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Stationary distributions for a class of Markov-modulated tandem fluid queues

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ABSTRACT

We consider a model consisting of two fluid queues driven by the same background continuous-time Markov chain, such that the rates of change of the fluid in the second queue depend on whether the first queue is empty or not: when the first queue is nonempty, the content of the second queue increases, and when the first queue is empty, the content of the second queue decreases.

We analyze the stationary distribution of this tandem model using operator-analytic methods. The various densities (or Laplace–Stieltjes transforms thereof) and probability masses involved in this stationary distribution are expressed in terms of the stationary distribution of some embedded process. To find the latter from the (known) transition kernel, we propose a numerical procedure based on discretization and truncation. For some examples we show the method works well, although its performance is clearly affected by the quality of these approximations, both in terms of accuracy and run time.

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1. Introduction

Markov-modulated fluid queues[6, 18, 25] can be considered as continuous-state counterparts to ordinary queues: fluid (rather than individual customers) arrives to some buffer where it is temporarily stored, before it is released again. The rates at which the fluid flows into and out of the buffer may vary over time in a stochastic sense, being regulated by some finite-state continuous-time Markov chain. Stationary distributions of such fluid queues have been studied extensively, first using spectral methods[6], later via more efficient matrix-analytic methods[11–14, 16, 23]. The analysis of networks of fluid queues[1, 7, 17, 21, 22] is much harder, and no simple expressions exist for the case when the inflow/outflow rates depend on the Markov chain as well as the fluid level in another queue, even for the special case of tandem queues. In[1] the marginal distributions of the individual fluid

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queues in some tandem model were found, see also[7], as well as correlations between the different queues, but to find the entire joint distribution of the queue contents is much harder, and results are only known for a few special two-node cases.

In[17] two fluid queues[1] X and Y are considered, where the first one is driven by a CTMC φ with two states $+$ and $-$ (for "on" and "off," say), and 2×2 generator matrix T. Thus, $\{\varphi(t), X(t)\}$ is a standard on/off fluid queue process, with fluid rates $r_+ > 0$ and $r_- < 0$. The second fluid queue Y behaves such that its content increases when $X(t) > 0$ at some fixed rate $\widehat{c} > 0$, and it decreases when $X(t) = 0$ at some fixed rate $\check{c} < 0$, so that by choosing parameters appropriately we can indeed model the case where the output of X is fed into Y (at constant rate as long as $X(t) > 0$), i.e., a tandem model. Note that it was assumed that $\widehat{c}$ is always the same, no matter what the current state ($+$ or $-$) of φ. For this model,[17] presented closed-form expressions for the stationary joint distribution of $\{\varphi(t), X(t), Y(t)\}$.

In this paper, we generalize this model in two ways. First, we replace the on/off assumption for the input of X, so we allow φ to have $n \geq 2$ states. Note that this is not trivial in the presence of the second buffer Y, which depends on X. Second, we allow the behavior of Y not only to depend on X (namely $X(t) = 0$ or $X(t) > 0$), but also on the state of φ. However, in order to analyze the model, we do need to retain the *assumption* that

- Y must increase (not decrease) when $X(t) > 0$ (at some rate $\widehat{c_i} \geq 0$ when $\varphi(t) = i$);
- Y must decrease when $X(t) = 0$ (at some rate $\check{c}_i < 0$ when $\varphi(t) = i$).

Note that $\check{c}_i$ is only defined for states i with $r_i \leq 0$, since for other states i the buffer X is nonempty with probability 1. Although the assumption means that we do not cover all possible two-node fluid queues, we argue that several important cases are included, see the examples in Figure 1.

In Figure 1(a), we consider X to be driven by some CTMC φ_X, with inflow rates $a_i > 0$ and a constant output capacity C. We assume that the output of X then arrives (instantaneously) at Y, but Y only allows a maximum inflow rate $b_j \leq C$, depending on the state of some (other) CTMC φ_Y. Thus, when $X(t) > 0$ and $\varphi_Y(t) = j$, fluid is lost at rate $C - b_j$, while the rest enters Y at rate b_j. On the other hand, at times when $X(t) = 0$ and $\varphi_X(t) = i$, the outflow of X has rate $a_i < C$ ("leaking through" an empty buffer), so the rate at which fluid flows into Y in that case is $\min(a_i, b_j)$ (when $\varphi_Y(t) = j$). For the output capacity of Y, we assume some fixed value D that should satisfy $D \leq \min_j b_j$ and $D > a_i$ for all i with $a_i \leq C$, in order to ensure that the content of Y will increase (not decrease) when $X(t) > 0$, and will decrease when $X(t) = 0$. Defining $\varphi(t) = (\varphi_X(t), \varphi_Y(t))$, we obtain a model that fits our description, with $r_{(i,\cdot)} = a_i - C, \widehat{c}_{(\cdot,j)} = b_j - D \geq 0$ and $\check{c}_{(i,j)} = \min(a_i, b_j) - D < 0$. Of course other variants can be considered as well, e.g., by also

[1]We use notation that corresponds to the sequel of the current paper, rather than the original notation in[17].

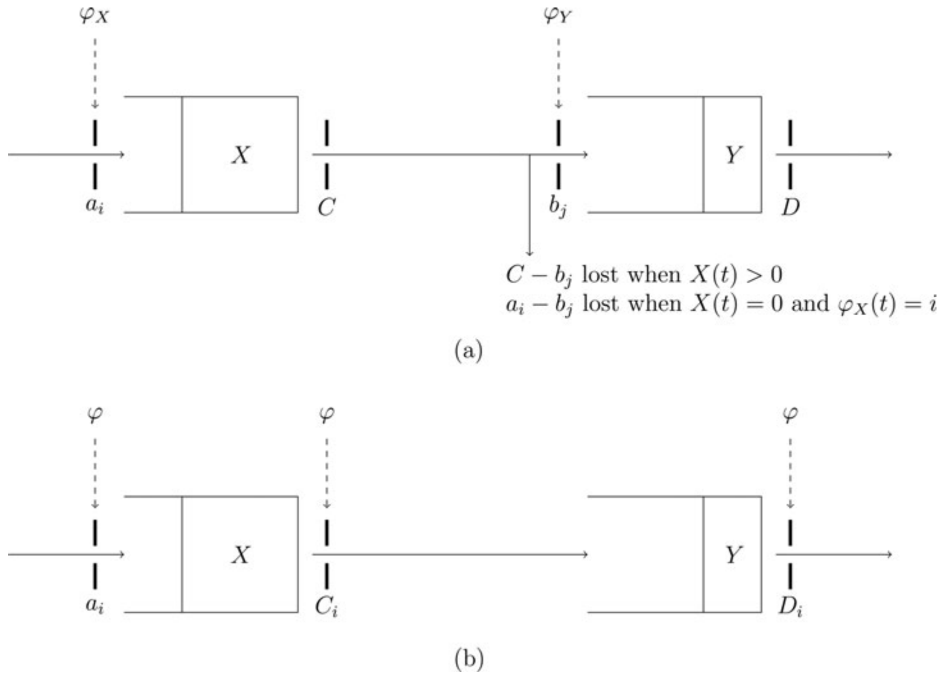

Figure 1. Two practical examples covered by our model. (a) Tandem fluid queue with fixed output capacities and modulated inputs; we assume $b_j \leq C$ and $\max_{i:a_i \leq C} a_i < D \leq \min_j b_j$. (b) Tandem fluid queue with (dependent) modulated inputs and outputs without loss; we assume $D_i \leq C_i$, and $D_i > a_i$ when $a_i \leq C_i$.

modulating the output capacity of Y, or introducing dependencies by taking $\varphi(t) = \varphi_X(t) = \varphi_Y(t)$.

Another even more natural model is in Figure 1(b) where we consider both the inflow and outflow rates of X to be driven by some CTMC φ, say with inflow rates $a_i > 0$ and outflow rates C_i. Denoting the (modulated) outflow rate of Y by D_i, we assume that $D_i \leq C_i$, and that $D_i > a_i$ when $a_i \leq C_i$. Then, the model fits the description with $r_i = a_i - C_i$, $\widehat{c}_i = C_i - D_i \geq 0$ and $\check{c}_i = a_i - D_i < 0$.

For the special on/off case model considered in[17], analytical results could be obtained by considering an embedded $M/G/1$ queue. Here, this seems to be much harder, and we derive expressions by embedding at times that X becomes empty; in fact, this is precisely the reason we need to restrict to the assumption mentioned above. In deriving the expressions, we rely heavily on using operator-analytic methods, which generalize the matrix-analytic methods for single queues, and were shown to be a promising approach in Bean and O'Reilly[8, 9] and also in Margolius and O'Reilly[20]. In[8], a tandem model was considered in which one of the buffers has no lower bounds, meaning that the queue contents can become negative. In[9], a tandem with lower bounds in both queues was considered in which the inflow/outflow rates of one queue depend on the underlying Markov chain as well as the level in the other queue. In Margolius and O'Reilly[20], a time-varying queue is analyzed. In this work, we show that operator-analytic methods can indeed lead to good results.

The remainder of the paper is structured as follows. In Section 2, we detail the model and give some preliminaries. Then, in the main Section 3, we analyze the model by first considering an embedded discrete-time Markov process (in Section 3.2), and then use it to provide expressions for the various densities involved in the stationary joint distribution (in Section 3.3). We provide some numerical results in Section 4 and conclude in Section 5.

2. Model and preliminaries

In this section, we first describe the model of interest and then give the stability condition. We end with some preliminary statements about the sample paths that can be taken by the model, and the implications for the shape (in particular the support) of the stationary distribution.

2.1. Model description

We consider two fluid queues, collecting fluid in buffers X and Y, with level variables recording the content at time t denoted by $X(t)$ and $Y(t)$, respectively, that are being driven by the same background continuous-time Markov chain $\{\varphi(t) : t \geq 0\}$ with some finite state space $\mathcal{S}$ and irreducible generator $\mathbf{T}$. The first queue behaves as a standard fluid queue $\{(\varphi(t), X(t)) : t \geq 0\}$ studied in[11], with a lower boundary at level 0, and real-valued fluid rates r_i collected in a diagonal matrix $\mathbf{R} = \text{diag}(r_i)_{i\in\mathcal{S}}$. Thus, the content $X(t)$ increases at rate r_i when $\varphi(t) = i$, unless r_i is negative and $X(t) = 0$. More precisely,

$$\frac{d}{dt}X(t) = r_{\varphi(t)} \qquad \text{when } X(t) > 0, \tag{1}$$

$$\frac{d}{dt}X(t) = \max(0, r_{\varphi(t)}) \qquad \text{when } X(t) = 0. \tag{2}$$

We partition the state space $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_\circ$, where $r_i > 0$ when $i \in \mathcal{S}_+$ (states in $\mathcal{S}_+$ will be called upstates), $r_i < 0$ when $i \in \mathcal{S}_-$ (states in $\mathcal{S}_-$ will be called downstates), and $r_i = 0$ when $i \in \mathcal{S}_\circ$ (states in $\mathcal{S}_\circ$ will be called zero-states). With the behavior at $X(t) = 0$ in mind, it will sometimes be helpful to use additional notation $\mathcal{S}_\ominus = \mathcal{S}_- \cup \mathcal{S}_\circ$ for the set of "zero-states at $X(t) = 0$." After appropriately ordering the states in $\mathcal{S}$, we can write $\mathbf{T}$ as 3×3 block matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{++} & \mathbf{T}_{+-} & \mathbf{T}_{+\circ} \\ \mathbf{T}_{-+} & \mathbf{T}_{--} & \mathbf{T}_{-\circ} \\ \mathbf{T}_{\circ+} & \mathbf{T}_{\circ-} & \mathbf{T}_{\circ\circ} \end{bmatrix}. \tag{3}$$

Further, we assume that the behavior of the second fluid queue depends on both $\varphi(t)$ and $X(t)$ in the following way. Assuming fluid rates $\widehat{c}_i > 0$ and $\check{c}_i < 0$ for all $i \in \mathcal{S}$, collected in $\widehat{\mathbf{C}} = \text{diag}(\widehat{c}_i)_{i\in\mathcal{S}}$ and $\check{\mathbf{C}} = \text{diag}(\check{c}_i)_{i\in\mathcal{S}}$, we have

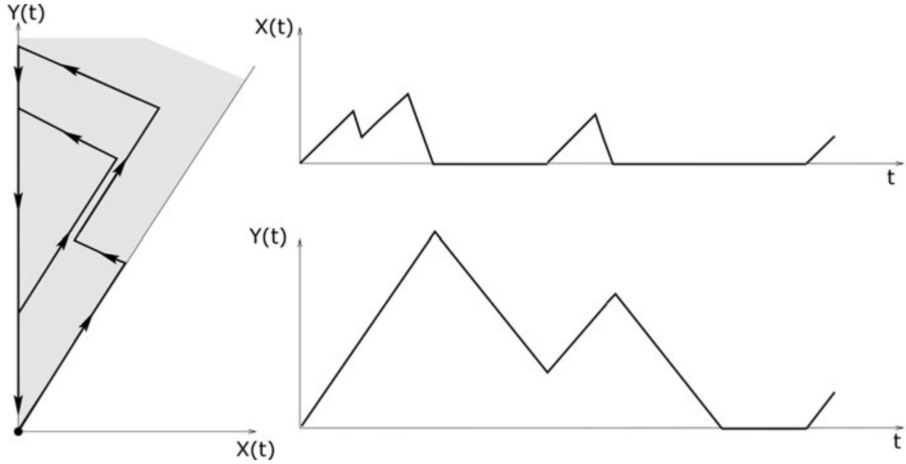

Figure 2. Example of a typical sample path in the tandem fluid queue, also see[17]. The level in buffer Y decreases when $X = 0$, and increases when $X > 0$, at rates dependent on the current phase $\varphi(t)$.

$$\frac{d}{dt}Y(t) = \widehat{c}_{\varphi(t)} > 0 \qquad \text{when } X(t) > 0, \tag{4}$$

$$\frac{d}{dt}Y(t) = \check{c}_{\varphi(t)} < 0 \qquad \text{when } X(t) = 0,\ Y(t) > 0, \tag{5}$$

$$\frac{d}{dt}Y(t) = \widehat{c}_{\varphi(t)} \cdot 1\{\varphi(t) \in \mathcal{S}_+\} \qquad \text{when } X(t) = 0,\ Y(t) = 0. \tag{6}$$

Thus, the fluid level $Y(t)$ increases when $X(t) > 0$, and decreases when $X(t) = 0$, unless both levels are at 0; in the latter case, $Y(t)$ (and $X(t)$) increases as soon as $\varphi(t)$ makes a transition from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$. We illustrate this in Figure 2.

Throughout we denote by $\mathbf{1}$, $\mathbf{0}$, $\mathbf{I}$, and $\mathbf{O}$ a column vector of ones, a row vector of zeros, an identity matrix, and a zero matrix of appropriate sizes, respectively. Also, for any matrix $\mathbf{A} = [A_{ij}]$, we use notation $|\mathbf{A}|$ for a matrix collecting absolute values of the elements of $\mathbf{A}$, with $|\mathbf{A}| = [\ |A_{ij}|\ ]$.

2.2. Stability condition

The stability condition for the first queue, $\{(\varphi(t), X(t)) : t \geq 0\}$, is well-known to be

$$\sum_{i \in \mathcal{S}} r_i P(\varphi = i) < 0, \tag{7}$$

where the random variable φ is distributed according to the stationary distribution of $\varphi(t)$. Assuming this condition is satisfied, the second queue (buffer Y) will be stable when the expected increase rate of $Y(t)$ is less than the expected decrease rate, i.e.,

$$\sum_{i \in \mathcal{S}} \widehat{c}_i P(\varphi = i, X > 0) < \sum_{i \in \mathcal{S}_\ominus} |\check{c}_i| P(\varphi = i, X = 0), \tag{8}$$

where the random vector (φ, X) is distributed according to the stationary distribution of $(\varphi(t), X(t))$.

2.3. Qualitative behavior

In this subsection, we give a short discussion of how the process $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$ behaves and what the stationary distribution looks like. Here, and in the sequel, we will sometimes write, e.g., "the process hits $x = 0$," which will be short for "the process $(\varphi(t), X(t), Y(t))$ hits the set $\mathcal{S} \times \{0\} \times [0, \infty)$," or we will speak of "the probability mass at $x = 0, y > 0$" meaning "the stationary probability that the process $(\varphi(t), X(t), Y(t))$ is in the set $\mathcal{S} \times \{0\} \times (0, \infty)$."

Typically, the process alternates, between

(i) periods on $x = 0$, with $Y(t)$ decreasing, possibly being halted at $x = 0, y = 0$, and $\varphi(t)$ in $\mathcal{S}_\ominus$; such a period starts at $x = 0, y > 0$, with $\varphi(t)$ in $\mathcal{S}_-$ and ends at $x = 0, y > 0$ or at $x = 0, y = 0$ as soon as $\varphi(t)$ makes a transition from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$;

(ii) periods on $x > 0$, with $Y(t)$ increasing, while $X(t)$ can either increase and decrease. Such a period starts where the previous type (i) period ended with $\varphi(t) \in \mathcal{S}_+$ and $X(t)$ increasing, and ends at $x = 0, y > 0$ with $\varphi(t)$ in $\mathcal{S}_-$ as soon as $X(t)$ decreases to 0.

Note that in stationarity, the process cannot be at $y = 0, x > 0$, since $Y(t) = 0$ implies $X(t) = 0$ (or alternatively, $X(t) > 0$ implies $Y(t) > 0$). In fact when a type (ii) period starts from $x = 0, y = 0$, due to a transition of $\varphi(t)$ to some phase $i \in \mathcal{S}_+$, the process will move with $\frac{d}{dt}X(t) = r_i > 0$ and $\frac{d}{dt}Y(t) = \widehat{c}_i > 0$, so it will stay on the line $\{(x, y) : y = x\widehat{c}_i/r_i\}$ until some future transition of $\varphi(t)$ to some other state i'. Note that the slope of any such path leaving the origin is at least $\min_{i\in\mathcal{S}_+}\{\widehat{c}_i/r_i\}$, and also after the path has been left, the slope of the ensuing path can never be less than this value (assuming $i' \in \mathcal{S}_+$, otherwise $X(t)$ will not increase). Thus, after the process has hit the origin for the first time (which it will, due to stability), the set $\{(x, y) : y < x \cdot \min_{i\in\mathcal{S}_+}\{\widehat{c}_i/r_i\}\}$ can never be reached.

As a consequence of the above, the stationary distribution will have the following form.

- Corresponding to (i), there will be a (one-dimensional) density at $x = 0, y > 0$, denoted by $\boldsymbol{\pi}(0, y)$, and a probability point mass at $(0, 0)$, denoted by $\mathbf{p}(0, 0)$.
- Corresponding to (ii), there will be a two-dimensional density on $\{(x, y) : x > 0, y > x \cdot \min_{i\in\mathcal{S}_+}\{\widehat{c}_i/r_i\}\}$, denoted as $\boldsymbol{\pi}(x, y)$, and there will be one-dimensional densities on each of the lines $y = x\widehat{c}_i/r_i$, $i \in \mathcal{S}_+$, denoted as $\pi^i(x, x\widehat{c}_i/r_i)$. Also, define $\boldsymbol{\pi}^j(x, x\widehat{c}_j/r_j) = [\delta_{ij}\pi^j(x, x\widehat{c}_j/r_j)]_{i\in\mathcal{S}}$ for all $j \in \mathcal{S}$. There will be no other probability masses or densities, in particular there is no density at $y = 0, x > 0$.

It is important to realize that the one- and two-dimensional densities just mentioned, as well as the point mass at (0,0), are all *vectors* with $|\mathcal{S}|$ components, where the ith component corresponds to $\varphi(t) = i$. Some of these components will be zero; in particular for $i \in \mathcal{S}_+$ we will have $[\mathbf{p}(0,0)]_i = 0$ and $[\boldsymbol{\pi}(0,y)]_i = 0$. Also $[\boldsymbol{\pi}^j(x, x\widehat{c_j}/r_j)]_i = 0$ for all $i \neq j$.

In the next section, we show how to proceed to find the stationary distribution.

3. Analysis

Roughly speaking, our analysis is based on the alternation between (i) stages during which $X(t) = 0$ and hence $Y(t)$ decreases, and (ii) stages during which $X(t) > 0$ and hence $Y(t)$ increases, as detailed in Section 2.3. For (parts of) both of these stages we will apply ideas from[8, 24], in order to keep track of the amount by which $Y(t)$ increases (or decreases), in much the same way as we can keep track of the amount of time that passes. We will review this in Section 3.1. In Section 3.2, we will look at the state $(\varphi(t), X(t))$ when the process hits the line $x = 0$, so that with these building blocks we can in Section 3.3 establish expressions for the stationary distribution.

3.1. Replacing time by shift

We are interested in certain behavior of buffer X, not during some amount of *time*, but while buffer Y experiences a certain (downward/upward, virtual) *shift*. For a motivation of the expressions below, we refer to[8], where the concept of shift was introduced, as well as to[24], where a generalization of this idea is discussed. We consider two cases, corresponding to the periods (i) and (ii) as introduced before.

(i) First, consider the behavior at $x = 0$, when the level in buffer Y is strictly decreasing, according to the rates in $\check{\mathbf{C}}$. Below we define matrices $\check{\mathbf{Q}}_{\ominus\ominus}$ and $\check{\mathbf{Q}}_{\ominus+}$, which are the key components of the analysis for this case.

Suppose $X(0) = 0$ and $\varphi(u) \in \mathcal{S}_\ominus$ for $0 \leq u \leq t$. Define the random variable $D(t)$,

$$D(t) = \int_{u=0}^{t} |\check{c}_{\varphi(u)}| du, \tag{9}$$

interpreted as the total *downward shift* $Y(0) - Y(t)$ in buffer Y at time t when $Y(t) > 0$. Also, for any $z > 0$, define the random variable

$$t_z = \inf\{t > 0 : D(t) = z\}, \tag{10}$$

which we interpret, for any $y \geq 0$, as the first time at which the level in buffer Y shifts from level $Y(0) = y + z$ to y.

Denote

$$\mathbf{T}_{\ominus\ominus} = \begin{bmatrix} \mathbf{T}_{--} & \mathbf{T}_{-\circ} \\ \mathbf{T}_{\circ-} & \mathbf{T}_{\circ\circ} \end{bmatrix} \tag{11}$$

and

$$\mathbf{T}_{\ominus+} = \begin{bmatrix} \mathbf{T}_{-+} \\ \mathbf{T}_{\circ+} \end{bmatrix}, \quad \mathbf{T}_{\pm\circ} = \begin{bmatrix} \mathbf{T}_{+\circ} \\ \mathbf{T}_{-\circ} \end{bmatrix}, \tag{12}$$

and let $\check{\mathbf{C}}_{\ominus} = \text{diag}(\check{c}_i)_{i \in \mathcal{S}_{\ominus}}$ be a diagonal matrix partitioned according to $\mathcal{S}_{\ominus} = \mathcal{S}_{-} \cup \mathcal{S}_{\circ}$.

We define the generator matrix

$$\check{\mathbf{Q}}_{\ominus\ominus} = (|\check{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus\ominus}, \tag{13}$$

which has the following physical interpretation. By the analysis in[11, Lemmas 1-2], for $i, j \in \mathcal{S}_{\ominus}$, and $z > 0$, we have

$$[e^{\check{\mathbf{Q}}_{\ominus\ominus} z}]_{ij} = P(\varphi(t_z) = j, \varphi(u) \in \mathcal{S}_{\ominus}, \quad 0 \le u \le t_z \mid \varphi(0) = i, X(0) = 0), \tag{14}$$

which, for any $y > 0$, we interpret as the probability that the process is in phase j at time t_z and the phase remains in the set $\mathcal{S}_{\ominus}$ at least until time t_z, given the process starts from phase i with empty buffer X and level $y + z$ in buffer Y.

Also, define

$$\check{\mathbf{Q}}_{\ominus+} = (|\check{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus+}, \tag{15}$$

which by[11, Lemma 2] is a matrix of transition rates, w.r.t. level, to phases in $\mathcal{S}_{+}$, corresponding to the moments at which the level in buffer Y starts to increase.

(ii) Second, consider the behavior at $x > 0$, when the level in buffer Y is strictly increasing according to the rates in $\widehat{\mathbf{C}}$. The key components of the analysis are matrices $\widehat{\mathbf{Q}}(s)$ and $\widehat{\mathbf{\Psi}}(s)$ to be defined below and interpreted afterwards.

Let

$$\theta = \inf\{t > 0 : X(t) = 0\}, \tag{16}$$

be the first time at which the level in buffer X reaches 0.

Suppose $X(0) > 0$, or $X(0) = 0$ and $\varphi(0) \in \mathcal{S}_{+}$; and $t \le \theta$. Define the random variable $U(t)$,

$$U(t) = \int_{u=0}^{t} \widehat{c}_{\varphi(u)} du, \tag{17}$$

interpreted as the total *upward shift* $Y(t) - Y(0)$ in buffer Y at time t. In particular, $U(\theta)$ is then the total upward shift in buffer Y during a "busy period" of buffer X.

Also, for any $z > 0$, define

$$\widehat{t_z} = \inf\{t > 0 : U(t) = z\}, \tag{18}$$

which we interpret, for any $y \ge 0$, as the first time at which the level in buffer Y shifts from level $Y(0) = y$ to $y + z$.

We define the key generator matrix $\widehat{\mathbf{Q}}(s)$,

$$\widehat{\mathbf{Q}}(s) = \begin{bmatrix} \widehat{\mathbf{Q}}(s)_{++} & \widehat{\mathbf{Q}}(s)_{+-} \\ \widehat{\mathbf{Q}}(s)_{-+} & \widehat{\mathbf{Q}}(s)_{--} \end{bmatrix}, \tag{19}$$

with

$$\widehat{\mathbf{Q}}(s)_{++} = (\mathbf{R}_{+})^{-1}(\mathbf{T}_{++} - s\widehat{\mathbf{C}}_{+} - \mathbf{T}_{+\circ}(\mathbf{T}_{\circ\circ} - s\widehat{\mathbf{C}}_{\circ})^{-1}\mathbf{T}_{\circ+}), \tag{20}$$

$$\widehat{\mathbf{Q}}(s)_{+-} = (\mathbf{R}_+)^{-1}(\mathbf{T}_{+-} - \mathbf{T}_{+\circ}(\mathbf{T}_{\circ\circ} - s\widehat{\mathbf{C}}_\circ)^{-1}\mathbf{T}_{\circ-}), \tag{21}$$
$$\widehat{\mathbf{Q}}(s)_{-+} = (|\mathbf{R}_-|)^{-1}(\mathbf{T}_{-+} - \mathbf{T}_{-\circ}(\mathbf{T}_{\circ\circ} - s\widehat{\mathbf{C}}_\circ)^{-1}\mathbf{T}_{\circ+}), \tag{22}$$
$$\widehat{\mathbf{Q}}(s)_{--} = (|\mathbf{R}_-|)^{-1}(\mathbf{T}_{--} - s\widehat{\mathbf{C}}_- - \mathbf{T}_{-\circ}(\mathbf{T}_{\circ\circ} - s\widehat{\mathbf{C}}_\circ)^{-1}\mathbf{T}_{\circ-}), \tag{23}$$

where $\mathbf{R}_+ = \text{diag}(r_i)_{i\in\mathcal{S}_+}$, $\mathbf{R}_- = \text{diag}(r_i)_{i\in\mathcal{S}_-}$, $\widehat{\mathbf{C}}_+ = \text{diag}(\widehat{c_i})_{i\in\mathcal{S}_+}$, $\widehat{\mathbf{C}}_- = \text{diag}(\widehat{c_i})_{i\in\mathcal{S}_-}$, and $\widehat{\mathbf{C}}_\circ = \text{diag}(\widehat{c_i})_{i\in\mathcal{S}_\circ}$.

The physical interpretation of $\widehat{\mathbf{Q}}(s)$ was established in [8, Theorem 2]. For completeness, we state this result in Theorem 3.1 below. Now, for any $s > 0$, we can find the minimum nonnegative solution $\widehat{\boldsymbol{\Psi}}(s)$ of the Riccati equation

$$\widehat{\mathbf{Q}}(s)_{+-} + \widehat{\mathbf{Q}}(s)_{++}\widehat{\boldsymbol{\Psi}}(s) + \widehat{\boldsymbol{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{--} + \widehat{\boldsymbol{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{-+}\widehat{\boldsymbol{\Psi}}(s) = \mathbf{O}, \tag{24}$$

which has the following interpretation, by the analysis in [8, Theorem 3]. For all $i \in \mathcal{S}_+$ and $j \in \mathcal{S}_-$,

$$[\widehat{\boldsymbol{\Psi}}(s)]_{ij} = E\left(e^{-sU(\theta)}1\{\varphi(\theta) = j\} \mid \varphi(0) = i, X(0) = 0\right), \tag{25}$$

i.e., $[\widehat{\boldsymbol{\Psi}}(s)]_{ij}$ is the Laplace–Stieltjes transform of the distribution of the upward shift in buffer Y at the moment the level in buffer X first returns to 0 and does so in phase j, given start from phase i and empty buffer X.

Note that by the upward homogeneity of $X(t)$, $[\widehat{\boldsymbol{\Psi}}(s)]_{ij}$ also is the Laplace–Stieltjes transform of the distribution of the upward shift in buffer Y at the moment the level in buffer X first returns to z and does so in phase j, given start from phase i and $X = z$, for any $z \geq 0$.

We can write

$$\widehat{\boldsymbol{\Psi}}(s) = \int_{z=0}^{\infty} e^{-sz}\widehat{\boldsymbol{\psi}}(z)dz, \tag{26}$$

where the entry $[\widehat{\boldsymbol{\psi}}(z)]_{ij}$, for $i \in \mathcal{S}_+$ and $j \in \mathcal{S}_-$, is the corresponding probability density, which can be derived by numerically inverting $[\widehat{\boldsymbol{\Psi}}(s)]_{ij}$ using the algorithm by Abate and Whitt[3], for any $z > 0$. That is, the matrix $\widehat{\boldsymbol{\psi}}(z)$ is an $|\mathcal{S}_+| \times |\mathcal{S}_-|$ matrix of densities, the (i, j)th component of which records the density of an upward shift of z in buffer Y, from some y to $y + z$, during a busy period of buffer X, ending in phase $j \in \mathcal{S}_-$, starting at phase $i \in \mathcal{S}_+$.

In the remainder of this section, we will give a slightly enhanced proof of Theorem 2 in[8]. This theorem gives the matrix recording the Laplace–Stieltjes transforms of the distribution of the shift in buffer Y, during the time that an amount x has flown into or out of buffer X, ending up in phase j given that it starts in i. In[8], this matrix is called[2] $\tilde{\Delta}^y(s)$, while in the current paper we write it as $\mathbf{U}^{(x)}(s)$. More importantly, we modify its definition somewhat, to reflect the fact that the value of the shift in buffer Y does not only depend on the initial phase i, the ending phase j, and the time duration, but on the *whole sample path* of $\varphi(t)$ in between. For the

[2] In[8], the superscript y is used rather than x, since the monotonously increasing (or decreasing) buffer in which the shift is measured is there called X, so the notations for X and Y are interchanged.

moment we will assume that, in our current context, $Y(t)$ can never decrease, so that the shift in buffer Y, expressed as $Y(t) - Y(0)$, is always nonnegative[3].

Let, as in[8], the random variable $f(t) = \int_0^t |r_{\varphi(u)}|du$ be the total amount of fluid that flowed into or out of buffer X during $(0, t)$, referred to as the *in-out fluid* of X, and let $\omega(x) = \inf\{t > 0 : f(t) = x\}$ be the first time (random) that this in-out fluid reaches level x. Moreover, let V_i^x formally be the set of all realizations of $\varphi(u)$ for $0 \leq u \leq \omega(x)$, starting from $\varphi(0) = i$, such that the total in–out fluid in buffer X is precisely x. Each path v is given by

- number of transitions of $\{\varphi(t)\}$ during v, denoted as $n(v)$;
- sequence of states visited by $\{\varphi(t)\}$, denoted as $s_0(v), s_1(v), \ldots, s_{n(v)}(v)$;
- sequence of sojourn times in these states, denoted as $t_0(v), t_1(v), \ldots, t_{n(v)}(v)$.

It will be convenient to denote the total time of a path v as $|v| = \sum_{\ell=0}^{n(v)} t_\ell(v)$. Note that for each $v \in V_i^x$ we have $s_0(v) = i$ and $\sum_{\ell=0}^{n(v)} |r_{s_\ell(v)}|t_\ell(v) = x$. Now for each path v let $U(v)$ be the total shift in the second buffer during $(0, |v|)$ (see [17]), given by

$$U(v) = \sum_{\ell=0}^{n(v)} \widehat{c}_{s_\ell(v)} t_\ell(v). \tag{27}$$

Note that for a random path $V \in V_i^x$, the shift $U(V)$ is a random variable that is completely determined by V. We formally define the matrix of Laplace–Stieltjes transforms of $U(V)$, calling it $\mathbf{U}^{(x)}(s)$, via its (i, j)th entry as follows:

$$\left[\mathbf{U}^{(x)}(s)\right]_{ij} = \int_{v \in V_i^x} e^{-sU(v)} 1\{\varphi(|v|) = j\} dP(V = v), \tag{28}$$

where the integral incorporates the (countable) number of all possible successive states that $\varphi(t)$ visits, as well as all the corresponding sojourn times during all of these visits.

In other words, (28) is a short-hand notation for the more exact expression:

$$\left[\mathbf{U}^{(x)}(s)\right]_{ij} = \sum_{n=0}^{\infty} \sum_{s_1,\ldots,s_{n-1} \in \mathcal{S}} \cdots \sum \int \cdots \int_{\{\sum_{\ell=0}^{n-1} |r_{s_\ell}| t_\ell < x\}} e^{-s \sum_{\ell=0}^{n} \widehat{c}_{s_\ell} t_\ell} \left(\prod_{\ell=0}^{n-1} e^{T_{s_\ell, s_\ell} t_\ell} T_{s_\ell, s_{\ell+1}} \right) \times e^{T_{s_n, s_n} t_k} dt_{n-1} \cdots dt_0, \tag{29}$$

where $s_0 = i$ and $s_n = j$ and $t_n = (x - \sum_{\ell=0}^{n-1} |r_{s_\ell}| t_\ell)/|r_{s_n}|$ so that $\sum_{\ell=0}^{n} |r_{s_\ell}| t_\ell = x$.

Using this definition, we can prove the following result.

Theorem 3.1 (Theorem 2 in Bean and O'Reilly[8]).

$$\mathbf{U}^{(x+h)}(s) = \mathbf{U}^{(x)}(s)\mathbf{U}^{(h)}(s),$$

from which it follows that

$$\mathbf{U}^{(x)}(s) = e^{\widehat{\mathbf{Q}}(s)x}.$$

[3]That is, we only consider the case $X(t) > 0$; the case $X(t) = 0$ is similar, except that we should replace the word "shift" by "virtual shift," as if buffer Y had no lower boundary at 0.

Proof. First note that any random path $V \in V_i^{x+h}$ can be seen as a concatenation of two paths, $V_1 \in V_i^x$, ending in some phase k (random), and $V_2 \in V_k^h$ representing the inflow/outflow increase in buffer X from x to $x+h$. Due to the Markov property in $\{\varphi(t)\}$, these paths are independent, conditional on V_2 starting in the same phase k as where V_1 finished. Since in that case clearly we also have $U(V) = U(V_1) + U(V_2)$, using notation as in (28), we arrive at

$$\begin{aligned} & e^{-sU(v)} 1\{\varphi(|v|) = j\} dP(V = v) \\ = \quad & \sum_k e^{-sU(v_1)} 1\{\varphi(|v_1|) = k\} dP(V_1 = v_1) e^{-sU(v_2)} 1\{\varphi(|v_2|) = j\} dP(V_2 = v_2), \end{aligned}$$

from which we find

$$\begin{aligned} & \int_{v \in V_i^{x+h}} e^{-sU(v)} 1\{\varphi(|v|) = j\} dP(V = v) \\ & = \sum_k \int_{v \in V_i^x} e^{-sU(v)} 1\{\varphi(|v|) = k\} dP(V = v) \\ & \quad \times \int_{v \in V_k^h} e^{-sU(v)} 1\{\varphi(|v|) = j\} dP(V = v), \end{aligned}$$

and hence the first statement follows. The second statement then simply follows from the first as in[8]. □

3.2. Embedded discrete-time Markov chain

Let θ_k be the kth time that $(\varphi(t), X(t), Y(t))$ hits the line $x = 0$, and let the discrete-time Markov chain $J_k = (\varphi(\theta_k), Y(\theta_k))$ with discrete/continuous state space $\mathcal{S}_- \times (0, \infty)$, record the position of $(\varphi(t), Y(t))$ at time θ_k. Also, let $\tau_k > \theta_k$ be the kth time the process *leaves* the boundary $x = 0$.

Lemma 3.1. The transition kernel of J_k is given by

$$\begin{aligned} \mathbf{P}_{z,y} = & \int_{u=[z-y]^+}^{z} [\mathbf{I} \quad \mathbf{O}] e^{\check{\mathbf{Q}}_{\ominus\ominus} u} \check{\mathbf{Q}}_{\ominus+} \widehat{\boldsymbol{\psi}}(y - z + u) du \\ & + [\mathbf{I} \quad \mathbf{O}] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (-\check{\mathbf{Q}}_{\ominus\ominus})^{-1} \check{\mathbf{Q}}_{\ominus+} \widehat{\boldsymbol{\psi}}(y). \end{aligned} \tag{30}$$

where $[x]^+$ denotes $\max(0, x)$, and $[\mathbf{IO}]$ is a $|\mathcal{S}_-| \times |\mathcal{S}_\ominus|$ matrix.

Proof. We apply the physical interpretations of the quantities analyzed in Section 3.1. Essentially, the process J_k satisfies a Lindley-type recursion, since for its second component $Y(\theta_k)$ we can write

$$Y(\theta_{k+1}) = [Y(\theta_k) - D_k]^+ + U_k, \tag{31}$$

where

$$D_k = \int_{u=\theta_k}^{\tau_k} |\check{c}_{\varphi(u)}| du, \quad U_k = \int_{u=\tau_k}^{\theta_{k+1}} \widehat{c}_{\varphi(u)} du \tag{32}$$

are appropriately chosen random variables. More precisely, starting from time θ_k, with $X(\theta_k) = 0$ and $\varphi(\theta_k) = i \in \mathcal{S}_-$, we recall the two consecutive stages described in Section 2.3.

First, (i) the process $Y(t)$ will make a negative shift of size $-D$, say, as long as $\varphi(t) \in \mathcal{S}_\ominus$ (while $X(t)$ remains at zero during this stage). Then, after a transition of $\varphi(t)$ from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$, the second stage (ii) commences, during which the process $Y(t)$ will make a positive shift of size U, say, during a busy period of the first queue (i.e., during a first return time of $X(t)$ back to level zero, starting at level zero).

There are two alternatives. The first alternative is that the chain J_k transitions from (i, z) to (j, y) without the level in buffer Y returning to 0 during time interval (θ_k, θ_{k+1}). Assume $y \geq z$. In this case,

- first, the phase remains in the set $\mathcal{S}_\ominus$ at least until the level in buffer Y shifts down by u units (from z to $z - u$), for some u with $0 \leq u \leq z$; this occurs according to the probability matrix $e^{\check{\mathbf{Q}}_{\ominus\ominus}u}$;
- then, the process makes a transition to some phase in $\mathcal{S}_+$, which starts the busy period in buffer X; this occurs according to the rate matrix $\check{\mathbf{Q}}_{\ominus+}$;
- finally, the busy process in buffer X ends and the level y is observed in buffer Y; this occurs according to the density matrix $\widehat{\psi}(y - z + u)$ since the shift in buffer Y during the busy period in X must be exactly $y - (z - u) = y - z + u$.

The transition kernel of the first alternative, when $y \geq z$, is therefore

$$I(y \geq z)[\mathbf{I} \quad \mathbf{O}] \int_{u=0}^{z} e^{\check{\mathbf{Q}}_{\ominus\ominus}u} \check{\mathbf{Q}}_{\ominus+} \widehat{\boldsymbol{\psi}}(y - z + u) du, \tag{33}$$

and by analogous argument, when $y < z$,

$$I(y < z)[\mathbf{I} \quad \mathbf{O}] \int_{u=z-y}^{z} e^{\check{\mathbf{Q}}_{\ominus\ominus}u} \check{\mathbf{Q}}_{\ominus+} \widehat{\boldsymbol{\psi}}(y - z + u) du. \tag{34}$$

The second alternative is that the chain J_k transitions from (i, z) to (j, y) with the level in buffer Y returning to 0 some time during time interval (θ_k, θ_{k+1}). In this case,

- first, the phase remains in the set $\mathcal{S}_\ominus$ at least until the level in buffer Y shifts down by z units (from z to 0); this occurs according to the probability matrix $\int_{u=z}^{\infty} e^{\check{\mathbf{Q}}_{\ominus\ominus}u} du = e^{\check{\mathbf{Q}}_{\ominus\ominus}z}(-\check{\mathbf{Q}}_{\ominus\ominus})^{-1}$;
- then, the process makes a transition to some phase in $\mathcal{S}_+$, which starts the busy period in buffer X; this occurs according to the rate matrix $\check{\mathbf{Q}}_{\ominus+}$;
- finally, the busy process of buffer X ends at level y; this occurs according to the density matrix $\widehat{\psi}(y)$.

The transition kernel of the second alternative is therefore

$$[\mathbf{I} \quad \mathbf{O}] e^{\check{\mathbf{Q}}_{\ominus\ominus}z}(-\check{\mathbf{Q}}_{\ominus\ominus})^{-1} \check{\mathbf{Q}}_{\ominus+} \widehat{\boldsymbol{\psi}}(y), \tag{35}$$

and so the result follows by summing (33)–(35). □

We could work with the above directly, but in Section 4 we prefer to determine the following Laplace–Stieltjes transforms, which can then be inverted using the algorithm in Abate and Whitt[3]. We note that $\mathbf{P}_{z,y}$ is continuous w.r.t. $y > 0$, and it is easy to check that $\int_{y=0}^{\infty} \mathbf{P}_{z,y} dy \mathbf{1} = \mathbf{1}$.

Corollary 3.1. *The Laplace–Stieltjes transform of* $\mathbf{P}_{z,y}$ *w.r.t. y is given by the matrix*

$$\mathbf{P}_{z,\cdot}(s) = \begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{-sz} \left(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I} \right)^{-1} \left(e^{\left(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I} \right) z} - \mathbf{I} \right) \check{\mathbf{Q}}_{\ominus +} \widehat{\boldsymbol{\Psi}}(s) \\ + \begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (-\check{\mathbf{Q}}_{\ominus\ominus})^{-1} \check{\mathbf{Q}}_{\ominus +} \widehat{\boldsymbol{\Psi}}(s). \tag{36}$$

Proof. By straightforward computation of $\int_{y=0}^{\infty} e^{-sy} \mathbf{P}_{z,y} dy$, or by using (31) directly as follows. Letting $Y_k = Y(\theta_k)$ and $\varphi_k = \varphi(\theta_k)$ for notational convenience, we have

$$\begin{aligned} & E[e^{-sY_{k+1}} 1\{\varphi_{k+1} = j\} \mid Y_k = z, \varphi_k = i] \\ & \quad = E[e^{-s(z - D_k + U_k)} 1\{\varphi_{k+1} = j\} 1\{D_k \le z\} \mid Y_k = z, \varphi_k = i] \\ & \qquad + E[e^{-sU_k} 1\{\varphi_{k+1} = j\} 1\{D_k > z\} \mid Y_k = z, \varphi_k = i]. \end{aligned} \tag{37}$$

By conditioning on the phases m and ℓ just before and after the time when the process leaves $x = 0$, we rewrite the first term as

$$\begin{aligned} & E\left[e^{-s(z - D_k + U_k)} 1\{\varphi_{k+1} = j\} 1\{D_k \le z\} \mid Y_k = z, \varphi_k = i\right] \\ & = \sum_{m \in \mathcal{S}_\ominus} \sum_{\ell \in \mathcal{S}_+} e^{-sz} \times E\left[e^{sD_k} 1\{\varphi(\tau_k -) = m\} \cdot 1\{D_k \le z\} \mid Y_k = z, \varphi_k = i\right] \\ & \quad \times E\left[1\{\varphi(\tau_k) = \ell\} \mid \varphi(\tau_k -) = m\right] \times E\left[e^{-sU_k} 1\{\varphi_{k+1} = j\} \mid \varphi(\tau_k) = \ell\right] \\ & = \sum_{m \in \mathcal{S}_\ominus} \sum_{\ell \in \mathcal{S}_+} e^{-sz} \int_{u=0}^{z} \left[\begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus} u}\right]_{im} e^{su} du \left[\check{\mathbf{Q}}_{\ominus +}\right]_{m\ell} \left[\widehat{\boldsymbol{\Psi}}(s)\right]_{\ell j} \\ & = \left[\begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} e^{-sz} (\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} (e^{(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}) z} - \mathbf{I}) \check{\mathbf{Q}}_{\ominus +} \widehat{\boldsymbol{\Psi}}(s)\right]_{ij}. \end{aligned} \tag{38}$$

A similar expression can be given for the second term, by which the statement follows. □

We denote the stationary distribution of J_k by a row vector $\boldsymbol{\xi}_z = [\xi_{i,z}]_{i \in \mathcal{S}_-}$ of densities, satisfying

$$\begin{cases} \int_{z=0}^{\infty} \boldsymbol{\xi}_z \mathbf{P}_{z,y} dz & = \boldsymbol{\xi}_y \\ \int_{y=0}^{\infty} \boldsymbol{\xi}_y dy \mathbf{1} & = 1, \end{cases} \tag{39}$$

and proceed in the next section to express the stationary distribution of the process $(\varphi(t), X(t), Y(t))$ at level $x = 0$ in terms of $\boldsymbol{\xi}_z$.

Remark 3.1. Instead of (31), we could also have worked with the true Lindley recursion

$$Y(\tau_{k+1}) = [Y(\tau_k) + U_k - D_{k+1}]^+. \tag{40}$$

This is the approach that was followed in[17]. There, the stationary distribution of the chain, embedded at these times, in fact gave immediately also the stationary distribution of the whole process at $x = 0$, due to a PASTA-like argument related to the workload in an $M/G/1$ queue. However, in the more general model at hand, with possibly multiple phases being visited while $X(t) = 0$, this need not be true; for example, there may be phases in $\mathcal{S}_\ominus$ from which it is impossible to jump to a state in $\mathcal{S}_+$. Moreover, one disadvantage would be that the stationary distribution of the embedded Markov chain besides having a density for $y > 0$ also has a mass at $y = 0$. Hence, we decided to embed at *hitting times* of $x = 0$, in a manner similar to the analysis in[9].

3.3. Stationary distribution

In the following subsections, we show how to find the various densities and probability masses that define the joint stationary distribution of the process.

3.3.1. Density at $\mathbf{x = 0, y > 0}$ and mass at $\mathbf{x = 0, y = 0}$

Recall from Section 2.3 that we need expressions for the vectors $\boldsymbol{\pi}(0, y)$ and $\mathbf{p}(0, 0)$, which we give in the following.

Lemma 3.2. We have $\boldsymbol{\pi}(0, y) = [\,\mathbf{0}\ \boldsymbol{\pi}(0, y)_\ominus\,]$, where

$$\boldsymbol{\pi}(0, y)_\ominus = \alpha \int_{z=y}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus}(z-y)} (|\check{\mathbf{C}}_\ominus|)^{-1} dz, \tag{41}$$

and $\mathbf{p}(0, 0) = [\,\mathbf{0}\ \mathbf{p}(0, 0)_\ominus\,]$, where

$$\mathbf{p}(0, 0)_\ominus = \alpha \int_{z=0}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1}. \tag{42}$$

Here, α is a normalization constant that satisfies

$$1 = \mathbf{p}(0, 0)\mathbf{1} + \int_{y=0}^{\infty} \boldsymbol{\pi}(0, y) dy \mathbf{1} + \sum_{j \in \mathcal{S}_+} \int_{x=0}^{\infty} \pi^j(x, x\widehat{c}_j/r_j) dx + \int_{x=0}^{\infty} \int_{y=0}^{\infty} \boldsymbol{\pi}(x, y) dy dx \mathbf{1},$$

given by

$$\alpha = \Big\{ \big[\,\boldsymbol{\xi}\ \mathbf{0}\,\big] (-\mathbf{T}_{\ominus\ominus})^{-1} \Big(\mathbf{1} + \mathbf{T}_{\ominus +} \mathbf{K}^{-1} [\,(\mathbf{R}_+)^{-1}\ \boldsymbol{\Psi}(|\mathbf{R}_-|)^{-1}\,] \times \big(\mathbf{1} + \mathbf{T}_{\pm\circ} (-\mathbf{T}_{\circ\circ})^{-1} \mathbf{1} \big) \Big) \Big\}^{-1}, \tag{43}$$

where, $\boldsymbol{\xi} = \int_{z=0}^{\infty} \boldsymbol{\xi}_z dz$, $\boldsymbol{\Psi} = \widehat{\boldsymbol{\Psi}}(s)|_{s=0}$ and $\mathbf{K} = \widehat{\mathbf{K}}(s)|_{s=0}$ with

$$\widehat{\mathbf{K}}(s) = \widehat{\mathbf{Q}}(s)_{++} + \widehat{\boldsymbol{\Psi}}(s) \widehat{\mathbf{Q}}(s)_{-+}. \tag{44}$$

Proof. In (i)–(iii), we prove (41)–(43), respectively.

(i) We apply argument analogous to [19, Theorem 4.1], adapted to the process here. With $D(t)$ as defined in (9), let D denotes the corresponding buffer collecting fluid $D(t)$. For $k,\ j \in \mathcal{S}_\ominus$, define the probability

$$\check{F}_{kj}(z,t) = P(D(t) \le z, \varphi(t) = j, \varphi(u) \in \mathcal{S}_\ominus, 0 \le u \le t \mid \varphi(0) = k, D(0) = 0), \tag{45}$$

and matrix $\check{\mathbf{F}}(z) = [\check{F}_{kj}(z)]_{k,j\in\mathcal{S}_\ominus}$ such that

$$\check{F}_{kj}(z) = \int_{t=0}^{\infty} \check{F}_{kj}(z,t)dt, \tag{46}$$

which we interpret as the expected time spent in phase j, $x = 0$ and $D \le z$, given start in phase k, $x = 0$, and $D = 0$.

By[5], or direct application of[4, equation (8) in Theorem 3.1.1] to Y observed when $X = 0$ and so the stochastic fluid process $(\varphi(t), Y(t))$ restricted to phases in $\mathcal{S}_\ominus$, we have

$$\frac{\partial}{\partial z}\check{\mathbf{F}}(z) = e^{\check{\mathbf{Q}}_{\ominus\ominus}z}(|\check{\mathbf{C}}_\ominus|)^{-1}. \tag{47}$$

Therefore, by conditioning on the last time the process hits $x = 0$, we have

$$\begin{aligned}
[\boldsymbol{\pi}(0,y)_\ominus]_j &= \sum_{i\in\mathcal{S}_-}\alpha\int_{z=y}^{\infty}\xi_{z,i}\left(\frac{\partial}{\partial u}\int_{t=0}^{\infty}\check{F}_{ij}(u,t)dt\right)\Bigg|_{u=z-y}dz \\
&= \alpha\sum_{i\in\mathcal{S}_-}\int_{z=y}^{\infty}\xi_{z,i}\left[\frac{\partial}{\partial u}\check{\mathbf{F}}(u)\right]_{ij}\Bigg|_{u=z-y}dz \\
&= \alpha\sum_{i\in\mathcal{S}_-}\int_{z=y}^{\infty}\xi_{z,i}[e^{\check{\mathbf{Q}}_{\ominus\ominus}(z-y)}(|\check{\mathbf{C}}_\ominus|)^{-1}]_{ij}dz,
\end{aligned} \tag{48}$$

and so equation (41) for $\boldsymbol{\pi}(0,y)$ follows.

We interpret (48) as follows. The process hits level 0 in buffer X and does so in some $Y = z$ and phase $i \in \mathcal{S}_-$, and then stays in $\mathcal{S}_\ominus$ for an amount of 'shift' (rather than time) of precisely $(z - y)$, in order to be at $Y = y$ in some phase j in stationarity.

(ii) Similar arguments show the expression for (42); for ending up in $(j, 0, 0)$ from $(i, 0, z)$ with $i \in \mathcal{S}_-$ and $z \ge 0$, the process $\varphi(t)$ now needs to stay in $\mathcal{S}_\ominus$ for an amount of "shift" (rather than time) of $z + w$ for some $w \ge 0$, and end up in phase $j \in \mathcal{S}_\ominus$. We have

$$\begin{aligned}
[\mathbf{p}(0,0)]_j &= \sum_{i\in\mathcal{S}_-}\alpha\int_{z=y}^{\infty}\xi_{z,i}\int_{w=0}^{\infty}\left(\frac{\partial}{\partial u}\int_{t=0}^{\infty}\check{F}_{ij}(u,t)dt\right)\Bigg|_{u=z+w}dw dz \\
&= \alpha\sum_{i\in\mathcal{S}_-}\int_{z=y}^{\infty}\xi_{z,i}\int_{w=0}^{\infty}[e^{\check{\mathbf{Q}}_{\ominus\ominus}(z+w)}(|\check{\mathbf{C}}_\ominus|)^{-1}]_{ij}dw dz \\
&= \alpha\sum_{i\in\mathcal{S}_-}\int_{z=y}^{\infty}\xi_{z,i}[e^{\check{\mathbf{Q}}_{\ominus\ominus}z}(-\mathbf{T}_{\ominus\ominus})^{-1}]_{ij}dz.
\end{aligned} \tag{49}$$

(iii) To find α, since this is a constant that does not depend on buffer Y, we only need to consider the process $(\varphi(t), X(t))$, together with the distribution of $\{\varphi(t)\}$ upon hitting $x = 0$, which is $\boldsymbol{\xi} = \int_{z=0}^{\infty} \boldsymbol{\xi}_z dz$. The vector $\boldsymbol{\xi}$ is the stationary distribution of the corresponding discrete-time Markov chain with state space $\mathcal{S}_-$ that records the position of $\varphi(t)$ at time θ_k. The vector $\boldsymbol{\xi}$ is the unique solution of the set of equations:

$$\begin{aligned} [\,\boldsymbol{\xi} \quad \mathbf{0}\,](-\mathbf{T}_{\ominus\ominus})^{-1}\mathbf{T}_{\ominus+}\boldsymbol{\Psi} &= \boldsymbol{\xi}, \\ \boldsymbol{\xi}\mathbf{1} &= 1. \end{aligned} \tag{50}$$

The stationary distribution for the standard fluid model has been derived in the literature in[9, 10, 14, 16, 19, 23] in slightly different contexts. For completeness, we summarize here the results required for the derivation of the stationary distribution of $(\varphi(t), X(t))$, including the probability mass vector at level zero, which we will denote as $\mathbf{p} = [\,\mathbf{0}\ \mathbf{p}_-\ \mathbf{p}_\circ\,]$, and the probability density vector for $x > 0$, which we will denote as $\boldsymbol{\pi}(x) = [\,\boldsymbol{\pi}(x)_+\ \boldsymbol{\pi}(x)_-\ \boldsymbol{\pi}(x)_\circ\,]$, for all $x > 0$. By conditioning on the last time the process $(\varphi(t), X(t))$ hits level zero from above, in a manner similar to [9, Theorem 2],

$$[\,\mathbf{p}_- \quad \mathbf{p}_\circ\,] = \alpha[\,\boldsymbol{\xi} \quad \mathbf{0}\,](-\mathbf{T}_{\ominus\ominus})^{-1}, \tag{51}$$

and

$$\begin{aligned} [\,\boldsymbol{\pi}(x)_+ \quad \boldsymbol{\pi}(x)_-\,] &= [\,\mathbf{p}_- \quad \mathbf{p}_\circ\,]\mathbf{T}_{\ominus+}e^{\mathbf{K}x}[\,(\mathbf{R}_+)^{-1} \quad \boldsymbol{\Psi}(|\mathbf{R}_-|)^{-1}\,], \\ \boldsymbol{\pi}(x)_\circ &= [\,\boldsymbol{\pi}(x)_+ \quad \boldsymbol{\pi}(x)_-\,]\mathbf{T}_{\pm\circ}(-\mathbf{T}_{\circ\circ})^{-1}. \end{aligned} \tag{52}$$

Alternatively, (51) can be found by integrating (41) w.r.t. y and adding to (42). Similarly, (52) can be found by integrating $\boldsymbol{\pi}(x, y)$ w.r.t. y and adding $\sum_{j\in\mathcal{S}_+} \boldsymbol{\pi}^j(x, x\widehat{c}_j/r_j)$; the expressions for these quantities will be derived in following sections.

Since α is a normalizing constant that solves

$$\mathbf{p}\mathbf{1} + \int_{x=0}^{\infty} \boldsymbol{\pi}(x)dx\mathbf{1} = 1, \tag{53}$$

we have

$$\begin{aligned} \alpha^{-1} = [\,\boldsymbol{\xi} \quad \mathbf{0}\,](-\mathbf{T}_{\ominus\ominus})^{-1}(\mathbf{1} + \mathbf{T}_{\ominus+}\mathbf{K}^{-1}[\,(\mathbf{R}_+)^{-1} \quad \boldsymbol{\Psi}(|\mathbf{R}_-|)^{-1}\,] \\ \times (\mathbf{1} + \mathbf{T}_{\pm\circ}(-\mathbf{T}_{\circ\circ})^{-1}\mathbf{1})), \end{aligned} \tag{54}$$

and so the expression (43) for α follows. □

Note that α can also be interpreted as the total (stationary) rate of leaving $x = 0$, since by (51),

$$\begin{aligned} [\,\mathbf{p}_- \quad \mathbf{p}_\circ\,]\mathbf{T}_{\ominus+}\mathbf{1} &= -[\,\mathbf{p}_- \quad \mathbf{p}_\circ\,]\mathbf{T}_{\ominus\ominus}\mathbf{1} \\ &= \alpha\left[\,\boldsymbol{\xi} \quad \mathbf{0}\,\right]\mathbf{1} \\ &= \alpha, \end{aligned} \tag{55}$$

and also as the total (stationary) rate of hitting $x = 0$, since by (52) and $\mathbf{\Psi 1} = \mathbf{1}$,

$$\lim_{x \to 0^+} \boldsymbol{\pi}(x)_- |\mathbf{R}_-| \mathbf{1} = [\,\mathbf{p}_- \quad \mathbf{p}_\circ\,] \mathbf{T}_{\ominus +} \mathbf{1} \\ = \alpha, \tag{56}$$

with the two forms equivalent, as expected in stationarity.

For the Laplace–Stieltjes transform vector of the density part, denoted as

$$\boldsymbol{\pi}(0,\cdot)(s) = \int_{z=0}^{\infty} e^{-sy} \boldsymbol{\pi}(0,y) dy, \tag{57}$$

we have the following.

Corollary 3.2. *We have* $\boldsymbol{\pi}(0,\cdot)(s) = [\,\mathbf{0}\ \boldsymbol{\pi}(0,\cdot)(s)_\ominus\,]$, *where*

$$\boldsymbol{\pi}(0,\cdot)(s)_\ominus = \alpha \int_{z=0}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \\ \times (\mathbf{I} - e^{-(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}) z}) (|\check{\mathbf{C}}_\ominus|)^{-1} dz. \tag{58}$$

Proof. Since

$$\begin{aligned} \boldsymbol{\pi}(0,\cdot)(s)_\ominus &= \int_{y=0}^{\infty} e^{-sy} \alpha \int_{z=y}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus}(z-y)} (|\check{\mathbf{C}}_\ominus|)^{-1} dz dy \\ &= \alpha \int_{z=0}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} \int_{y=0}^{z} e^{-(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}) y} (|\check{\mathbf{C}}_\ominus|)^{-1} dy dz \\ &= \alpha \int_{z=0}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} \\ &\quad \times \left(- e^{-(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}) y} (\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \Big|_{y=0}^{z} \right) (|\check{\mathbf{C}}_\ominus|)^{-1} dz, \end{aligned}$$

the result follows. □

3.3.2. Density at $x > 0, y > 0$

We now proceed to the density vector $\boldsymbol{\pi}(x,y)$ as a function of y for fixed value of x. Define the Laplace–Stieltjes transform $\boldsymbol{\pi}(x,\cdot)(s)$ such that, for $i \in \mathcal{S}_\ominus$,

$$[\boldsymbol{\pi}(x,\cdot)(s)]_i = \int_{y=0}^{\infty} e^{-sy} [\boldsymbol{\pi}(x,y)]_i dy, \tag{59}$$

and for $i \in \mathcal{S}_+$,

$$[\boldsymbol{\pi}(x,\cdot)(s)]_i = \int_{y=0}^{\infty} e^{-sy} [\boldsymbol{\pi}(x,y)]_i dy + e^{-sx\widehat{c_i}/r_i} \pi^i(x, x\widehat{c_i}/r_i). \tag{60}$$

Lemma 3.3. We have

$$\boldsymbol{\pi}(x,\cdot)(s) = [\,\boldsymbol{\pi}(x,\cdot)(s)_+ \ \boldsymbol{\pi}(x,\cdot)(s)_- \ \boldsymbol{\pi}(x,\cdot)(s)_\circ\,]$$

with

$$[\,\boldsymbol{\pi}(x,\cdot)(s)_{+}\ \boldsymbol{\pi}(x,\cdot)(s)_{-}\,] = (\boldsymbol{\pi}(0,\cdot)(s)_{\ominus} + \mathbf{p}(0,0)_{\ominus}) \times \mathbf{T}_{\ominus+}e^{\widehat{\mathbf{K}}(s)x} \times [\,(\mathbf{R}_{+})^{-1}\ \widehat{\boldsymbol{\Psi}}(s)(|\mathbf{R}_{-}|)^{-1}\,], \tag{61}$$

and

$$\boldsymbol{\pi}(x,\cdot)(s)_{\circ} = [\,\boldsymbol{\pi}(x,\cdot)(s)_{+}\ \boldsymbol{\pi}(x,\cdot)(s)_{-}\,]\mathbf{T}_{\pm\circ}(s\widehat{\mathbf{C}}_{\circ} - \mathbf{T}_{\circ\circ})^{-1}. \tag{62}$$

Proof. The result (61) follows immediately by a partitioning of the sample paths argument, analogous to the one used in the derivation of (52). We also apply argument analogous to[19, Theorem 4.1], adapted to the process here. For $k \in \mathcal{S}_{+}$, $j \in \mathcal{S}_{+} \cup \mathcal{S}_{-}$, and $\widehat{t_z}$ as defined in (18), define the probability

$$\widehat{F_{kj}}(u,z) = P(X(\widehat{t_z}) \le u, \varphi(\widehat{t_z}) = j, \widehat{t_z} < \theta \mid \varphi(0) = k, X(0) = 0), \tag{63}$$

and matrix $\widehat{\mathbf{F}}(s;u) = [\widehat{F_{kj}}(s;u)]_{k\in\mathcal{S}_{+},\, j\in\mathcal{S}_{+}\cup\mathcal{S}_{-}}$ such that

$$\widehat{F_{kj}}(s;u) = \int_{z=0}^{\infty} e^{-sz}\widehat{F_{kj}}(u,z)dz, \tag{64}$$

which we interpret as the Laplace–Stieltjes transform of the total upward shift in buffer Y accumulated during total time spent in phase j and $x \le u$ with a taboo on hitting $x = 0$, given start in phase k and $x = 0$.

By[5], or direct application of[4, equations (8)–(10) in Theorem 3.1.1] to the stochastic fluid process $(\varphi(t), X(t))$, and consideration of the Laplace–Stieltjes transforms w.r.t. upward shift in Y when $X > 0$ using generator $\widehat{\mathbf{Q}}(s)$ of Section 3.1, we have

$$\frac{\partial}{\partial x}\widehat{\mathbf{F}}(s;x) = e^{\widehat{\mathbf{K}}(s)x}[\,(\mathbf{R}_{+})^{-1}\ \widehat{\boldsymbol{\Psi}}(s)(|\mathbf{R}_{-}|)^{-1}\,]. \tag{65}$$

Therefore, for $i \in \mathcal{S}_{+} \cup \mathcal{S}_{-}$,

$$\begin{aligned}
[\boldsymbol{\pi}(x,\cdot)(s)]_i &= \int_{y=0}^{\infty} e^{-sy}[\boldsymbol{\pi}(x,y)]_i dy + e^{-sx\widehat{c_i}/r_i}\pi^i(x, x\widehat{c_i}/r_i)\\
&= \sum_{j\in\mathcal{S}_{\ominus}} \left[\int_{z=0}^{\infty} e^{-sz}\left(\boldsymbol{\pi}(0,z)_{\ominus} + \mathbf{p}(0,0)_{\ominus}\right)dz\right]_j \sum_{k\in\mathcal{S}_{+}} [\mathbf{T}_{\ominus+}]_{jk}\\
&\quad \times \frac{\partial}{\partial x}\int_{z=0}^{\infty} e^{-sz}\widehat{F_{ki}}(x,z)dz\\
&= \sum_{j\in\mathcal{S}_{\ominus}}\sum_{k\in\mathcal{S}_{+}} \left[\boldsymbol{\pi}(0,\cdot)(s)_{\ominus} + \mathbf{p}(0,0)_{\ominus}\right]_j [\mathbf{T}_{\ominus+}]_{jk}\\
&\quad \times \sum_{\ell\in\mathcal{S}_{+}} \left[e^{\widehat{\mathbf{K}}(s)x}\right]_{k\ell} [\,(\mathbf{R}_{+})^{-1}\ \widehat{\boldsymbol{\Psi}}(s)(|\mathbf{R}_{-}|)^{-1}\,]_{\ell i}.
\end{aligned} \tag{66}$$

The result (62) follows by a similar argument, with for $i \in \mathcal{S}_\circ$,

$$
\begin{aligned}
[\boldsymbol{\pi}(x,\cdot)(s)]_i &= \int_{y=0}^{\infty} e^{-sy}[\boldsymbol{\pi}(x,y)]_i dy \\
&= \sum_{j\in\mathcal{S}_+\cup\mathcal{S}_-} \left[\int_{z=0}^{\infty} e^{-sz}\boldsymbol{\pi}(x,z)_+ dz \ \int_{z=0}^{\infty} e^{-sz}\boldsymbol{\pi}(x,z)_- dz\right]_j \sum_{k\in\mathcal{S}_\circ}[\mathbf{T}_{\pm\circ}]_{jk} \\
&\quad \times \int_{t=0}^{\infty}\int_{z=0}^{\infty} e^{-sz} dP(Y(t)\le z, \varphi(t)=i, \varphi(u)\in\mathcal{S}_\circ, 0\le u\le t \\
&\quad \times \mid \varphi(0)=k, Y(0)=0))dt \\
&= \sum_{j\in\mathcal{S}_+\cup\mathcal{S}_-} \left[\boldsymbol{\pi}(x,\cdot)(s)_+ \ \boldsymbol{\pi}(x,\cdot)(s)_-\right]_j \sum_{k\in\mathcal{S}_\circ}[\mathbf{T}_{\pm\circ}]_{jk} \\
&\quad \times \left[\int_{t=0}^{\infty} e^{(\mathbf{T}_{\circ\circ}-s\widehat{\mathbf{C}}_\circ)t} dt\right]_{ki},
\end{aligned}
\tag{67}
$$

since by a direct application of[8, Theorem 1] to the stochastic fluid process $(\varphi(t), Y(t))$ observed during times when $X > 0$, for all $k, i \in \mathcal{S}_\circ$ we have,

$$
\begin{aligned}
\left[e^{(\mathbf{T}_{\circ\circ}-s\widehat{\mathbf{C}}_\circ)z}\right]_{ki} &= \int_{z=0}^{\infty} e^{-sz} dP\Big(Y(t)\le z, \varphi(t) \\
&= i, \varphi(u)\in\mathcal{S}_\circ, 0\le u\le t \mid \varphi(0)=k, Y(0)=0\Big).
\end{aligned}
\tag{68}
$$

□

Corollary 3.3. *Letting* $\boldsymbol{\pi}(\cdot,\cdot)(v,s) = \int_{x=0}^{\infty} e^{-vx}\boldsymbol{\pi}(x,\cdot)(s)dx$, *we have*

$$
\boldsymbol{\pi}(\cdot,\cdot)(v,s) = [\,\boldsymbol{\pi}(\cdot,\cdot)(v,s)_+ \ \boldsymbol{\pi}(\cdot,\cdot)(v,s)_- \ \boldsymbol{\pi}(\cdot,\cdot)(s)_\circ\,]
$$

with

$$
\begin{aligned}
[\,\boldsymbol{\pi}(\cdot,\cdot)(v,s)_+ \ \boldsymbol{\pi}(\cdot,\cdot)(v,s)_-\,] &= (\boldsymbol{\pi}(0,\cdot)(s)_\ominus + \mathbf{p}(0,0)_\ominus) \\
&\quad \times \mathbf{T}_{\ominus+}(-\widehat{\mathbf{K}}(s)+v\mathbf{I})^{-1}[\,(\mathbf{R}_+)^{-1}\ \widehat{\boldsymbol{\Psi}}(s)(|\mathbf{R}_-|)^{-1}\,],
\end{aligned}
\tag{69}
$$

and

$$
\boldsymbol{\pi}(\cdot,\cdot)(s)_\circ = [\,\boldsymbol{\pi}(\cdot,\cdot)(s)_+ \ \boldsymbol{\pi}(\cdot,\cdot)(s)_-\,]\mathbf{T}_{\pm\circ}(s\widehat{\mathbf{C}}_\circ - \mathbf{T}_{\circ\circ})^{-1}. \tag{70}
$$

3.3.3. Density at $y = x\widehat{c}_i/r_i$

Finally, we state the result for the one-dimensional densities on each of the lines $y = x\widehat{c}_i/r_i$, $i \in \mathcal{S}_+$.

Lemma 3.4. For all $i \in \mathcal{S}_+$,

$$
\pi^i(x, x\widehat{c}_i/r_i) = \sum_{j\in\mathcal{S}_\ominus}[\mathbf{p}(0,0)]_j T_{ji}\exp(xT_{ii}/r_i)/r_i. \tag{71}
$$

Proof. This result essentially follows by arguments analogous to the proof of the first equation in (52), in a slightly different environment.

By conditioning on the most recent time the process leaves the point $(0, 0)$, in order to observe the process in stationarity at the point $(x, x\widehat{c}_i/r_i)$, the following must occur.

- First, the process starts from state $(j, 0, 0)$ for some $j \in \mathcal{S}_\ominus$, with probability $\mathbf{p}_j(0, 0)$, and instantaneously transitions to phase i with probability $-T_{ji}/T_{ii}$.
- Next, the process remains in phase i at least for the duration of time x/r_i, with probability $\exp(xT_{ii}/r_i)$.

Therefore,

$$\pi^i(x, x\widehat{c}_i/r_i) = \sum_{j\in\mathcal{S}_\ominus}[\mathbf{p}(0,0)]_j(-T_{ji}/T_{ii})\frac{d}{dx}P(\varphi(u) = i, 0 \le u \le x/r_i \mid \varphi(0) = i)$$

$$= \sum_{j\in\mathcal{S}_\ominus}[\mathbf{p}(0,0)]_j(-T_{ji}/T_{ii})\frac{d}{dx}(1 - e^{xT_{ii}/r_i}), \tag{72}$$

and the result (71) follows. □

3.4. Main result

We now summarize the results for the stationary distribution of the process $\{(\varphi(t), X(t), Y(t)) : t \ge 0\}$.

Theorem 3.2. *The probability mass components of the stationary distribution, corresponding to $x = 0$, are*

$$\boldsymbol{\pi}(0, y) \quad \text{and} \quad \mathbf{p}(0, 0),$$

given in Lemma 3.2. The Laplace–Stieltjes transforms of $\boldsymbol{\pi}(0, y)$ w.r.t. y are given in Corollary 3.2.

The one-dimensional density components of the stationary distribution, corresponding to $y = x\widehat{c}_j/r_j$, collected in the vector

$$\boldsymbol{\pi}^j(x, x\widehat{c}_j/r_j) = [\delta_{ij}\pi^j(x, x\widehat{c}_j/r_j)]_{i\in\mathcal{S}}, \quad j \in \mathcal{S}_+,$$

are given in Lemma 3.4.

The Laplace–Stieltjes transforms of the two-dimensional density components of the stationary distribution, $\boldsymbol{\pi}(x, y)$, corresponding to $x > 0$, w.r.t. y, are

$$[\boldsymbol{\pi}(x, \cdot)(s)]_i, \quad i \in \mathcal{S}_\ominus$$

and

$$[\boldsymbol{\pi}(x, \cdot)(s)]_i - e^{-sx\widehat{c}_i/r_i}\pi^i(x, x\widehat{c}_i/r_i), \quad i \in \mathcal{S}_+,$$

given in Lemma 3.3. The corresponding Laplace–Stieltjes transforms w.r.t. x and y are given in Corollary 3.3.

4. Numerical treatment

In order to evaluate the stationary distribution of the model using the theoretical results of Section 3, we apply discretization and truncation with appropriate parameters Δu, and L, $\ell = 1, 2, \ldots L$. The key points of the methodology are summarized below.

Step 1. Construct a discretized version of the process J_k discussed in Section 3.2, with a truncated level variable as follows.

(1a) Fix some small $\Delta u > 0$ and some large integer $L > 0$, and consider a discrete-time Markov chain $\{\bar{J}_k : k = 0, 1, 2, \ldots\}$ with state space $\{(i, \ell) : i \in \mathcal{S}_-, \ell = 1, 2, \ldots L\}$, with the interpretation that when $J_k = (j, z)$ for some z with $(\ell - 1)\Delta u \le z < \ell \Delta u$, $\ell = 1, 2, \ldots L - 1$, then we have $\bar{J}_k = (j, \ell)$, and when $J_k = (j, z)$ with $z \ge (L - 1)\Delta u$, we let $\bar{J}_k = (j, L)$.

(1b) Approximate the corresponding transition probabilities

$$\bar{P}_{i,\ell;j,m} = P(\bar{J}_{k+1} = (j, m) \mid \bar{J}_k = (i, \ell)), \tag{73}$$

to be collected in a matrix $\bar{\mathbf{P}} = [\bar{\mathbf{P}}_{\ell m}]_{\ell,m=0,1,2,\ldots,L}$ made up of block matrices $\bar{\mathbf{P}}_{\ell m} = [\bar{P}_{i,\ell;j,m}]_{i,j\in\mathcal{S}_-}$, as follows. First use, for $\ell, m = 1, 2, \ldots L$,

$$\begin{aligned}\bar{\mathbf{P}}_{\ell m} &= \int_{y=(m-1)\Delta u}^{m\Delta u} \mathbf{P}_{\ell\Delta u, y} dy \\ &\approx \Delta u \mathbf{P}_{\ell \Delta u, m\Delta u}.\end{aligned} \tag{74}$$

Then, normalize $\bar{\mathbf{P}}$ using

$$\bar{P}_{i,\ell;j,m} = \frac{\bar{P}_{i,\ell;j,m}}{\sum_{j'\in\mathcal{S}_-}\sum_{m'=1}^{L} \bar{P}_{i,\ell;j',m'}} \tag{75}$$

so that $\bar{\mathbf{P}}\mathbf{1} = \mathbf{1}$.

(1c) Next, with the notation $\lim_{k\to\infty} P(\bar{J}_k = (j, \ell)) = \bar{\xi}_{j;\ell}$, where the limits exist due to the stability condition of Section 2.2, denote by $\bar{\boldsymbol{\xi}} = [\bar{\boldsymbol{\xi}}_\ell]_{\ell=1,2,\ldots,L}$, $\bar{\boldsymbol{\xi}}_\ell = [\bar{\xi}_{j;\ell}]_{j\in\mathcal{S}_-}$, the stationary distribution vector of the process $\{\bar{J}_k : k = 0, 1, 2, \ldots\}$. Derive $\bar{\boldsymbol{\xi}}$ by solving the set of equations, using standard methods,

$$\bar{\boldsymbol{\xi}}\bar{\mathbf{P}} = \bar{\boldsymbol{\xi}}, \quad \bar{\boldsymbol{\xi}}\mathbf{1} = \mathbf{1}. \tag{76}$$

Step 2. Approximate the stationary distribution of the process $\{(\varphi(t), X(t), Y(t)) : t \ge 0\}$ as follows.

(2a) For any z with $(\ell - 1)\Delta u \le z < \ell\Delta u$, $\ell = 1, 2, \ldots L$, approximate

$$\boldsymbol{\xi}_z \approx \frac{\bar{\boldsymbol{\xi}}_\ell}{\Delta u}. \tag{77}$$

(2b) Using (42), apply

$$\begin{aligned}
\mathbf{p}(0,0)_{\ominus} &= \alpha \int_{z=0}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1} \\
&= \alpha \sum_{\ell=1}^{\infty} \int_{z=(\ell-1)\Delta u}^{\ell \Delta u} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1} \\
&\approx \alpha \sum_{\ell=1}^{L} \int_{z=(\ell-1)\Delta u}^{\ell \Delta u} [\,\frac{\bar{\boldsymbol{\xi}}_\ell}{\Delta u} \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1} \\
&\approx \alpha \sum_{\ell=1}^{L} [\,\bar{\boldsymbol{\xi}}_\ell \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} \ell \Delta u} (-\mathbf{T}_{\ominus\ominus})^{-1}. \qquad (78)
\end{aligned}$$

(2c) Apply analogous ideas to approximate $\boldsymbol{\pi}(0,y)$, $y>0$, and $\boldsymbol{\pi}(x,y)$, $x>0$, $y>0$, using (58), (69), (70), and the inversion method of Abate and Whitt in[2, 3], with

$$\begin{aligned}
\boldsymbol{\pi}(0,\cdot)(s)_{\ominus} &= \alpha \int_{z=0}^{\infty} [\,\boldsymbol{\xi}_z \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \\
&\quad \times (\mathbf{I} - e^{-(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}) z}) (|\check{\mathbf{C}}_{\ominus}|)^{-1} dz \\
&\approx \alpha \sum_{\ell=1}^{L} [\,\bar{\boldsymbol{\xi}}_\ell \quad \mathbf{0}\,] e^{\check{\mathbf{Q}}_{\ominus\ominus} \ell \Delta u} (\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \\
&\quad \times (\mathbf{I} - e^{-(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}) \ell \Delta u}) (|\check{\mathbf{C}}_{\ominus}|)^{-1}. \qquad (79)
\end{aligned}$$

Example 4.1. We consider a process with the following parameters: $\mathcal{S} = \{1, 2\}$, $r_1 = 2$, $r_2 = -6$, $\widehat{c}_1 = \widehat{c}_2 = 2$, $\check{c}_1 = \check{c}_2 = -3$, and

$$\mathbf{T} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}.$$

This simple process is similar to the model studied in[17] and [25, Chapter 4] (but different from the numerical example analyzed there). The numerical evaluations confirm that the stability conditions of Section 2.2 are met as required, with

$$\sum_{i \in \mathcal{S}} r_i P(\varphi = i) = -3.3333 < 0,$$

$$\sum_{i \in \mathcal{S}} \widehat{c}_i P(\varphi = i, X > 0) - \sum_{i \in \mathcal{S}_{\ominus}} |\check{c}_i| P(\varphi = i, X = 0) = -0.7778 < 0.$$

We apply the approximation with parameters $L = 1000$ and $\Delta u = 0.05$, and so truncate the values of y to the interval $[0, 50]$. The choice of the truncation level 50 in the buffer Y is motivated by the simulation of the process, the output of which is recorded in Figure 3.

The values $\boldsymbol{\xi}_z$ estimated with (77) are plotted in Figure 4. Using (78), we estimate $\mathbf{p}(0,0)_{-} = 0.255987$.

By the explicit expressions (50) and (51), we obtain $\mathbf{p}_{-} = 0.555556$, which implies that the stationary probability mass at $x = 0$ and $y > 0$ is $\int_{y=0}^{\infty} \boldsymbol{\pi}(0,y) dy = \mathbf{p}_{-} -$

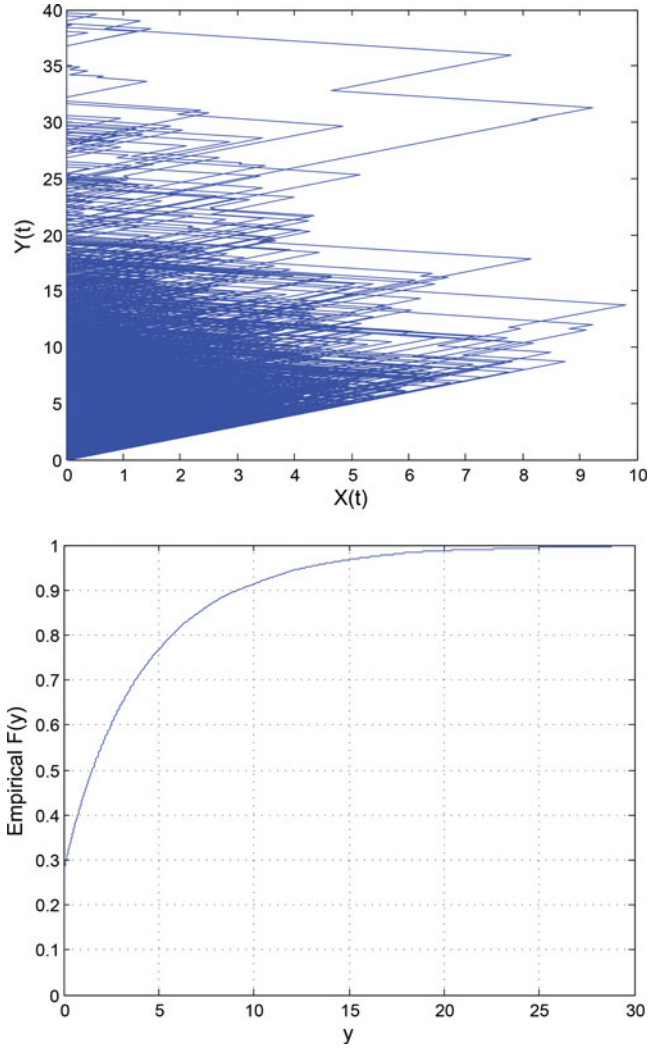

Figure 3. Simulated values $(X(t), Y(t))$, $0 \leq t \leq 10^5$, and the corresponding empirical values of $F(y) = P(Y \leq y)$, in Example 4.1.

$\mathbf{p}(0,0)_- = 0.555556 - 0.255987 = 0.299569$. We use (79) and the Euler inversion method of Abate and Whitt in[3] to estimate the values $\boldsymbol{\pi}(0, y)$ and plot them in Figure 5.

It follows that $\int_{y=0}^{\infty}\int_{x=0}^{\infty}\boldsymbol{\pi}(x,y)dxdy = 1 - \mathbf{p}_- = 0.444444$, giving the stationary probability of both buffers being non-empty. Selected values $\boldsymbol{\pi}(x, y)$ are estimated using (69), (70), (79), and the Euler–Euler inversion method of Abate and Whitt in[2] (with some care due to the mass at $\pi^i(x, x\widehat{c_i}/r_i)$ for $i \in \mathcal{S}_+$), and plotted in Figure 6.

In comparison to[25], by the explicit expressions in [25, equation (4.34) in Section 4.5] applied to our example, we have $\mathbf{p}(0,0)_- = 0.259259$. That is, our approximation of $\mathbf{p}(0,0)_-$ is underestimated by $0.259259 - 0.255987 = 0.003272$, or 1.262058% of the exact value.

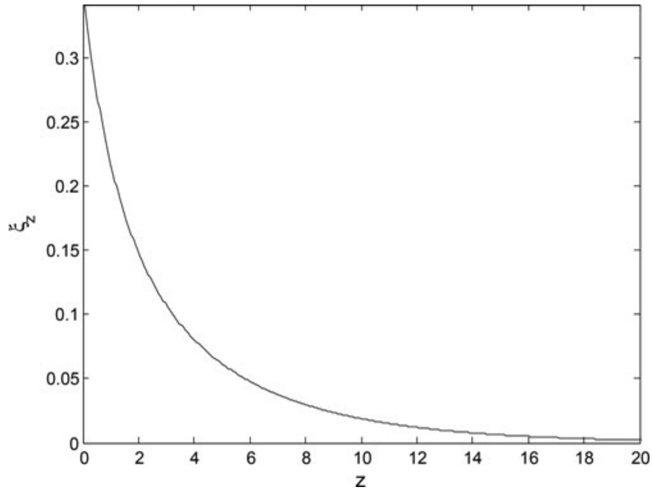

Figure 4. The estimated values $[\boldsymbol{\xi}_z]_j$ for $j = 2$ in Example 4.1.

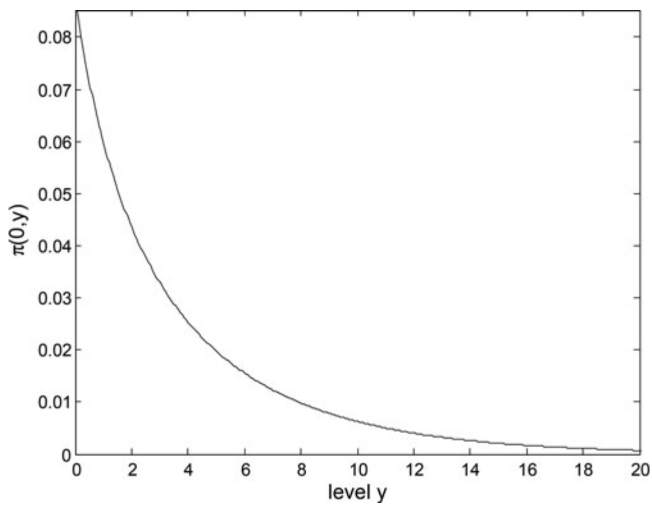

Figure 5. The estimated values $[\boldsymbol{\pi}(0, y)]_j$ for $j = 2$ in Example 4.1.

We consider the percentage error of the estimate denoted $\mathbf{p}(0, 0)_-$, given by

$$\delta = 100\% \times \frac{|0.259259 - \mathbf{p}(0, 0)_-|}{0.259259}, \tag{80}$$

for $\Delta u = n^{-1}$ with $n = 2, 4, 10, 20, 50$, $L = Kn$ and $K = 50$, see Table 1.

To further improve the quality of the approximation, we recommend to choose Δu as small as practically possible, and also to vary the values of Δu so that

Table 1. Error δ as defined in (80), in relation to n, with $\Delta u = n^{-1}$.

n	2	4	10	20	50
δ (%)	12.4816	6.4070	2.5697	1.2620	0.4711

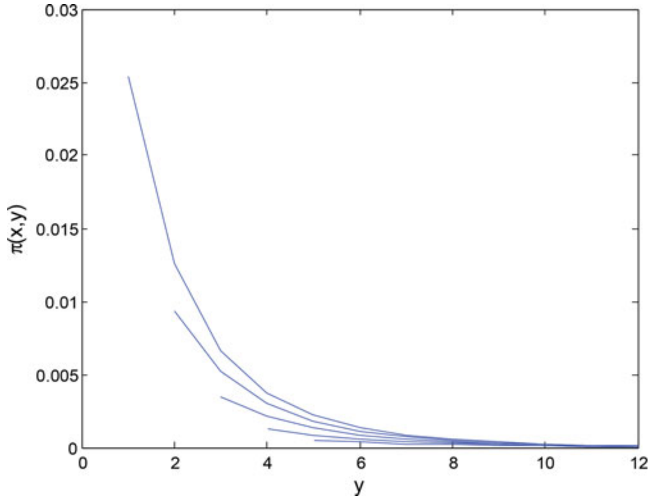

Figure 6. The estimated values $[\boldsymbol{\pi}(x, y)]_j$ for $j = 2$ and selected values of x, y in Example 4.1. The lines (top to bottom) correspond to $x = 1, 2, \ldots, 5$, with always $y = x + 0.01, x + 1, x + 2, \ldots, 12$. We note that the range of x and y is chosen such that it meets the condition $y > x\widehat{c}_1/r_1$ (here equivalent to $y > x$ since $\widehat{c}_1/r_1 = 1$), as detailed in Section 2.3.

the intervals are smaller when closer to zero, where more probability mass is accumulated.

5. Conclusion

We considered a tandem fluid queue model consisting of two queues, in which the first queue $\{(\varphi(t), X(t)) : t \geq 0\}$ is a standard stochastic fluid model with an infinite buffer and rates r_i, and the second queue $\{(\varphi(t), Y(t)) : t \geq 0\}$ is also a stochastic fluid model with an infinite buffer and rates $\widehat{c}_i > 0$ and $\check{c}_i < 0$, such that the rates of change of level depend on whether the first queue is empty or not. Specifically, we assumed that the rates of change of level in the second queue are negative ($dY(t)/dt = \check{c}_i$) when the first queue is empty, and positive ($dY(t)/dt = \widehat{c}_i$) otherwise.

We derived theoretical results for the stationary analysis of such a tandem fluid queue and, based on these results, summarized the key points of the methodology for the numerical evaluation of the stationary distribution.

As future work, we are also interested in the analysis of a fluid queue model that is dual to the current one, in the sense that the rates of change of level in the second queue are positive ($dY(t)/dt = \widehat{c}_i$) when the first queue is empty, and negative ($dY(t)/dt = \check{c}_i$) otherwise. This dual model requires slightly different techniques than the current model, since the process may then hit the boundary in either of the buffers, which needs to be considered in the analysis. Work on the theoretical analysis of the dual model is in progress. Also, in progress is work on a time-varying variant of the tandem fluid queue, motivated by the extension of the results in[20].

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