 1$, $\varepsilon^\alpha \leqq y \leqq 1$, where $0 < \alpha < 1$. These estimates follow from analysis of corner singularities that are present in $R_0^{(1,2)}$ (equations (5.19), (5.14), (5.15)). Using the inequality

$$\left| \frac{x}{\sqrt{4\varepsilon y}} \exp\left[-\frac{x^2}{4\varepsilon y}\right] \right| < 1,$$

valid uniformly in $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, we find the following estimates in $\overline{G^*}$:

$$R_0^{(1,2)} = O(\varepsilon^{1-\alpha}) \quad \text{if (5.35) are satisfied,} \tag{5.36}$$

$$R_0^{(1,2)} = O(\varepsilon^{\frac{1}{2}(1-\alpha)}) \quad \text{if (5.35) are not satisfied,} \tag{5.37}$$

$$R_0^{(1,2)} = O(\varepsilon^{\min(\frac{1}{2},\, 1-\alpha)}) \quad \text{if } b=0 \text{ while one other condition (5.35) is not satisfied.} \tag{5.38}$$

These estimates will be seen to be very important in establishing the proof of asymptotic properties.

Presently we turn to the analysis of the boundary layer solution along the boundary $y=1$. For this purpose, in analogy to Section 4.5, we consider the following problem:

$$L_\varepsilon(\Phi^*)=0 \tag{5.39}$$

for $y\leqq 1$ and with the boundary condition

$$\Phi^*(x,1)=\psi(x). \tag{5.40}$$

We study the case in which there exist points $x=x_i$ such that

$$\frac{d^n\psi}{dx^n}=O(1) \qquad \text{for } x\neq x_i,$$
$$\frac{d^n\psi}{dx^n}=O(\varepsilon^{-\frac{1}{2}n}) \qquad \text{for } x=x_i. \tag{5.41}$$

We introduce the local coordinate

$$\eta=\frac{1-y}{\varepsilon} \tag{5.42}$$

and, moreover, define

$$c(x,1)=\frac{1}{\omega(x)}. \tag{5.43}$$

Equation (5.39) transforms into

$$\frac{1}{\omega}\frac{\partial^2\Phi^*}{\partial\eta^2}+\frac{\partial\Phi^*}{\partial\eta}=-\varepsilon L_1(\Phi^*)-\varepsilon^2 L_2(\Phi^*), \tag{5.44}$$

$$L_1=-2b\frac{\partial^2}{\partial x\,\partial\eta}-e\frac{\partial}{\partial\eta}+\frac{c\,\omega-1}{\varepsilon\,\omega}\frac{\partial^2}{\partial\eta^2}-g, \tag{5.45}$$

$$L_2=a\frac{\partial^2}{\partial x^2}+d\frac{\partial}{\partial x}+f. \tag{5.46}$$

We shall study the development

$$\Phi^*(x,y,\varepsilon)=\sum_{n=0}^{N}\varepsilon^n V_n(x,\eta,\varepsilon)+Z_N^*(x,y,\varepsilon) \tag{5.47}$$

which we define as follows:

$$V_0(x,\eta)=\psi(x)\,e^{-\omega\eta}, \tag{5.48}$$

$$\frac{1}{\omega}\frac{\partial^2 V_1}{\partial\eta^2}+\frac{\partial V_1}{\partial\eta}=-L_1(V_0), \tag{5.49}$$

$$\frac{1}{\omega}\frac{\partial^2 V_n}{\partial\eta^2}+\frac{\partial V_n}{\partial\eta}=-L_1(V_{n-1})-L_2(V_{n-2}). \tag{5.50}$$

The boundary conditions on the functions V_n, $n>0$, are

$$V_n(x,0)=0, \qquad \lim_{\eta\to+\infty} V_n=0. \tag{5.51}$$

For the function Z_n^* we then have

$$L_\varepsilon(Z_N^*) = R_N^*, \tag{5.52}$$

$$R_0^* = -L_1(V_0) - \varepsilon L_2(V_0), \tag{5.53}$$

$$R_N^* = -\varepsilon^N \{L_2(V_{N-1}) + L_1(V_N)\} - \varepsilon^{N+1} L_2(V_N). \tag{5.54}$$

The explicit calculation of the functions V_1, V_n, *etc.* is complicated, but these functions can be shown to be of the following general form:

$$V_1(x,\eta,\varepsilon) = \eta \left\{ G_0^{(1)}(x,\eta,\varepsilon)\psi(x) + G_1^{(1)}(x,\eta,\varepsilon)\frac{d\psi}{dx} \right\} e^{-\omega\eta}, \tag{5.55}$$

$$V_2(x,\eta,\varepsilon) = \eta \left\{ G_0^{(2)}(x,\eta,\varepsilon)\psi(x) + G_1^{(2)}(x,\eta,\varepsilon)\frac{d\psi}{dx} + G_2^{(2)}(x,\eta,\varepsilon)\frac{d^2\psi}{dx^2} \right\} e^{-\omega\eta} \tag{5.56}$$

where the functions $G_0^{(1)}$, $G_1^{(1)}$, $G_0^{(2)}$ *etc.* are all uniformly bounded and of the order of magnitude of unity in the region under consideration, as are their derivatives with respect to x and η. Moreover, the function $G_1^{(1)}$ is identically zero if $b=0$.

We are now in the position to investigate the functions R_N^*. R_1^* is of the order ε if $x \neq x_i$, but

$$R_1^* = O(1) \qquad \text{for } y \geqq 0 \text{ and } x = x_i. \tag{5.57}$$

Proceeding to $N=2$, we have finally, for $\eta \geqq 0$,

$$\begin{aligned} R_2^* &= O(\sqrt{\varepsilon}) && \text{if } b \neq 0, \\ R_2^* &= O(\varepsilon) && \text{if } b = 0, \end{aligned} \tag{5.58}$$

the estimates being uniformly valid, including the points $x = x_i$.

Thus it appears that we shall need the boundary layer approximation $N=2$ in order to be able to give proofs of asymptotic behavior that are of uniform validity. We therefore define the boundary layer function

$$V_0^*(x,\eta,\varepsilon) = V_0(x,\eta,\varepsilon) + \varepsilon V_1(x,\eta,\varepsilon) + \varepsilon^2 V_2(x,\eta,\varepsilon), \tag{5.59}$$

$$\psi(x) = f_2(x) - w_0(x,1) - u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, 1\right) - u_0^{(2)}\left(\frac{1-x}{\sqrt{\varepsilon}}, 1\right). \tag{5.60}$$

Returning to the boundary value problem (5.20), (5.21), we summarize our results by writing

$$\begin{aligned} \Phi(x,y,\varepsilon) = w_0(x,y) &+ u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}}, y\right) + u_0^{(2)}\left(\frac{1-x}{\sqrt{\varepsilon}}, y\right) + \\ &+ V_0^*\left(x, \frac{1-y}{\varepsilon}, \varepsilon\right) + \bar{Z}_0(x,y,\varepsilon). \end{aligned} \tag{5.61}$$

The remainder term $\bar{Z}_0$ is a solution of the equation

$$L_\varepsilon(\bar{Z}_0)=\bar{R}_0; \tag{5.62}$$

$\bar{R}_0$ is given by

$$\bar{R}_0=\rho_0+R_0^{(1)}+R_0^{(2)}+R_2^*. \tag{5.63}$$

In the subregion, obtained by excluding from the region $0\leqq x\leqq 1$, $0\leqq y\leqq 1$ two fixed arbitrarily small neighborhoods of the corner points $x=0$, $y=0$ and $x=1$, $y=0$, we have the following estimates: If

$$\begin{aligned} b&=0, & d&=0,\\ a&=a(y), & g&=g(y), \end{aligned} \tag{5.64}$$

then

$$\bar{R}_0=O(\varepsilon). \tag{5.65}$$

If one of conditions (5.64) is not satisfied, then

$$\bar{R}_0=O(\sqrt{\varepsilon}). \tag{5.66}$$

Moreover, considering a subregion $0\leqq x\leqq 1$, $\varepsilon^\alpha\leqq y\leqq 1$, with $0<\alpha<1$, we have

$$\bar{R}_0=O(\varepsilon^{1-\alpha}) \tag{5.67}$$

if conditions (5.64) are satisfied;

$$\bar{R}_0=O(\varepsilon^{\min(\frac{1}{2},1-\alpha)}) \qquad \text{if } b=0. \tag{5.68}$$

Finally

$$\bar{R}_0=O(\varepsilon^{\frac{1}{2}(1-\alpha)}) \qquad \text{if } b\neq 0. \tag{5.69}$$

We proceed to analyse the boundary conditions for the function $\bar{Z}_0$. These are

$$\begin{aligned} &\bar{Z}_0(x,0)=-V_0^*\left(x,\frac{1}{\varepsilon},\varepsilon\right),\\ &\bar{Z}_0(x,1)=0,\\ &\bar{Z}_0(0,y)=-u_0^{(2)}\left(\frac{1}{\sqrt{\varepsilon}},y\right)-V_0^*\left(0,\frac{1-y}{\varepsilon},\varepsilon\right),\\ &\bar{Z}_0(1,y)=-u_0^{(1)}\left(\frac{1}{\sqrt{\varepsilon}},y\right)-V_0^*\left(1,\frac{1-y}{\varepsilon},\varepsilon\right). \end{aligned} \tag{5.70}$$

Utilising the explicit results for the boundary layer functions, we find

$$\begin{aligned} &[\bar{Z}_0]_\Gamma=O(\sqrt{\varepsilon}) && \text{if } b\neq 0\\ &[\bar{Z}_0]_\Gamma=O(\varepsilon) && \text{if } b=0, \end{aligned} \tag{5.71}$$

where $[\bar{Z}_0]_\Gamma$ symbolizes the values of $\bar{Z}_0$ along the boundary of the region $0\leqq x\leqq 1$, $0\leqq y\leqq 1$.

5.3. *Proof of the Asymptotic Properties*

We rewrite the definition (5.61) of the function $\bar{Z}_0$ as follows:

$$\bar{Z}_0=[\Phi(x,y,\varepsilon)-f_1(x)]-[w_0(x,y)-f_1(x)]-u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}},y\right)-\\ -u_0^{(2)}\left(\frac{1-x}{\sqrt{\varepsilon}},y\right)-V_0^*\left(x,\frac{1-y}{\varepsilon},\varepsilon\right) \tag{5.72}$$

where the function Φ is defined by the boundary value problem (5.20), (5.21), while for all other functions appearing on the right hand side of equation (5.72) explicit definitions have been given in Section 5.2.

We remark now that Theorem IV of Chapter 2 applies to the boundary value problem (5.20), (5.21), so that we have the following estimate:

$$|\Phi(x,y,\varepsilon)-f_1(x)|\leqq My \tag{5.73}$$

where M is a constant independent of ε.

Let us define the subregion $\bar{G}^{**}$:

$$(x,y)\in G^{**} \quad \text{if } 0\leqq x\leqq 1,\ 0\leqq y\leqq \varepsilon^\alpha,\ 0<\alpha<1. \tag{5.74}$$

It follows from estimate (5.73) that

$$\Phi-f_1=O(\varepsilon^\alpha) \quad \text{in } G^{**}. \tag{5.75}$$

Moreover, using in equation (5.72) the explicit definition of the function w_0, $u_0^{(1,2)}$ and V_0^*, one finds without difficulty that

$$\bar{Z}_0=O(\varepsilon^\alpha) \quad \text{in } G^{**}. \tag{5.76}$$

We next consider the subregion $\bar{G}^*$:

$$(x,y)\in\bar{G}^* \quad \text{if } 0\leqq x\leqq 1,\ \varepsilon^\alpha\leqq y\leqq 1,\ 0<\alpha<1. \tag{5.77}$$

In this region from equation (5.62) for $\bar{Z}_0$ we have the equation

$$L_\varepsilon(\bar{Z}_0)=\bar{R}_0=O(\varepsilon^\mu) \tag{5.78}$$

where μ is given in estimates (5.67), (5.68), (5.69).

Moreover, result (5.76) also holds at the boundary $y=\varepsilon^\alpha$ of G^*. Together with the results (5.71) we obtain

$$\bar{Z}_0=O(\varepsilon^{\min(\nu,\alpha)}) \quad \text{on } \Gamma^*, \qquad \nu=\begin{Bmatrix}\frac{1}{2} \text{ if } b\neq 0\\ 1 \text{ if } b=0\end{Bmatrix} \tag{5.79}$$

where Γ^* is the boundary of G^*.

To the boundary value problem (5.78), (5.79) we apply Theorem III of Chapter 2, and it follows that

$$\bar{Z}_0=O(\varepsilon^{\min(\nu,\alpha,\mu)}) \quad \text{in } G^*. \tag{5.80}$$

We first consider the case in which conditions (5.64) are satisfied. In that case according to equation (5.67) $\mu=1-\alpha$, $\nu=1$, and the optimal choice of α is

(5.81) $$\alpha=\tfrac{1}{2}.$$

It follows that in G^*

(5.82) $$\bar{Z}_0=O(\sqrt{\varepsilon}) \qquad \text{if (5.64) is satisfied}.$$

Next we consider the case $b=0$, while one other condition (5.64) is not satisfied. From equation (5.68) we then have $\mu=\min(\tfrac{1}{2}, 1-\alpha)$. One easily finds that (5.81), (5.82) are again valid.

Finally we consider the case $b\neq 0$. Then from equations (5.69), (5.79) we have $\mu=\tfrac{1}{2}(1-\alpha)$, $\nu=\tfrac{1}{2}$. The optimal choice of α now appears to be

(5.83) $$\alpha=\tfrac{1}{3}.$$

It follows that in G^*

(5.84) $$\bar{Z}_0=O(\varepsilon^{\frac{1}{3}}) \qquad \text{if } b\neq 0.$$

Combining results (5.76), (5.81), (5.82), (5.83), (5.84), we obtain

(5.85) $$\bar{Z}_0=O(\varepsilon^{\nu}) \qquad \text{in } 0\leqq x\leqq 1,\ 0\leqq y\leqq 1.$$

(5.86) $$\begin{aligned} \nu&=\tfrac{1}{2} \qquad \text{if } b=0\\ \nu&=\tfrac{1}{3} \qquad \text{if } b\neq 0. \end{aligned}$$

This completes the proof of the asymptotic properties of the expansion (5.61). In a final step we analyse more closely the boundary layer function V_0^* appearing in that expansion. From the explicit formulas one easily finds

(5.87) $$V_0^*=\psi(x)\exp\left[-\omega(x)\frac{1-y}{\varepsilon}\right]+\xi_0,$$

(5.88) $$\xi_0=\begin{cases} O(\sqrt{\varepsilon}) & \text{if } b\neq 0,\\ O(\varepsilon) & \text{if } b=0. \end{cases}$$

Hence we have established the following theorem:

Theorem XI. *If the function Φ satisfies the boundary value problem* (5.20), (5.21), *then uniformly in* $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, *including all four corner points,*

$$\Phi=w_0(x,y)+u_0^{(1)}\left(\frac{x}{\sqrt{\varepsilon}},y\right)+u_0^{(2)}\left(\frac{1-x}{\sqrt{\varepsilon}},y\right)+\psi(x)\exp\left[-\omega(x)\frac{1-y}{\varepsilon}\right]+O(\varepsilon^{\nu})$$

where

$$\nu=\tfrac{1}{2} \qquad \textit{if } b=0,$$

and

$$\nu=\tfrac{1}{3} \qquad \textit{if } b\neq 0.$$

We remark that in problems studied in Chapter 4 conditions (5.64) were satisfied and thus also $b=0$. It is seen that in that case Theorem XI is in agreement with Theorem IX.

Let us consider finally the special case in which corner singularities are absent. One easily shows that these singularities do not appear if

$$(5.89)\qquad \begin{aligned} g_1'(0)+h(0,0)+g_1(0)\,g(0,0)&=0\,,\\ g_2'(0)+h(1,0)+g_2(0)\,g(1,0)&=0\,. \end{aligned}$$

In that case Theorem III can be applied directly to the boundary value problem (5.62), (5.71), with the estimates (5.65), (5.66). One then obtains the following modification of Theorem XI:

Theorem XI$^{\text{bis}}$**.** *The asymptotic error in Theorem XI is defined by* $\nu=\frac{1}{2}$ *if conditions* (5.89) *are satisfied, and by* $\nu=1$ *if conditions* (5.89) *and* (5.64) *are satisfied.*

5.4. Other Applications of the Theory of Parabolic Boundary Layers

The results of the preceding section can be utilised for the analysis of the problem described in Section 5.1. The asymptotic properties of the parabolic boundary layer in any finite subregion of the region $x\geqq 0$, $y\geqq 0$ can again be proved. The reasoning follows very closely the analysis of Section 4.6 and will therefore not be reproduced here. Instead we shall discuss briefly the application of the preceding theory to boundary value problems defined in more general

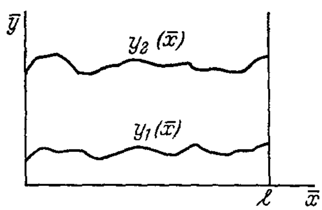

Fig. 6

bounded convex regions. Consider, in the coordinates system $\bar{x}, \bar{y}$ the region $0\leqq\bar{x}\leqq l$, $\bar{y}_1(\bar{x})\leqq\bar{y}\leqq\bar{y}_2(\bar{x})$, $\bar{y}_1(x)<\bar{y}_2(\bar{x})$, and let the function $\Phi(\bar{x},\bar{y},\varepsilon)$ be a solution of the equation

$$(5.90)\qquad \begin{aligned} \bar{L}_\varepsilon(\Phi)=\varepsilon\Big\{\bar{a}(\bar{x},\bar{y})\frac{\partial^2}{\partial\bar{x}^2}+2\bar{b}(\bar{x},\bar{y})\frac{\partial^2}{\partial\bar{x}\,\partial\bar{y}}+\bar{c}(\bar{x},\bar{y})\frac{\partial^2}{\partial\bar{y}^2}+\bar{d}(\bar{x},\bar{y})\frac{\partial}{\partial\bar{x}}+\\ +\bar{e}(\bar{x},\bar{y})\frac{\partial}{\partial\bar{x}}+\bar{f}(\bar{x},\bar{y})\Big\}\Phi-\frac{\partial\Phi}{\partial\bar{y}}-\bar{g}(\bar{x},\bar{y})\Phi=\bar{h}(\bar{x},\bar{y})\,. \end{aligned}$$

Along the boundary Γ of the region continuous boundary conditions are given. Equation (5.90) is of elliptic type, the coefficients are continuous, and, in particular, $\bar{a}>0$ and $\bar{g}-\varepsilon\bar{f}\geqq 0$. We assume sufficient differentiability of the parameters of the problem. The region of the $\bar{x},\bar{y}$ plane under consideration can in general be mapped into a rectangle by a transformation suggested by Višik & Lyusternik:

$$(5.91)\qquad y=\frac{\bar{y}-\bar{y}_1}{\bar{y}_2-\bar{y}_1},\qquad x=\frac{\bar{x}}{l}\,.$$

It is easily seen that equation (5.90) transforms into equation (5.1), (5.2). It follows that asymptotic solutions of problems defined in regions described above can be obtained by transformation from the asymptotic solution of the problem for the rectangular region. Of course, for this to be true the functions $\bar{y}_1(\bar{x})$ and $\bar{y}_2(\bar{x})$ must satisfy certain differentiability conditions which can be obtained from the requirement that the boundary value problem obtained by transformation (5.91) should satisfy all the conditions needed for the validity of the theory developed in Sections 5.2 and 5.3.

6. Conclusions

In the preceding chapters we have presented the general theory of singular perturbation problems associated with linear elliptic differential equations of second order. However, in the course of our analysis we have left unanswered (and sometimes not even explicitly formulated) a number of questions whose elucidation was not strictly needed for the development of the general theory. It is the purpose of this concluding chapter to summarize these as yet "open questions".

In Chapter 3 we have studied problems in which the boundary Γ is a smooth convex curve having only two points A and B in which the boundary is tangent to a characteristic line of the operator L_1 (Fig. 7). We have proved the uniform validity of the asymptotic expansion in a region $\overline{G - V(A) - V(B)}$ where $V(A)$ and $V(B)$ are arbitrarily small fixed neighborhoods of A and B. Thus naturally there arises the question what is the asymptotic approximation of the function Φ

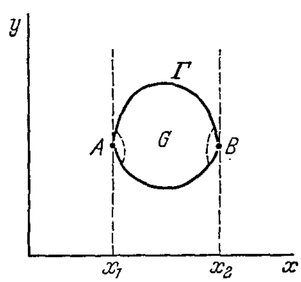

Fig. 7

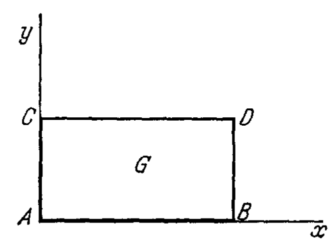

Fig. 8

in the immediate neighborhood of these points? To elucidate this question, the method of the analysis used must be modified. We shall consider this matter in a subsequent paper.

A similar question arises in the theory of parabolic boundary layers. In Chapters 4 and 5 we have analysed the corner singularity that appears at points A and B of the regions under consideration (Fig. 8). Somewhat to our surprise we have succeeded in proving the uniform validity of the *first* asymptotic approximation in the whole region $\bar{G}$, *including the cornerpoints A and B*. However, we have seen that in the exceptional case in which the corner singularities are absent the order of magnitude of the asymptotic error is $\varepsilon^{\frac{1}{2}}$ or ε, depending on the case that is considered, while in the presence of the corner singularities the corresponding asymptotic error is of the order $\varepsilon^{\frac{1}{3}}$ or $\varepsilon^{\frac{1}{2}}$. This indicated that if one should wish to develop higher order asymptotic approximations, the difficulties due to the corner singularity would reappear and would have to be reconsidered. A modification of the method of analysis may then be needed. Similarly, at corner points C and D, the difficulties due to overlapping of two boundary layers have been removed in the first asymptotic approximation by an improved construction of the boundary layer along CD. In higher approximations this aspect of the problem would also have to be reconsidered. Thus, the extension of the theory involving parabolic boundary layers to higher order approximation should not be considered a trivial matter.

Throughout our analysis we have explicitly required the boundary Γ to be convex. This condition assures to some extent that the asymptotic approximation will not contain the so-called "free boundary layer". A free boundary layer in a domain G can be defined as a function which is asymptotically equivalent to zero everywhere in G except for a neighborhood of a line l that lies in G. The derivatives of the free boundary layer in the direction normal to l are of the order of magnitude of inverse powers of ε in the neighborhood of l.

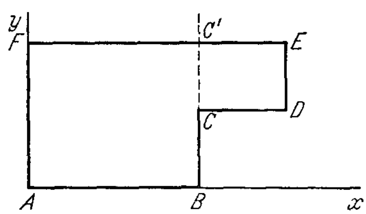

Fig. 9

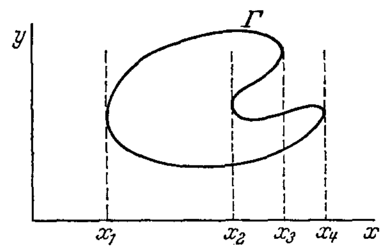

Fig. 10

The occurrence of the free boundary layer can be illustrated by the boundary value problem in a domain $ABCDEF$ as sketched in Fig. 9. The reader may convince himself that it is impossible to treat this problem as an application of the theory of Chapters 4, 5. The partially concave boundary leads to difficulties along the line CC', which can only be overcome by introducing free boundary layer terms in the solution.

Similarly, in the case of a smooth boundary, possessing more than two points at which the tangent coincides with the characteristics of L_1 (Fig. 10), the theory of Chapter 3 cannot be applied and free boundary layers appear. Free boundary layers should also be expected to occur in some cases in which the differentiability conditions, given in the preceding chapters, are violated. This, for example, is the case, when in Fig. 9 the boundary condition assigned along AB is discontinuous at some point or has a discontinuous derivative. Similarly, free boundary layers should be expected if in the problem studied in Chapter 3 the boundary Γ^- is not sufficiently smooth.

The phenomenon of free boundary layers has various interesting aspects which merit a detailed investigation. Very little has been published on this subject to date; our results on the theory of free boundary layers will be presented in a subsequent paper.

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