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On dominating and spanning circuits in graphs

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Abstract

An eulerian subgraph of a graph is called a circuit. As shown by Harary and Nash–Williams, the existence of a Hamilton cycle in the line graph $L(G)$ of a graph G is equivalent to the existence of a dominating circuit in G, i.e., a circuit such that every edge of G is incident with a vertex of the circuit. Important progress in the study of the existence of spanning and dominating circuits was made by Catlin, who defined the reduction of a graph G and showed that G has a spanning circuit if and only if the reduction of G has a spanning circuit. We refine Catlin's reduction technique to obtain a result which contains several known and new sufficient conditions for a graph to have a spanning or dominating circuit in terms of degree-sums of adjacent vertices. In particular, the result implies the truth of the following conjecture of Benhocine et al.: If G is a connected simple graph of order n such that every cut edge of G is incident with a vertex of degree 1 and $d(u)+d(v)>2(\frac{1}{5}n-1)$ for every edge uv of G, then, for n sufficiently large, $L(G)$ is hamiltonian.

1. Introduction

We use [2] for terminology and notation not defined here and consider only loopless graphs. Let G be a graph. An eulerian subgraph of G will be called a *circuit*. A circuit may consist of a single vertex. A *spanning circuit* or *S-circuit* of G is a circuit containing all vertices of G. A *dominating circuit* or *D-circuit* of G is a circuit such that every edge of G is incident with at least one vertex of the circuit. G is *almost bridgeless* if every cut edge of G is incident with a vertex of degree 1. If G is noncomplete, then $\sigma_2(G)$ denotes

$$\min\{d(u)+d(v)\mid uv\notin E(G)\}.$$

If $|E(G)|>0$, then $\bar{\sigma}_2(G)$ denotes

$$\min\{d(u)+d(v)\mid uv\in E(G)\}.$$

As shown by Harary and Nash–Williams [11], there is a close relationship between D-circuits in graphs and Hamilton cycles in line graphs.

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Theorem 1 [11]. *Let G be a graph with $|E(G)|\geq 3$. Then the line graph $L(G)$ of G is hamiltonian if and only if G has a D-circuit.*

Various sufficient conditions for the existence of S- and D-circuits in a graph G and the existence of Hamilton cycles in $L(G)$ in terms of $\sigma_2(G)$ or $\bar{\sigma}_2(G)$ have been derived. See, e.g., [1, 4–7, 9, 10]. The following result was independently established by Catlin [5] and Benhocine, Clark, Köhler and Veldman [1].

Theorem 2 [1, 5]. *Let G be a connected simple almost bridgeless graph of order $n\geq 4$ such that $\bar{\sigma}_2(G)\geq\frac{1}{3}(2n+1)$. Then $L(G)$ is hamiltonian.*

In [1] it was conjectured that, for n sufficiently large, the requirement $\bar{\sigma}_2(G)\geq\frac{1}{3}(2n+1)$ in Theorem 2 could be weakened to $\bar{\sigma}_2(G)>2(\frac{1}{5}n-1)$. Our main result (Theorem 7 in Section 3) implies the truth of this conjecture, as well as several other known and new results on D-circuits in graphs and S-circuits in graphs of minimum degree at least 3. The proof of Theorem 7 uses a refinement of a powerful reduction technique in the study of circuits in graphs, developed by Catlin [3].

2. A refinement of Catlin's reduction technique

We start with a description of Catlin's reduction technique. If H is a connected subgraph of a graph G, then G/H denotes the graph obained from G by *contracting H*, i.e., replacing H by a vertex v_H such that the number of edges in G/H joining any $v\in V(G)-V(H)$ to v_H in G/H equals the number of edges joining v in G to H. A graph G is *contractible* to a graph G' if G contains pairwise vertex-disjoint connected subgraphs $H_1,\dots,H_k$ with $\bigcup_{i=1}^k V(H_i)=V(G)$ such that G' is obtained from G by successively contracting $H_1,\dots,H_k$. The subgraph H_i of G is called the *preimage* of the vertex v_{H_i} of G'; the vertex v_{H_i} is called *trivial* if H_i is trivial ($i=1,\dots,k$). A graph G is *collapsible* if for every even subset X of $V(G)$ there exists a spanning connected subgraph F of G such that $X=\{v\in V(G)\,|\,d_F(v)\text{ is odd}\}$. In particular, K_1 is collapsible. In [3], Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1,\dots,H_k$ such that $\bigcup_{i=1}^k V(H_i)=V(G)$. The *reduction* of G is the graph obtained from G by successively contracting $H_1, H_2,\dots,H_k$. A graph is *reduced* if it is the reduction of some graph.

As an example, consider the graph G_0 with $V(G_0)=\{v_i\,|\,1\leq i\leq 14\}$ and $E(G_0)=\{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_{13}, v_4v_5, v_4v_7, v_5v_6, v_5v_7, v_6v_7, v_7v_8, v_8v_9, v_8v_{11}, v_9v_{10}, v_{10}v_{11}, v_{10}v_{12}, v_{12}v_{13}, v_{13}v_{14}\}$. The nontrivial maximal collapsible subgraphs of G_0 are $H_1=G_0[\{v_1,v_2,v_3\}]$ and $H_2=G_0[\{v_4,v_5,v_6,v_7\}]$. Hence the reduction G_0' of G has

$$V(G_0')=\{v_i\,|\,8\leq i\leq 14\}\cup\{v_{H_1}, v_{H_2}\}$$

and

$$E(G_0')=E(G_0[\{v_i|8\leqslant i\leqslant 14\}])\cup\{v_{H_1}v_{H_2}, v_{H_1}v_{13}, v_{H_2}v_8\}.$$

A number of results in [3] are summarized in the following lemmas.

Lemma 3 [3]. *Let G be a connected graph and G' the reduction of G. Then each of the following holds.*

(a) *G has an S-circuit if and only if G' has an S-circuit.*

(b) *G has a D-circuit if and only if G' has a D-circuit containing all nontrivial vertices of G'.*

Lemma 4 [3]. *Let G be a connected graph. Then each of the following holds.*

(a) *G is reduced if and only if G contains no nontrivial collapsible subgraphs.*

(b) *If G is reduced, then every subgraph of G is reduced.*

(c) *If G is reduced, then either $G\in\{K_1, K_2\}$ or $|E(G)|\leqslant 2|V(G)|-4$.*

We now describe a refinement of Catlin's reduction method. Let G be a simple graph and define $D(G)=\{v\in V(G)|d(v)\in\{1,2\}\}$. For an independent subset X of $D(G)$, define $I_X(G)$ as the graph obtained from G by contracting one edge incident with each vertex of X. In other words, $I_X(G)$ is obtained from G by deleting the vertices in X of degree 1 and replacing each path of length 2 whose internal vertex is a vertex in X of degree 2 by an edge. Note that $I_X(G)$ need not be simple. We call G *X-collapsible* if $I_X(G)$ is collapsible. A subgraph H of G is an *X-subgraph* of G if $d_H(x)=d_G(x)$ for all $x\in X\cap V(H)$. An X-subgraph H of G is called X-collapsible if H is $(X\cap V(H))$-collapsible. By $R(X)$ we denote the set of vertices in X which are not contained in an X-collapsible X-subgraph of G. Since the graph $I_X(G)$ has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $F_1,\ldots,F_k$ such that $\bigcup_{i=1}^k V(F_i)=V(I_x(G))$, the graph G has a unique collection of pairwise vertex-disjoint maximal X-collapsible X-subgraphs $H_1,\ldots,H_k$ such that $\bigcup_{i=1}^k V(H_i)\cup R(X)=V(G)$. The *$X$-reduction* of G is the graph obtained from G by contracting $H_1,\ldots,H_k$. The graph G is *X-reduced* if there exists a graph G_1 and an independent subset X_1 of $D(G_1)$ such that $X=R(X_1)$ and G is the X_1-reduction of G_1. An X-subgraph H of G is called X-reduced if H is $(X\cap V(H))$-reduced.

As an example, again consider the graph G_0 defined before. Set $X=D(G_0)=\{v_1,v_6, v_9,v_{11},v_{12},v_{14}\}$. Then the nontrivial maximal X-collapsible X-subgraphs of G_0 are $H_1=G_0[\{v_1,v_2,v_3\}]$, $H_2=G_0[\{v_4,v_5,v_6,v_7\}]$, $H_3=G_0[\{v_8,v_9,v_{10},v_{11}\}]$ and $H_4=G_0[\{v_{13},v_{14}\}]$, while $R(X)=\{v_{12}\}$. Hence the X-reduction G_0'' of G_0 has $V(G_0'')=\{v_{H_1},v_{H_2},v_{H_3},v_{12},v_{H_4}\}$ and $E(G_0'')=\{v_{H_1}v_{H_2},v_{H_2}v_{H_3},v_{H_3}v_{12},v_{12}v_{H_4}, v_{H_4}v_{H_1}\}$.

Note that a graph G is $\emptyset$-collapsible if and only if G is collapsible, every subgraph of G is an $\emptyset$-subgraph, the $\emptyset$-reduction of G equals the reduction of G, and G is $\emptyset$-reduced if and only if G is reduced. Hence Lemmas 3(b), 4(a), 4(b) and 4(c) are special cases of Lemmas 5, 6(b), 6(c) and 6(d) below, respectively.

Lemma 5. *Let G be a connected simple graph, X an independent subset of $D(G)$, and G' the X-reduction of G. Then G has a D-circuit if and only if G' has a D-circuit containing all nontrivial vertices of G'.*

Proof. Clearly, if G has a D-circuit, then G' has a D-circuit containing all nontrivial vertices of G'. Conversely, assume G' has a D-circuit C' containing all nontrivial vertices of G'. Set $G'_1 = G'[V(C')]$, $U = V(G') - V(C')$ and $X^* = \{x \in X \mid d_G(x) = 1\}$. Then U is an independent subset of both $V(G')$ and $V(G)$, $U \cap X^* = \emptyset$ and C' is an S-circuit of G'_1. Set $G_1 = G - (U \cup X^*)$, $G_1^* = G - U$, $W = X - U$, $G_2 = I_W(G_1)$ and let G'_2 be the reduction of G_2. Then G'_1 is the W-reduction of both G_1 and G_1^*. Also, by our definitions, G'_1 is a subdivision of G'_2. Hence, since G'_1 has an S-circuit, G'_2 has an S-circuit. By Lemma 3(a), G_2 also has an S-circuit. Since G_1 is a subdivision of $G_2 = I_W(G_1)$ with each edge of G_2 subdivided at most once, it follows that G_1, and hence G_1^* also, has a D-circuit C. We have $V(G_1^*) - V(C) \subseteq W$, hence $V(G) - V(C) \subseteq U \cup W$. Since $U \cup W$ is an independent set, C is a D-circuit of G. □

Lemma 6. *Let G be a connected simple graph and X an independent subset of $D(G)$. Then each of the following holds.*

(a) *G is X-reduced if and ony if $I_X(G)$ is reduced.*

(b) *G is X-reduced if and only if G contains no nontrivial X-collapsible X-subgraphs.*

(c) *If G is X-reduced, then every X-subgraph of G is X-reduced.*

(d) *If G is X-reduced, then $d(x) = 2$ for all $x \in X$ and exactly one of the following holds.*

(d1) *$G \in \{K_1, K_2\}$ and $X = \emptyset$.*

(d2) *$G = P_3$ and $|X| = 1$.*

(d3) *$|E(G)| \leqslant 2|V(G)| - |X| - 4$.*

Proof. (a) Suppose G is X-reduced. Let G_1 and $X_1 \subseteq D(G_1)$ be such that $X = R(X_1)$ and G is the X_1-reduction of G_1. Then our definitions imply that $I_X(G)$ is the reduction of $I_{X_1}(G_1)$. Hence $I_X(G)$ is reduced. Conversely, assume $I_X(G)$ is reduced. By Lemma 4(a), $I_X(G)$ contains no nontrivial collapsible subgraphs, whence G contains no nontrivial X-collapsible X-subgraphs. Thus G coincides with the X-reduction of G, implying that G is X-reduced.

(b) This is an immediate consequence of Lemmas 4(a) and 6(a).

(c) This follows immediately from Lemma 6(b).

(d) If G is X-reduced, then by definition, X contains no vertex of degree 1. The rest follows from Lemmas 4(c) and 6(a) and the relations

$$|E(I_X(G))| = |E(G)| - |X|$$

and

$$|V(I_X(G))| = |V(G)| - |X|. \qquad \square$$

3. Main result and consequences

Using Lemmas 5 and 6 we now prove our main result.

Theorem 7. *Let G be a connected simple graph of order n and* $p \geqslant 2$ *an integer such that*

$$\bar{\sigma}_2(G) \geqslant 2(\lfloor n/p \rfloor - 1). \tag{1}$$

If n is sufficiently large relative to p, then

$$|V(G')| \leqslant \max\{p, \tfrac{3}{2}p - 4\}, \tag{2}$$

where G' *is the* $D(G)$*-reduction of G. Moreover, for* $p \leqslant 7$, (2) *holds with equality only if* (1) *holds with equality.*

Proof. Let G be a connected simple graph of order n and $p \geqslant 2$ an integer such that (1) holds. If n is sufficiently large relative to p, henceforth abbreviated $n \gg p$, then $D(G)$ is an independent set by (1), so the $D(G)$-reduction G' of G is well defined. Set $c = 5p + 20$ and define the following subsets of $V(G')$:

$$S = \{v \in V(G') \mid d_{G'}(v) < c\},$$

$$L = \{v \in V(G') \mid d_{G'}(v) \geqslant c\},$$

$$T = \{v \in S \mid v \text{ is trivial}\},$$

$$M = \{v \in S \mid v \text{ is nontrivial}\},$$

$$L' = L \cup M,$$

$$T_1 = \{v \in T \mid d_{G'}(v) \leqslant 2\},$$

$$M_1 = \{v \in M \mid N_{G'}(v) \cap T \neq \emptyset\},$$

$$M_2 = \{v \in M \mid N_{G'}(v) \cap T = \emptyset\}.$$

Set $n' = |V(G')|$, $s = |S|$, $l = |L|$, $t = |T|$, $m = |M|$, $l' = |L'|$, $t_1 = |T_1|$, $m_1 = |M_1|$ and $m_2 = |M_2|$. We state a sequence of assertions which hold for $n \gg p$, each followed by a proof.

If H is the preimage of a vertex in M, then $|V(H)| \geqslant \lfloor n/p \rfloor$ and $|V(H)| = \lfloor n/p \rfloor$ only if (1) holds with equality. (3)

Let H be the preimage of a vertex in M. Suppose $N_G(x) \cap (V(G) - V(H)) \neq \emptyset$ for all $x \in V(H)$. Then $|V(H)| \leqslant c$ and, for an arbitrary edge yz of H,

$$2(\lfloor n/p \rfloor - 1) \leqslant \bar{\sigma}_2(G) \leqslant d_G(y) + d_G(z) \leqslant 2(|V(H)| - 1) + c \leqslant 3c - 2,$$

a contradiction if $n \gg p$. Hence, if $n \gg p$, H contains a vertex x_0 with $N_G(x_0) \subseteq V(H)$. If x_0 is adjacent to a vertex y_0 of H with $N_G(y_0) \subseteq V(H)$, then

$$2(\lfloor n/p \rfloor - 1) \leqslant \bar{\sigma}_2(G) \leqslant d_G(x_0) + d_G(y_0) \leqslant 2(|V(H)| - 1),$$

implying that $|V(H)|\geqslant\lfloor n/p\rfloor$ and $|V(H)|=\lfloor n/p\rfloor$ only if (1) holds with equality. If $N_G(y)\cap(V(G)-V(H))\neq\emptyset$ for all $y\in N_G(x_0)$, then, for an arbitrary neighbor y_1 of x_0,

$$2(\lfloor n/p\rfloor-1)\leqslant d_G(x_0)+d_G(y_1)<c+|V(H)|-1+c,$$

implying that $|V(H)|>\lfloor n/p\rfloor$ if $n\gg p$. Now (3) follows.

$$m\leqslant p \text{ and } m=p \text{ only if (1) holds with equality.} \tag{4}$$

This is an easy consequence of (3) if $n\gg p$.

$$\text{If } H \text{ is the preimage of a vertex in } M_1\text{, then } |V(H)|>2\lfloor n/p\rfloor-2c-1. \tag{5}$$

Let H be the preimage of a vertex in M_1. Then H contains a vertex x which is adjacent to a vertex y in T. By (1),

$$2(\lfloor n/p\rfloor-1)\leqslant d_G(x)+d_G(y)<|V(H)|-1+c+c,$$

whence (5) follows.

$$m_1\leqslant\tfrac{1}{2}p. \tag{6}$$

This is an easy consequence of (5) if $n\gg p$.

$$2m_1+m_2\leqslant p. \tag{7}$$

By (3), (5) and (6),

$$n\geqslant m_1(2\lfloor n/p\rfloor-2c-1)+m_2\lfloor n/p\rfloor\geqslant(2m_1+m_2)\lfloor n/p\rfloor-\tfrac{1}{2}p(2c-1),$$

whence (7) follows for $n\gg p$.

$$S=V(G'). \tag{8}$$

Set $F'=G'[L'\cup T_1]$. If $n\gg p$, then T is an independent set by (1), so that, in particular, F' is a T_1-subgraph of G'. By Lemma 6(c), F' is T_1-reduced since G' is. Thus by Lemma 6(d),

$$2t_1\leqslant|E(F')|\leqslant 2(l'+t_1)-t_1,$$

whence $t_1\leqslant 2l'$. From Lemma 4(c) and the fact that T is an independent set of vertices of degree at least 2 we conclude that

$$2n'>|E(G')|\geqslant 2t_1+3(t-t_1)=3t-t_1\geqslant 3t-2l'. \tag{9}$$

Also by Lemma 4(c) and by the definition of l,

$$4n'>\sum_{v\in V(G')}d_{G'}(v)\geqslant cl,$$

so $l<(4/c)n'$. Hence by (4) and the fact that $t+l'=n'$,

$$l'<\frac{4}{c}n'+p \quad\text{and}\quad t>\frac{c-4}{c}n'-p. \tag{10}$$

By combining (9) and (10) we obtain

$$2n' > 3\left(\frac{c-4}{c}n' - p\right) - 2\left(\frac{4}{c}n' + p\right),$$

or, equivalently,

$$\frac{c-20}{c}n' < 5p. \tag{11}$$

If we assume $L \neq \emptyset$, then $n' > c$, whence $c-20 < 5p$ by (11), contradicting $c = 5p+20$. Thus $L = \emptyset$ and (8) follows.

If $M_1 \neq \emptyset$, then $t \leqslant 2m_1 - 4$ unless $t = 1$ and $m_1 = 2$. (12)

Assume $M_1 \neq \emptyset$. Set $H' = G'[M_1 \cup T]$. By (8) and the fact that T is an independent set if $n \geqslant p$, H' is a T_1-subgraph of G'. By Lemma 6(c), H' is T_1-reduced. Every vertex of T has degree at least 2, so $|V(H')| \geqslant 3$ and $|V(H')| = 3$ if and only if $t = t_1 = 1$ and $m_1 = 2$. If $|V(H')| \geqslant 4$, then by Lemma 6(d),

$$2t_1 + 3(t - t_1) \leqslant |E(H')| \leqslant 2(m_1 + t) - t_1 - 4,$$

whence $t \leqslant 2m_1 - 4$. This completes the proof of (12).

If $M_1 = \emptyset$, then $n' \leqslant p$ and $n' = p$ only if (1) holds with equality. (13)

Assume $M_1 = \emptyset$. Then by (8) and since G' is connected, $T = \emptyset$. Now by (8), $n' = m$, so (13) follows from (4).

If $M_1 \neq \emptyset$, then $n' \leqslant \max\{p-1, \frac{3}{2}p - 4\}$. (14)

Assume $M_1 \neq \emptyset$. By (8), $n' = m_1 + m_2 + t$. If $t = 1$ and $m_1 = 2$, then by (7), $m_2 \leqslant p - 4$, so that $n' \leqslant p - 1$. Otherwise by (12), (7) and (6),

$$n' \leqslant m_1 + m_2 + 2m_1 - 4 \leqslant m_1 + p - 4 \leqslant \tfrac{3}{2}p - 4,$$

proving (14).

The conclusions of the theorem now follow from (13), (14) and the observation that $p > \frac{3}{2}p - 4$ for $p \leqslant 7$. □

We now show that Theorem 7 is best possible in the sense that for every integer $p \geqslant 2$ there exist infinitely many connected simple graphs G with

$$\bar{\sigma}_2(G) \geqslant 2\left(\left\lfloor \frac{1}{p}|V(G)| \right\rfloor - 1\right)$$

such that

$$|V(G')| = \max\{p, \lfloor \tfrac{3}{2}p - 4 \rfloor\},$$

where G' is the $D(G)$-reduction of G. A vertex v of a graph G is said to be *k-enlarged* ($k \geqslant 1$) if $k-1$ new pairwise adjacent vertices are added to $|V(G)|$ and joined to v.

First assume $2 \leqslant p \leqslant 7$. For a positive integer k, obtain the graph $G_{p,k}$ from an arbitrary connected reduced graph G_p of order p by k-enlarging every vertex of G_p. If $k \geqslant 4$, then $D(G_{p,k}) = \emptyset$ and hence the $D(G_{p,k})$-reduction $G'_{p,k}$ of $G_{p,k}$ coincides with the reduction of $G_{p,k}$, which is G_p. Thus for $k \geqslant 4$ we have

$$|V(G'_{p,k})| = |V(G_p)| = p = \max\{p, \tfrac{3}{2}p - 4\},$$

while

$$\bar{\sigma}_2(G_{p,k}) = 2k - 2 = 2\left(\left\lfloor \frac{1}{p}\, |V(G_{p,k})| \right\rfloor - 1\right).$$

Now assume $p \geqslant 8$. Suppose first p is even, $p = 2q$ say. Let G_p be the graph obtained from $K_{2,q-2}$ by subdividing each edge. If $p = 8$, let X_p be one of the two independent sets of cardinality 4 in G_p. Otherwise, let X_p be the unique maximal independent set of G_p containing the two vertices of degree $q-2$. For a positive integer k, obtain the graph $G_{p,k}$ from G_p by k-enlarging each vertex of X_p. If $k \geqslant 4$, then the $D(G_{p,k})$-reduction $G'_{p,k}$ of $G_{p,k}$ is well defined and $G'_{p,k} = G_p$. For $k \geqslant 5$ we have

$$|V(G'_{p,k})| = |V(G_p)| = 3q - 4 = \tfrac{3}{2}p - 4,$$

while

$$\bar{\sigma}_2(G_{p,k}) = k + 3 > k - \frac{4}{q} = 2\left(\frac{kq + 2q - 4}{2q} - 1\right) = 2\left(\frac{1}{p}|V(G_{p,k})| - 1\right).$$

Suppose next p is odd, say $p = 2q + 1$. Let v_p be a vertex of $K_{2,q-1}$ of degree 2 and G_p the graph obtained from $K_{2,q-1}$ by subdividing all edges not incident with v_p. Let X_p be the maximal independent set of G_p containing the two vertices of degree $q-1$. For a positive integer k, obtain the graph $G_{p,k}$ from G_p by k-enlarging each vertex of X_p and $(\lfloor \tfrac{1}{2}k \rfloor + 3)$-enlarging v_p. If $k \geqslant 4$, then the $D(G_{p,k})$-reduction $G'_{p,k}$ of $G_{p,k}$ is well defined and $G'_{p,k} = G_p$. For $k \geqslant 5$ we have

$$|V(G'_{p,k})| = |V(G_p)| = 3q - 3 = \lfloor \tfrac{3}{2}p - 4 \rfloor,$$

while

$$\bar{\sigma}_2(G_{p,k}) = k + 3 > k - \frac{4}{2q+1} = 2\left(\frac{kq + \frac{1}{2}k + 3 + 2q - 4}{2q+1} - 1\right)$$

$$\geqslant 2\left(\frac{1}{p}|V(G_{p,k})| - 1\right).$$

Note that if $p \geqslant 8$, (2) may hold with equality even if (1) does not, as shown by the graphs $G_{p,k}$ with $p \geqslant 8$ and $k \geqslant 5$.

We mention a number of consequences of Theorem 7, some known, some new.

Corollary 8. *Let G be a connected simple graph of order n and $p \geqslant 2$ an integer such that*

$$\bar{\sigma}_2(G) > 2\left(\left\lfloor \frac{n}{p} \right\rfloor - 1\right).$$

If n is sufficiently large relative to p, then either G has a D-circuit or the D(G)-reduction G′ of G satisfies

$$|V(G')| \leqslant \max\{p-1, \tfrac{3}{2}p-4\}$$

and G′ has no D-circuit containing all nontrivial vertices of G′.

Proof. Corollary 8 is an immediate consequence of Theorem 7 and Lemma 5. □

Corollary 9 [10]. *Let G be a connected simple graph of order n such that $\bar{\sigma}_2(G) \geqslant n-1-\varepsilon(n)$, where $\varepsilon(n)=0$ if n is even and $\varepsilon(n)=1$ if n is odd. If n is sufficiently large, then L(G) is hamiltonian.*

Proof. Apply Theorem 1 and the case $p=2$ of Corollary 8. □

The smallest possible lower bound on n in Corollary 9 is 6 [10].

Corollary 10. *Let G be a connected almost bridgeless simple graph of order n such that $\bar{\sigma}_2(G) > 2(\lfloor \frac{1}{5}n \rfloor - 1)$. If n is sufficiently large, then L(G) is hamiltonian.*

Proof. If G is connected, simple and almost bridgeless, then the $D(G)$-reduction of G is either 2-edge-connected or trivial. Corollary 10 now follows from Theorem 1, the case $p=5$ of Corollary 8 and the fact that every 2-edge-connected graph of order at most 4 has an S-circuit. □

Corollary 10 settles the conjecture of Benhocine et al. [1] mentioned in Section 1 in the affirmative.

Corollary 11. *Let G be a connected almost bridgeless simple graph of order n such that $\bar{\sigma}_2(G) > 2(\lfloor \frac{1}{7}n \rfloor - 1)$. If n is sufficiently large, then either L(G) is hamiltonian or G is contractible to $K_{2,3}$ in such a way that all vertices of degree 2 in $K_{2,3}$ are nontrivial.*

Proof. Corollary 11 follows from Theorem 1, the case $p=7$ of Corollary 8 and the fact that the only 2-edge-connected graphs of order at most 6 without an S-circuit are $K_{2,3}$ and the graph obtained from $K_{2,3}$ by subdividing an edge, which is contractible to $K_{2,3}$. □

Corollary 11 is best possible in the sense that there exist infinitely many connected almost bridgeless simple graphs G with $\bar{\sigma}_2(G) = 2(\lfloor \frac{1}{7}|V(G)| \rfloor - 1)$ such that $L(G)$ is nonhamiltonian and G is not contractible to $K_{2,3}$. Examples of such graphs can be found among the graphs contractible to $K_{2,5}$ or the 3-cube minus a vertex.

Corollary 12. *Let G be a connected simple graph of order n and $p \geqslant 2$ an integer such that $\delta(G) \geqslant 3$ and $\bar{\sigma}_2(G) > 2(\lfloor n/p \rfloor - 1)$. If n is sufficiently large relative to p, then either G has an S-circuit or the reduction G' of G satisfies*

$$|V(G')| \leqslant \max\{p-1, \tfrac{3}{2}p-4\}$$

and G' has no S-circuit.

Proof. If $\delta(G) \geqslant 3$, then $D(G) = \emptyset$, so that the $D(G)$-reduction of G coincides with the reduction of G. Now Corollary 12 is an immediate consequence of Theorem 7 and Lemma 3(a). □

Corollary 13. *Let G be a 2-edge-connected simple graph of order n such that $\delta(G) \geqslant 3$ and $\bar{\sigma}_2(G) > 2(\lfloor \frac{1}{5}n \rfloor - 1)$. If n is sufficiently large, then G has an S-circuit.*

The proof of Corollary 13 is similar to the proof of Corollary 10 and is hence omitted. Within the class of 'large' graphs with minimum degree at least 3, Corollary 13 improves the following best possible result of Catlin [6].

Theorem 14 [6]. *Let G be a 2-edge-connected simple graph of order n such that $\bar{\sigma}_2(G) \geqslant \frac{2}{5}(n+1)$. Then either G has an S-circuit or $G = K_{2,n-2}$ and n is odd.*

The case $\delta(G) \geqslant 4$ of Corollary 13 was recently established by Catlin and Li [7] (without restrictions on n).

Corollary 15. *Let G be a 2-edge-connected simple graph of order n such that $\delta(G) \geqslant 3$ and $\bar{\sigma}_2(G) > 2(\frac{1}{7}n - 1)$. If n is sufficiently large, then either G has an S-circuit or G is contractible to $K_{2,3}$.*

The proof of Corollary 15, being similar to the proof of Corollary 11, is omitted.

Corollary 16 [9]. *Let G be a 3-edge-connected simple graph of order n such that $\bar{\sigma}_2(G) \geqslant 2(\lfloor \frac{1}{10}n \rfloor - 1)$. If n is sufficiently large, then either G has an S-circuit or G is contractible to the Petersen graph.*

Proof. If G is 3-edge-connected, then the reduction of G is either 3-edge-connected or trivial. Chen [8] observed that the Petersen graph is the only 3-edge-connected reduced graph of order at most 11. Combination of these facts with Lemma 3(a) and the case $p = 10$ of Theorem 7 yields the desired result. □

In [9], Corollary 16 is obtained as a consequence of the following (slightly reformulated) result, which is closely related to Theorem 7.

Theorem 17 [9]. *Let G be a 3-edge-connected simple graph of order n and $p \geqslant 2$ an even integer such that $\bar{\sigma}_2(G) \geqslant 2(n/p-1)$. If $n > 3p(p-2)$, then the reduction G' of G satisfies $|V(G')| \leqslant \frac{3}{2}p-4$ and $\alpha'(G') \leqslant \frac{1}{2}p$, where $\alpha'(G')$ denotes the size of a maximum matching in G'.*

We close by mentioning a result of Catlin [4] which is analogous to Corollaries 8 and 12.

Theorem 18 [4]. *Let G be a connected simple graph of order n and let $p \geqslant 2$. If $\sigma_2(G) > 2(n/p-1)$ and $n \geqslant 4p^2$, then exactly one of the following holds.*
(a) *G has an S-circuit.*
(b) *The reduction G' of G satisfies $|V(G')| < p$ and G' has no S-circuit.*
(c) *$p=2$ and $G-x=K_{n-1}$ for some $x \in V(G)$ with $d(x)=1$.*

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