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On continuity of solutions for parabolic control systems and input-to-state stability

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Abstract

We study minimal conditions under which mild solutions of linear evolutionary control systems are continuous for arbitrary bounded input functions. This question naturally appears when working with boundary controlled, linear partial differential equations. Here, we focus on parabolic equations which allow for operator-theoretic methods such as the holomorphic functional calculus. Moreover, we investigate stronger conditions than continuity leading to input-to-state stability with respect to Orlicz spaces. This also implies that the notions of input-to-state stability and integral-input-to-state stability coincide if additionally the uncontrolled equation is dissipative and the input space is finite-dimensional.

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Keywords: Abstract parabolic control system; Admissible operator; Orlicz space; Bounded functional calculus; H^∞ calculus; Input-to-state stability

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1. Introduction

Many evolutionary systems and linear pde's can be modelled by abstract differential equations of the form

$$\Sigma(A, B): \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$

where A generates a strongly continuous semigroup $T(\cdot)$ on the Banach space X, $x_0 \in X$ and the control input $u : [0, t] \to U$ enters through the inhomogeneity Bu. Typical examples where B, as linear operator from U to X, is unbounded are given by boundary control systems, see e.g. [33, Chapter 11]. In that case, Eq. (1) is viewed on the extrapolation space $X_{-1} \supset X$ a-priori and B is bounded as operator from U to X_{-1}. This setting immediately gives rise to the (formal) mild solution $x : [0, \infty) \to X_{-1}$,

$$x(t) = T_{-1}(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,ds, \quad t > 0,$$

for every input u from a considered class Z of U-valued functions, e.g., $Z(0, t; U) = L^2(0, t; U)$. Here T_{-1} denotes the extension of the semigroup to the extrapolation space. This abstract lifting argument comes at a price: It is not even clear if the solution x is an X-valued continuous function. A central point of this article is to study 'minimal' conditions on A and B under which the solution x is continuous for any $u \in Z$. Necessary for the latter is that x is X-valued, or equivalently that

$$\int_0^t T_{-1}(s)Bu(s)\,ds \in X \quad \forall t > 0, u \in Z, \tag{2}$$

provided that Z is invariant under translations which will always be the case here. In this case, we call B a *Z-admissible control operator*. The question now is whether Z-admissibility already implies that solutions x are continuous. Admissible operators, in particular for $Z = L^2$, have been studied intensively, e.g., [15,26,33,37,38]. For $Z = L^p$ with $p \in [1, \infty)$, the above question has an affirmative answer which follows rather directly. This was already shown by Weiss in [38]. Intriguingly, the case $Z = L^\infty$ is still an open problem, see [37, Problem 2.4] and the discussion in [13, Sec. 6]. With results of the following type, we give a partial answer in the case of parabolic equations, (see Corollary 8, Theorem 10).

Theorem 1. *Let A generate an exponentially stable semigroup T on a Hilbert space X. Then the following assertions are equivalent.*

(i) Any bounded, linear operator $B : \mathbb{C} \to X_{-1}$ is L^∞-admissible.
(ii) The solutions x of (1) are continuous X-valued functions, for all $u \in L^\infty_{loc}(0, \infty; U)$, and $B \in \mathcal{L}(U, X_{-1})$ with $\dim U < \infty$.
(iii) T is a bounded analytic semigroup similar to a contraction semigroup.

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We remark that the similarity to a contraction in Theorem 1*(iii)* is a condition which is satisfied in many applications. It means nothing else than that A is dissipative with respect to an equivalent Hilbertian norm.

Identifying admissible operators is an interesting task in its own right: by the closed graph theorem this is the same as characterizing the operators $B \in \mathcal{L}(U, X_{-1})$ for which

$$\forall t > 0 \, \exists K > 0 : \quad \left\| \int_0^t T_{-1}(s) B u(s)\, ds \right\| \le K \|u\|_{Z(0,t;U)} \quad \forall u \in Z(0,t;U),$$

at least if the space $Z(0,t;U)$ is continuously embedded in $L^1(0,t;U)$. In 1991, Weiss [39] posed the question whether L^2-admissibility, i.e. $Z = L^2$, is equivalent to

$$\sup_{\operatorname{Re}\lambda > \omega_0} \|\sqrt{\operatorname{Re}\lambda}(\lambda - A)^{-1} B\|_{\mathcal{L}(U,X)} < \infty$$

for sufficiently large $\omega_0 > 0$. By setting $u(s) = e^{-s\lambda}$, the necessity of this condition is easy to see. Counterexamples where U is not a Hilbert space were already mentioned in [39]. However, the question for Hilbert spaces U and X was the starting point of intensive research around what has become known as the *Weiss conjecture*, see [15,33] for surveys. Although even in this case, counterexamples were found [16,17,41], there are situations with positive answers — most prominently, the case of contraction semigroups and $U = \mathbb{C}$, [14], in which a connection with deep results in complex analysis appears. In [26] Le Merdy characterized when the Weiss conjecture for bounded analytic semigroups and any space U is true — by drawing a link to the H^∞-functional calculus. For bounded analytic semigroups on Hilbert spaces, the latter can be rephrased as follows, see [25]: The Weiss conjecture is valid for A and A^* if and only if A is similar to a contraction semigroup.

Versions of the Weiss conjecture for $Z = L^p$, $p \in [1, \infty)$ and more general spaces have also been studied in the past, see e.g. [2,11,8,10] (for the particular case of analytic semigroups). However, the somewhat 'exotic' case $p = \infty$ has not gained a lot attention so far. To the best of the authors' knowledge, the only results in that direction are in [2] and [11] which imply that the Weiss conjecture for $Z = L^\infty$ and any input space U holds if and only if A_{-1} itself is L^∞-admissible, see Theorem 15. However, we point out that the latter condition is very restrictive and still not fully understood, see e.g. Proposition 14.

Summarizing, for the Weiss conjecture we distinguish the following parameters:

- the choice of Z: e.g., $Z = L^p$
- assumptions on the semigroup (bounded analytic, contraction,..)
- assumptions on the space X (Hilbert space, reflexive,..)
- assumptions on the space U ($\dim U < \infty$, $\dim U = \infty$)

Here, we will mainly consider bounded analytic semigroups. We will show that the assertion in Theorem 1 are equivalent to

(iv) The Weiss conjecture for $Z = L^\infty$ and any finite-dimensional U holds true.

The interest in L^∞-admissibility and the existence of continuous solutions comes from studying the notions of *input-to-state stability*, well-known from finite-dimensional system theory, that

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combine internal and external stability, in infinite-dimensions. Recently, this subject has attained growing interest, see e.g. [4,18,22,23,29,30]. In particular, the existence of continuous solution is an axiom in the paper [30]. For linear systems (1), input-to-state stability (ISS) is nothing else than exponential stability of the semigroup together with L^∞-admissibility of B. The relation to so-called *integral input-to-state stability*, a variant of ISS, is more involved. For systems of the form (1), the following implication holds:

$$\text{integral input-to-state stability} \quad \Longrightarrow \quad \text{input-to-state-stability.} \tag{3}$$

In general, it is not known whether the converse holds in (3). Using the characterization of integral input-to-state stability in terms of admissibility derived in [13], in this paper, we show that it indeed holds in the situation of Theorem 1, which covers a broad class of applications. In particular, we prove (see Corollary 21)

Theorem 2. *The converse in* (3) *holds for systems* $\Sigma(A, B)$ *provided that* $B : U \to X_{-1}$ *is bounded,* $\dim U < \infty$ *and* A *generates an exponentially stable, analytic semigroup on a Hilbert space which is similar to a contraction semigroup.*

Theorem 2 generalizes results for parabolic diagonal systems derived in [13]. We will further discuss how Theorems 1 and 2 can be generalized to more general spaces X, and how the exponential stability can be weakened.

In Section 2 we give sufficient conditions such that continuity of mild solutions holds for the extremal set of all input operator B with finite-dimensional input space. Moreover, in this situation, we even obtain admissibility with respect to Orlicz spaces, which is a stronger property. This enables us to infer consequences for the converse of (3), Section 5. Section 3 deals with optimality of the conditions supposed in Section 2 — this is done by establishing the converse of the results in Section 2 in terms of the H^∞-calculus.

In Section 4 we elaborate on the relation of the results to the Weiss conjecture. Finally, we conclude with an outlook, Section 6, including a detailed discussion of related (open) problems, which may be of interest in their own right.

1.1. Notions

In the following we will always denote the generator of a C_0-semigroup T on a Banach space X by A. We will consider the spaces X_{-1}^A and X_1^A which are defined by the completion of X using the norm $\|(\beta - A)^{-1} \cdot \|$ for some $\beta \in \rho(A)$ and by equipping $D(A)$ with the graph norm of A, respectively. If the operator A is clear from the context, we will simply write X_{-1} and X_1. By $R(\lambda, A)$ we denote the resolvent $(\lambda - A)^{-1}$. If the dual operator $A^* : D(A^*) \subset X' \to X'$ generates a strongly continuous semigroup — that is, if $D(A^*)$ is dense, e.g. when X is reflexive — then

$$(X_{-1}^A)' \simeq X_1^{A^*} \quad \text{and} \quad (X_1^A)' \simeq X_{-1}^{A^*},$$

see e.g. [34, p. 43-46] or [38]. If moreover X is reflexive, we have that an element $x \in X_{-1}^A$ lies in X if and only if the evaluation functional

$$f_x : X_1^{A^*} \to \mathbb{C}, \;\; y \mapsto \langle y, x \rangle_{X_1^{A^*}, X_{-1}^A}, \tag{4}$$

can be continuously extended to X'. We denote the extension of A to X_{-1}^A by A_{-1} and the C_0-semigroup generated by A_{-1} by T_{-1}.

For Banach spaces X, Y, the bounded operators from X to Y will be denoted by $\mathcal{L}(X, Y)$. A semigroup T is called bounded analytic semigroup if it can be extended to a sector $\{0\} \cup \Sigma_\alpha$ where $\Sigma_\alpha = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$, $\alpha \in (0, \pi]$, such that T is bounded and analytic on Σ_α. By an *exponentially stable analytic semigroup* we refer to bounded analytic semigroup which in addition is exponentially stable on $[0, \infty)$. There is a natural correspondence between bounded analytic semigroups and sectorial operators. In fact, bounded analytic semigroups are characterized by the property that there exists a $\omega \in (0, \frac{\pi}{2})$ such that

$$\sigma(-A) \subset \{0\} \cup \Sigma_\omega \quad \text{and} \quad \sup_{z \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}} \|z(z + A)^{-1}\| < \infty, \quad \forall \omega' \in (\omega, \pi).$$

Operators $-A$ of the latter form are called *sectorial* (of angle less than $\frac{\pi}{2}$). For sectorial operators $-A$, the holomorphic functional calculus is a well-studied subject. Very roughly speaking, this calculus is a way to make sense of "$f(-A)$" for scalar-valued functions that are holomorphic on a domain that "strictly contains" the spectrum of $-A$. This is done by using an operator version of the Cauchy formula, the Riesz–Dunford integral,

$$f(-A) = \frac{1}{2\pi i} \int_{\partial \Sigma_{\omega'}} f(z)(\lambda + A)^{-1}\, dz,$$

where $\omega < \omega' < \theta$ and f is a bounded analytic function on Σ_θ decaying suitably at 0 and ∞. This construction can be extended to the whole of $H^\infty(\Sigma_\theta)$, the space of bounded analytic functions on Σ_θ, but will in general lead to unbounded operators $f(-A)$. If indeed $f(-A) \in \mathcal{L}(X)$ for all $f \in H^\infty(\Sigma_\theta)$, then the $H^\infty(\Sigma_\theta)$-calculus is called *bounded*. For a detailed description of the construction we refer to the excellent monograph by Haase [12]. When dealing with analytic semigroups, it is controversial whether "A" or "$-A$" denotes the generator — the latter being common in the study of maximal regularity. In this paper, we have decided to stick to the convention "A" as this is the usual choice in systems theory.

Let U be a Banach space and $I \subset [0, \infty)$ be an interval. In this paper the function space $Z(I; U)$ — using the notation $Z(I) = Z(I; \mathbb{C})$ for $U = \mathbb{C}$ — will always refer to either of the following Banach spaces of U-valued functions: the continuous functions $\mathrm{C}(I; U)$, $L^\infty(I; U)$, $L^1(I; U)$ or an U-valued Orlicz space $E_\Phi(I; U)$ with respect to the Lebesgue measure on the interval I. This includes the L^p-spaces with $p \in (1, \infty)$. Let us briefly introduce the spaces $E_\Phi(I; U)$ — a detailed exposition may e.g. be found in [24]. In what follows, let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function, i.e. Φ is continuous, convex and increasing with $\lim_{x \to 0} \frac{\Phi(x)}{x} = 0$ and $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$. Then the set of Bochner measurable functions $f : I \to U$ such that there exists $k > 0$ with $\Phi(\|kf(\cdot)\|)$ integrable becomes a Banach space, denoted by $L_\Phi(I; U)$, with the norm

$$\|u\|_{L_\Phi(I;U)} = \inf \left\{ k > 0 \colon \int_I \Phi\left(\frac{\|u(s)\|}{k}\right) ds \leq 1 \right\}. \tag{5}$$

Although L_Φ is commonly referred to as *Orlicz space* in the literature, we rather would like to call the closure of boundedly supported $L^\infty(I; U)$ functions,

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$$E_\Phi(I;U) = \overline{\{f \in L^\infty(I;U) \colon \text{ess supp}(f) \text{ bounded}\}}^{\|\cdot\|_{L_\Phi(I;U)}},$$

the *Orlicz space with Young function* Φ and let $\|\cdot\|_{E_\Phi(I;U)} = \|\cdot\|_{L_\Phi(I;U)}$. Note that the inclusion $E_\Phi(I;U) \subset L_\Phi(I;U)$ is strict in general, but, for example, becomes an equality in the case of L^p spaces, $p \in (1,\infty)$. Also note that for any $u \in E_\Phi(I;U)$, $\Phi \circ \|u(\cdot)\|$ is integrable. Note that the following definition of Z*-admissible operators* particularly covers the common case of L^2-admissible operators.

Definition 3 *(Admissibility).* Let U, Y be Banach spaces and let Z be either E_Φ, L^1, L^∞ or C (see the notation above).

(1) An operator $B \in \mathcal{L}(U, X_{-1})$ is called *(finite-time) Z-admissible (control) operator* for A if, for all $t > 0$, the operator

$$\Phi_t := \Phi_t^B : Z(0,t;U) \to X_{-1}, u \mapsto \int_0^t T_{-1}(s)Bu(s)\,ds$$

has range in X, i.e. ran $\Phi_t \subset X$ and is thus bounded from $Z(0,t;U)$ to X.

(2) An operator $C \in \mathcal{L}(X_1, Y)$ is called (finite-time) Z-admissible observation operator for A if, for all $t > 0$,

$$\Psi_t := \Psi_t^C : X_1 \to Z(0,t;Y), x \mapsto CT(\cdot)x$$

has a bounded extension to X, which we denote again by Ψ_t.

If $\sup_{t>0} \|\Phi_t\| < \infty$ or $\sup_{t>0} \|\Psi_t\| < \infty$, then B or C is called infinite-time Z-admissible, respectively, in which case we write $\|B\|_{adm} := \sup_{t>0} \|\Phi_t\|_{\mathcal{L}(Z,X)}$ and $\|C\|_{adm} := \sup_{t>0} \|\Psi_t\|$.

Note that $B \in \mathcal{L}(U, X_{-1})$ is Z-admissible if and only if ran $\Phi_t \subset X$ for some $t > 0$. This can be seen using the semigroup property and that $u \in Z(0,t;U)$ implies $u(\tau + \cdot)|_{[0,s]} \in Z(0,s;U)$ for all $\tau \geq 0$, $s > 0$ and where u is identified with its extension on $\mathbb{R}$ by zero. It is easily seen that the latter holds true for our choices of Z. For exponentially stable C_0-semigroups, finite-time Z-admissibility is equivalent to infinite-time Z-admissibility, see [13, Lemma 8]. The following proposition confirms the intuition that the results which hold for input spaces $U = \mathbb{C}$ generalize to finite-dimensional spaces U — for choices like $Z = L^2$ this is folklore.

Proposition 4. *Let U be a finite dimensional Banach space and let Z be either E_Φ, L^1, L^∞ or* C. *Then for any semigroup generator A and $B \in \mathcal{L}(U, X_{-1})$ it holds that B is (infinite-time) Z-admissible if and only if Bf is (infinite-time) Z-admissible for every $f \in U$.*

Proof. Let $t > 0$. For $f \in U$, $u \mapsto u \cdot f$ is continuous from $Z(0,t;\mathbb{C})$ to $Z(0,t;U)$. Thus, if $\Sigma(A,B)$ is (infinite-time) Z-admissible then $\Sigma(A,Bf)$ is (infinite-time) Z-admissible. In order to prove the converse implications, we choose a basis $e_1,\dots,e_n$ of U. Assume that $\Sigma(A,Bf)$ is Z-admissible for every $f \in U$. Let $t > 0$ and $u \in Z(0,t;U)$. Then u can be written as $u = \sum_{k=1}^n u_k e_k$ with $u_k \in Z(0,t)$ for $k = 1,\dots,n$ and

$$\int_0^t T_{-1}(s)Bu(s)\,ds = \sum_{k=1}^n \int_0^t T_{-1}(s)Be_k u_k(s)\,ds \in X, \tag{6}$$

which implies that $\Sigma(A, B)$ is Z-admissible. That the implication also holds true for infinite-time admissibility, now follows from the uniform boundedness principle. In fact, through (6), pointwise boundedness of the family $(\Phi_t^B)_{t>0} \subset \mathcal{L}(Z(0,t;U), X)$ is implied by pointwise boundedness of $(\Phi_t^{Be_k})_{t>0,k\in\{1,..,n\}}$, where the notation from Definition 3 is used. □

Note that Proposition 4 does not generalize to the case where U is infinite-dimensional, see e.g. Proposition 14 or [17] (for the special cases of $Z = L^2$).

2. Orlicz space admissibility for finite-dimensional input spaces

As mentioned in the introduction, if B is L^p-admissible with $p \in [1,\infty)$, then the mild solutions of (1) are continuous. The analogous result holds for Z-admissibility if Z is some Orlicz space, see Proposition 5 below. In this section we give sufficient conditions for such Z-admissibility. Note that there exist operators that are L^∞-admissible but not L^p-admissible for any $p < \infty$, [13, Ex. 5.2].

Proposition 5. *Let X, U be Banach spaces and let A generate a C_0-semigroup on X. If $B : U \to X_{-1}$ is E_Φ-admissible for some Young function Φ, then B is L^∞-admissible and the mild solution $x : [0,\infty) \to X$ of (1) is continuous for any $u \in L^\infty_{loc}(0,\infty;U)$ and any initial value $x_0 \in X$.*

Proof. This follows from [13, Prop. 2.4] and the fact that E_Φ-admissibility of B implies that B is zero-class L^∞-admissibility, that is, B is L^∞-admissible and $\lim_{t\to 0^+} \|\Phi_t\| = 0$ with Φ_t from Definition 3. The latter can be argued by [13, Thm. 3.1 and Prop. 2.12]. □

The next technical result is at the core of what follows. Recall that although E_Φ-admissibility always implies L^∞-admissibility, the corresponding implication is no longer true for *infinite-time* admissibility, see e.g. [31].

Proposition 6. *Let A generate a bounded C_0-semigroup T on a Banach space X and let $x_0 \in X$. Suppose that $(-A_{-1})^{\frac{1}{2}}$ is an infinite-time L^2-admissible control operator, and let $f = (-A)^{\frac{1}{2}} T(\cdot)x_0$ either satisfy*

$$\text{(a) } f \in L^2(0,\infty;X), \quad \text{or} \quad \text{(b) } \|f(\cdot)\|_X^2 \in L_\Psi(0,\infty) \text{ for some Young function } \Psi.$$

If (a) holds, then $B = A_{-1}x_0$ is infinite-time L^∞-admissible. If (b) holds, then there exists a Young function Φ and $C > 0$ such that

$$\left\| \int_0^t T_{-1}(s)A_{-1}x_0 u(s)\,ds \right\|_X \le C\|u\|_{E_\Phi(0,t)}, \quad \forall u \in E_\Phi(0,t), t > 0.$$

Thus $B = A_{-1}x_0 \in \mathcal{L}(\mathbb{C}, X_{-1})$ is infinite-time E_Φ-admissible.

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Proof. Recall that A even generates a bounded analytic semigroup since $(-A_{-1})^{\frac{1}{2}}$ is L^2-admissible, [2, Prop. 2.7]. We first show that B is (finite-time) L^∞-admissible. Therefore, we need to prove that $\Phi_t^B u \in X_{-1}$ lies in X for all $u \in L^\infty(0,t)$ for $t > 0$. From either of *(a)* or *(b)* it follows that (the restriction of) f lies in $L^2(0, \frac{t}{2}; X)$ and thus also $u(2\cdot) f \in L^2(0, \frac{t}{2}; X)$. By L^2-admissibility of $(-A_{-1})^{\frac{1}{2}}$ this implies that

$$\Phi_{\frac{t}{2}}^{(-A_{-1})^{\frac{1}{2}}}(u(2\cdot)f) \in X.$$

By definition of $\Phi_{\frac{t}{2}}^{(-A_{-1})^{\frac{1}{2}}}$ and f, also using that $A_{-1}T_{-1}(s) = T_{-1}(s)A_{-1}$ on X,

$$\begin{aligned}
\Phi_t^B u &= \int_0^t T_{-1}(s)Bu(s)\,ds \\
&= -\int_0^t T_{-1}(\tfrac{s}{2})(-A_{-1})^{\frac{1}{2}} u(s)(-A)^{\frac{1}{2}} T(\tfrac{s}{2})x_0\,ds \\
&= -\frac{1}{2}\int_0^{\frac{t}{2}} T_{-1}(s)(-A_{-1})^{\frac{1}{2}} u(2s) f(s)\,ds \\
&= -\frac{1}{2}\Phi_{\frac{t}{2}}^{(-A_{-1})^{\frac{1}{2}}}(u(2\cdot)f) \in X.
\end{aligned}$$

We conclude that $B = A_{-1}x_0$ is L^∞-admissible and that

$$\begin{aligned}
\|\Phi_t^B u\|_X &\le \frac{1}{2}\|\Phi_{\frac{t}{2}}^{(-A_{-1})^{\frac{1}{2}}}\| \|u(2\cdot)f\|_{L^2(0,\frac{t}{2};X)} \\
&\le \frac{1}{2\sqrt{2}}\|(-A_{-1})^{\frac{1}{2}}\|_{adm}\|uf(\tfrac{\cdot}{2})\|_{L^2(0,t;X)}.
\end{aligned}$$

In case of *(a)*, that is, $f \in L^2(0,\infty;X)$, we can estimate the right-hand side by $c\|u\|_{L^\infty(0,t)}\|f\|_{L^2(0,\infty;X)}$ for some $c > 0$ independent of t. This shows that $B = A_{-1}x_0$ is infinite-time L^∞-admissible.

Now assume that *(b)* holds, hence $g : s \mapsto \|f(\frac{s}{2})\|^2$ lies in $L_\Psi(0,\infty)$. Hence, by Hölder's inequality for Orlicz spaces see e.g. [13,24],

$$\begin{aligned}
\|u(\cdot)f(\tfrac{\cdot}{2})\|_{L^2(0,t;X)} &\le \|u^2\|_{L_{\tilde\Phi}(0,t)}^{\frac{1}{2}} \|g\|_{L_\Psi(0,t)}^{\frac{1}{2}} \\
&\le \|u\|_{L_\Phi(0,t)} \|g\|_{L_\Psi(0,\infty)}^{\frac{1}{2}},
\end{aligned}$$

where $\tilde\Phi$ and Ψ are complementary Young functions and $\Phi(x) = \tilde\Phi(x^2)$, which is a Young function since it is the composition of two Young functions. Therefore,

$$\exists C > 0\ \forall t > 0, u \in L^\infty(0,t): \quad \left\| \Phi_t^B u \right\|_X \leq C\|u\|_{L_\Phi(0,t)} = C\|u\|_{E_\Phi(0,t)}.$$

Thus, B is infinite-time E_Φ-admissible since $L^\infty(0,t)$ is dense in $E_\Phi(0,t)$. □

By an important property of Orlicz spaces, condition *(a)* always implies *(b)* in Proposition 6. This enables us to prove the main result of this section.

Theorem 7. *Let A generate a bounded C_0-semigroup T on a Banach space X. Suppose that $(-A)^{\frac{1}{2}}$ is an infinite-time L^2-admissible observation operator and that $(-A_{-1})^{\frac{1}{2}}$ is an infinite-time L^2-admissible control operator.*

Then for any $B \in \mathcal{L}(U, X_{-1})$ with $\dim U < \infty$ and $\operatorname{ran} B \subset \operatorname{ran} A_{-1}$, it holds that

(i) B is infinite-time L^∞-admissible, and
(ii) there exists a Young function Φ such that B is infinite-time E_Φ-admissible.

Proof. By Proposition 4, it suffices to consider $B \in \operatorname{ran} A_{-1}$. We choose $x \in X$ such that $B = A_{-1}x$. Since $(-A)^{\frac{1}{2}}$ is an infinite-time L^2-admissible observation operator, $f(s) = (-A)^{\frac{1}{2}}T(s)x \in L^2(0,\infty;X)$ and since $\|(-A)^{\frac{1}{2}}T(\cdot)x\|$ is bounded on (τ,∞), for all $\tau > 0$, Lemma 29 in the Appendix implies that $\|f(\cdot)\|_X^2 \in L_\Psi(0,\infty)$ for some Young function Ψ. Therefore, the assumptions of Proposition 6, particularly both *(a)* and *(b)*, are satisfied. Hence the assertions (i) and (ii) follow. □

It is well-known that the conditions on T in Theorem 7 are naturally linked with an equivalent norm of the space X, [26]. In fact, for $x \in X$ and $u(s) = (-A)^{\frac{1}{2}}T(s)x$, the infinite-time L^2-admissibility gives, for $t > 0$,

$$\|T(t)x - x\|_X = \|\Phi_t^{(-A_{-1})^{\frac{1}{2}}} u\|_X \leq C_1 \|\Psi_t^{(-A)^{\frac{1}{2}}} x\|_{L^2(0,t;X)} \leq C_2\|x\|,$$

where C_1 and C_2 do not depend on t. If we additionally assume that T is strongly stable, i.e. $\lim_{t\to\infty} T(t)x = 0$ for all $x \in X$, then $x \mapsto \|\Psi_t^{(-A)^{\frac{1}{2}}} x\|_{L^2(0,\infty;X)}$ becomes an equivalent norm — which is Hilbertian if X is a Hilbert space — with respect to which T is contractive. From a system theoretic view-point this equivalence of norms means nothing else than the property that the system

$$\dot{x} = Ax, \qquad x(0) = x_0,$$
$$y = (-A)^{\frac{1}{2}}x$$

is *exactly observable*. We point out that since T is a bounded analytic semigroup in this situation, strong stability of T is equivalent to A having dense range, see e.g. [5, Cor. III.3.17]. Moreover, it is known that infinite-time L^2-admissibility of $(-A)^{\frac{1}{2}}$ and $(-A_{-1})^{\frac{1}{2}}$ together with A having dense range is equivalent to A satisfying square function estimates of the form[1]

[1] In the literature, the notion 'A satisfies square function estimates' typically refers to the version of (7) where only the second inequality is satisfied.

$$\forall \phi_0 \in H_0^\infty(\mathbb{C}_-) \exists k, K > 0 \, \forall x \in X \quad k\|x\|^2 \leq \int_0^\infty \|\phi_0(tA)x\|^2 \frac{dt}{t} \leq K\|x\|^2, \tag{7}$$

where $H_0^\infty(\mathbb{C}_-) = \{f \in H^\infty(\mathbb{C}_-) \colon \exists c, s > 0 : \|f(z)| \leq \frac{c|z|^s}{(1+|z|)^{2s}}\}$, see e.g. [11,26,27]. Note that for $\phi_0(z) = (-z)^{\frac{1}{2}} e^{-z}$, the inequality in (7) is nothing else than the previously mentioned equivalence of the norms $\|\cdot\|$ and $\|\Psi_t^{(-A)^{\frac{1}{2}}} \cdot \|_{L^2(0,\infty;X)}$. It can also be shown that T is contractive with respect to any of the equivalent norms induced by ϕ_0 via (7). For Hilbert spaces X there is a converse: Any generator, with dense range, of a bounded analytic semigroup that is similar to a contractive semigroup satisfies (7). This is a consequence of McIntosh's theorem stating that square function estimates of the form (7) characterize a bounded H^∞-calculus when X is a Hilbert space [12, Theorem 7.3.1] and results due to Le Merdy [25], and independently derived by Franks and Grabowski–Callier (see [12] and the references therein). With this and Proposition 5, Theorem 7 leads to the following corollary.

Corollary 8. *Let A generate a bounded analytic semigroup T on a Hilbert space and suppose that A has dense range. If A has a bounded H^∞-calculus, or equivalently if T is similar to a contraction semigroup, then for every $B \in \mathcal{L}(U, X_{-1})$ with* $\dim U < \infty$ *and* $\operatorname{ran} B \subset \operatorname{ran} A_{-1}$,

(i) there exists a Young function Φ such that B is infinite-time E_Φ-admissible.
(ii) the mild solutions of (1) are continuous for any $u \in L^\infty_{loc}(0, \infty; U)$.

By rescaling the results in this section can be adapted to analytic semigroups (instead of bounded analytic semigroups). Then, however, only finite-time admissibility is obtained in general.

3. L^∞-admissibility and bounded H^∞-calculus

The boundedness of the H^∞-calculus, the crucial condition in Theorem 7 and Corollary 8, may look artificially chosen in order to make the proofs, particularly of Proposition 6, work. The goal of this section is to demonstrate that this is not the case. In fact, we show that the converses of Theorem 7 and Corollary 8 hold for Hilbert spaces and explain what can be said in the case of more general Banach spaces. This reveals that the boundedness of the H^∞-calculus appears naturally in this context.

It will be convenient to use the following notation for spaces of admissible operators.

$$\mathfrak{b}_\infty(A) = \{B \in \mathcal{L}(\mathbb{C}, X_{-1}) \colon B \text{ is infinite-time } L^\infty\text{-admissible for } A\}$$
$$\mathfrak{c}_1(A) = \{C \in \mathcal{L}(X_1, \mathbb{C}) \colon C \text{ is infinite-time } L^1\text{-admissible for } A\}.$$

Upon identification these spaces are contained in X_{-1} and $(X_1)'$, respectively. Note that under the conditions of Theorem 7, it holds that $\mathfrak{b}_\infty(A) \supseteq \operatorname{ran} A_{-1}$. We will shortly see that this inclusion is in fact an equality.

First we show that analyticity of the semigroup is necessary under the condition that $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$.

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Proposition 9. *Let A generate a semigroup T on a Banach space X. If $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$, then A generates a bounded analytic semigroup.*

Proof. We first show that T is bounded. For $x \in X$, $b = A_{-1}x$ is infinite-time L^∞-admissible, hence with $u(s) = 1$,

$$\sup_{t>0} \|T(t)x - x\| = \sup_{t>0} \left\| \int_0^t T_{-1}(s)A_{-1}xu(s)\, ds \right\| \le \sup_{t>0} \|\Phi_t^{A_{-1}x}\|_{\mathcal{L}(L^\infty(0,t),X)} < \infty,$$

whence T is bounded by the uniform boundedness principle. With $u(s) = e^{-i\omega s - \varepsilon s}$, $\omega \in \mathbb{R}$, $\varepsilon > 0$ we obtain for every $x \in X$

$$\begin{aligned}\int_0^\infty T_{-1}(s)A_{-1}xu(s)\, ds &= \int_0^\infty e^{-(i\omega+\varepsilon)s} T_{-1}(s)A_{-1}x\, ds \\ &= (i\omega + \varepsilon - A_{-1})^{-1} A_{-1}x \in X.\end{aligned}$$

By the assumption that $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$, we get $\|(i\omega + \varepsilon - A_{-1})^{-1}A_{-1}x\| \le C\|x\|_X$ for a constant C independent of x, ω and ε. Thus,

$$\begin{aligned}\|i\omega(i\omega + \varepsilon - A)^{-1}x\| &= \|(i\omega - (A_{-1} - \varepsilon))^{-1}(A_{-1} - \varepsilon)x + x\| \\ &\le \|(i\omega - (A_{-1} - \varepsilon))^{-1}A_{-1}x\| + \varepsilon\|(i\omega - (A - \varepsilon))^{-1}x\| + \|x\| \\ &\le (C + M + 1)\|x\|,\end{aligned}$$

where $M \ge 1$ is such that $\|(\lambda - A)^{-1}\| \le \frac{M}{\operatorname{Re}\lambda}$ for all $\operatorname{Re}\lambda > 0$. Therefore, $A - \varepsilon$ generates a bounded analytic semigroup, see [6,32]. Moreover, since the sectorality constant

$$\sup_{\operatorname{Re} z>0} \|z(z - (A - \varepsilon))^{-1}\| \le C + M + 1,$$

is bounded independently of ε, and thus

$$(1 - A)^{-1} - (1 - (A - \varepsilon))^{-1} = \varepsilon(1 - (A - \varepsilon))^{-1}(1 - A)^{-1} \to 0,$$

as $\varepsilon \to 0$, the sequence of operators $(-A + \frac{1}{n})_{n\in\mathbb{N}}$ forms a sectorial approximation for $-A$ (on some sector Σ_θ, $\theta \in (0, \frac{\pi}{2})$), see [12, Sec. 2.1.2]. Thus A also generates a bounded analytic semigroup. □

Theorem 10. *Let A generate a semigroup on a Banach space X and let A have dense range. Consider the following assertions.*

(i) $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$
(ii) *A generates a bounded analytic semigroup T and $-A$ has a bounded $H^\infty(\Sigma_\theta)$-calculus for any $\theta \in (\frac{\pi}{2}, \pi)$*

(iii) $\mathfrak{c}_1(A) = \{\langle y', A\cdot\rangle_{X',X} : y' \in X'\}$

Then (i) ⇒ (ii) ⇔ (iii). If X is reflexive, then also (ii) ⇒ (i).

Before we give a proof of Theorem 10, let us discuss the relation to the results in Section 2. Note that a bounded $H^\infty(\Sigma_\theta)$-calculus for $\theta > \frac{\pi}{2}$ will in general not imply a bounded $H^\infty(\Sigma_\theta)$-calculus for any $\theta \le \frac{\pi}{2}$ (as e.g. required in Corollary 8), see [19] for a counterexample on a reflexive space. However, for Hilbert spaces it is known that any generator of a bounded analytic semigroup has a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta < \frac{\pi}{2}$ if and only if the calculus is bounded on some sector, see [27, Thm. 8, p. 225] for the original result by McIntosh or [12, Thm. 7.3.1]. This gives the following characterization, Corollary 11. If X is not a Hilbert space, we need an additional assumption: From a fundamental result by Kalton and Weis [21, Prop. 5.1], it follows that if $-A$ is R-sectorial of R-type $\omega_R < \frac{\pi}{2}$, i.e. $\sigma(-A) \subset \overline{\Sigma}_{\omega_R}$ and the set

$$\{\lambda R(\lambda, -A) : \lambda \in \overline{\Sigma_{\omega'}}^c\} \quad \text{is } R\text{-bounded} \quad \forall \omega' > \omega_R,$$

then the boundedness of the H^∞-calculus for some angle implies that the $H^\infty(\Sigma_\theta)$-calculus is bounded for $\theta > \omega_R$. Here, R-boundedness is a generalization of usual boundedness of sets of operators in $\mathcal{L}(X)$, see e.g. [21] for definitions. Note that since R-boundedness coincides with boundedness in the operator-norm on Hilbert spaces, the notions of R-sectorial of R-type ω_R and sectorality of angle ω_R coincide then. Another example of an R-sectorial operator with $\omega_R < \frac{\pi}{2}$ is given by analytic contraction semigroups of positive operators on L^p spaces with $p \in (1, \infty)$, see [21, Cor. 5.2].

Corollary 11. *Let A generate a bounded analytic semigroup on a Banach space X. Further assume that $-A$ is R-sectorial of angle $\omega_R \in (0, \frac{\pi}{2})$ — which is particularly satisfied if X is a Hilbert space — and A has dense range. Then the following assertions are equivalent.*

(i) $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$.
(ii) *$-A$ has a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta < \frac{\pi}{2}$.*
(iii) $\mathfrak{c}_1(A) = \{\langle y', A\cdot\rangle_{X',X} : y' \in X'\}$.

Proof. It remains to show that *(ii)* implies *(i)* (note that this automatically follows from Theorem 10 if X is reflexive) — the rest follows from Theorem 10 as argued above the corollary. For that we use a standard argument for the H^∞-calculus. For $u \in L^\infty(0,t)$ let $g \in H^\infty(\Sigma_\theta)$, $\theta \in (0, \frac{\pi}{2})$, be defined by $g(z) = \int_0^t h(zs)u(s)\frac{ds}{s}$, where $h(z) = ze^{-z}$. Since the $H^\infty(\Sigma_\theta)$-calculus is bounded by assumption, we have that $g(-A) \in \mathcal{L}(X)$ and in particular

$$g(-A)x = \int_0^t h(-sA)x\, u(s)\,\frac{ds}{s} = \int_0^t T(s)A_{-1}x\, u(s)\, ds \in X$$

for all $x \in X$. Since moreover $\|g(-A)\| \le C\|g\|_{H^\infty(\Sigma_\theta)} \le C_\theta \|u\|_{L^\infty(0,t)}$ where C, C_θ are independent of t, we conclude that $A_{-1}x \in \mathfrak{b}_\infty(A)$. □

For the proof of Theorem 10, we need the following lemma.

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Lemma 12. *Let A generate a semigroup T on a Banach space X. If either $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$ or $\mathfrak{c}_1(A) = \{\langle y', A\cdot\rangle_{X',X} : y' \in X'\}$, then T is bounded analytic and there exists $C > 0$ such that*

$$\int_0^\infty |\langle y, AT(s)x\rangle|\, ds \le C\, \|x\|\, \|y\|, \quad x \in X, y \in X'. \tag{8}$$

Proof. Assume first that $\mathfrak{c}_1(A) = \{\langle y', A\cdot\rangle_{X',X} : y' \in X'\}$. Then, by definition of $\mathfrak{c}_1(A)$, the mapping $y' \mapsto \Psi_\infty^{\langle y', A\cdot\rangle} x = \langle y', AT(\cdot)x\rangle_{X',X}$ is linear and well-defined from X' to $L^1(0,\infty)$ for any $x \in X$. An application of the closed graph theorem, yields that this is even a bounded operator for every x and thus, by the uniform boundedness principle, there exists $C > 0$ such that (8) holds. That T is bounded analytic follows by a similar argument as in the proof of Proposition 9. Now assume that $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$, which implies that T is bounded analytic by Proposition 9 and furthermore

$$\Psi_{u,t} : X \mapsto X, x \mapsto \int_0^t AT(s)x\, u(s)\, ds,$$

is well-defined for any $u \in L^\infty(0,t)$ and $t > 0$. Let $x_n \to x$ and $\Psi_{u,t}x_n \to y$ in X. Since $\Psi_{u,t}x_n \to \Psi_{u,t}x$ in X_{-1} by the boundedness of the integrand (bounded in X_{-1}), we conclude that $\Psi_{u,t}x = y$. Thus, $\Psi_{u,t}$ is bounded by the closed graph theorem. We further claim that $\Psi_{u,t}$ is uniformly bounded for $t > 0$ and $u \in \mathcal{U}_t = \{u \in L^\infty(0,t) : \|u\|_\infty = 1\}$. In fact, for fixed $x \in X$ it follows again by $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$ that $\sup_{u \in \mathcal{U}_t, t>0} \|\Psi_{u,t}x\| < \infty$. Hence, by the uniform boundedness principle, $C := \sup_{u\in\mathcal{U}_t, t>0} \|\Psi_{u,t}\| < \infty$.

Let $x \in X$ and $y \in X'$. Define $u(s) = \exp(-i \arg\langle y, AT(s)x\rangle)$, $s \in (0,t)$ which lies in $\mathcal{U}_t$. With this, we obtain

$$\begin{aligned}
\int_0^t |\langle y, AT(s)x\rangle|\, ds &= \int_0^t \langle y, AT(s)x\rangle\, u(s)\, ds \\
&= \langle y, \int_0^t AT(s)x\, u(s) ds\rangle \\
&\le \|y\|\, \|\Psi_{u,t}x\| \\
&\le C\, \|y\|\, \|x\|. \quad \square
\end{aligned}$$

Proof of Theorem 10. By Lemma 12, both *(i)* or *(iii)* respectively imply that A generates a bounded analytic semigroup and that

$$\exists C > 0 \quad \forall x \in X, y' \in X' \quad \|\langle y, AT(\cdot)x\rangle\|_{L^1(0,\infty)} \le C\, \|x\|\, \|y\|. \tag{9}$$

The latter condition is known as *weak square function estimate* and first appeared in the seminal paper by Cowling, Doust, McIntosh and Yagi [3]. By [3, Cor. 4.5 and Ex. 4.8], it is equivalent to $-A$ having a bounded $H^\infty(\Sigma_\theta)$-calculus for $\theta > \frac{\pi}{2}$. Thus, *(i)* ⇒ *(ii)* and *(iii)* ⇒ *(ii)*.

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By definition of $\mathfrak{c}_1(A)$, *(ii)* implies via (9) that $\mathfrak{c}_1(A) \supseteq \{\langle y', A\cdot\rangle : y' \in X'\}$. To see the other inclusion, let $c \in \mathfrak{c}_1(A)$ and define $y' \in X'$ by $\langle y', x\rangle_{X',X} = -\int_0^\infty \langle c, T(s)x\rangle\, ds$ for $x \in X_1$. Using that $\lim_{t\to\infty} T(t)x = 0$, which follows by analyticity and as A has dense range, one easily deduces that $\langle c, x\rangle = \langle y', Ax\rangle$ for $x \in D(A^2)$ and thus all $x \in X_1$. This shows that *(ii)* $\Rightarrow$ *(iii)*.

As discussed above, *(ii)* is equivalent to (9) which also implies that $\mathfrak{c}_1(A^*) \supseteq \{\langle x, A^*\cdot\rangle_{X,X'} : x \in X\}$. If X is reflexive, then the latter inclusion of sets is even an equality by a similar argument as for the implication *(ii)* $\Rightarrow$ *(iii)*. By [38, Thm. 6.9(ii)], using again reflexivity, we conclude that $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$ — in fact, for $b \in X_{-1}$, we have that $b \in \mathfrak{b}_\infty(A)$ if and only if $b^* \in \mathfrak{c}_1(A^*)$ by [38, Thm. 6.9(ii)]. Hence *(ii)* $\Rightarrow$ *(i)*. □

In the proof of Theorem 10 weak square function estimates seem to be the right choice to characterize bounded H^∞-calculus. However, these are somehow 'exotic' compared to the classic square functions (in the context of functional calculus). We refer to [9] for a detailed discussion of their relations.

Remark 13. As we have seen, the condition $\mathfrak{b}_\infty(A) = X_{-1}$, i.e. L^∞-admissibility for any possible scalar input, can be rewritten as

$$\int_0^t AT(s)xu(s)\, ds \in X, \tag{10}$$

for all $x \in X$, and $u \in L^\infty(0,t)$. As $\|AT(s)\|$ behaves like s^{-1} for analytic semigroups, (10) can be seen as a condition on the convergence of a singular integral. In other words, it is a 'unconditionality' condition, which is a natural phenomenon for a bounded H^∞-calculus, see [12, Theorem 5.6.2].

4. The Weiss conjecture and some counterexamples

Corollary 8 shows that for bounded analytic semigroups on Hilbert spaces with bounded H^∞-calculus the set of L^∞-admissible operators with finite-dimensional input space U is as large as possible — it equals $\mathcal{L}(U, X_{-1})$ if additionally $0 \in \rho(A)$. One may ask what does happen for infinite-dimensional spaces U. For that, let us draw the connection to the *Weiss-conjecture* for control operators: As indicated in the introduction, G. Weiss formulated the problem whether infinite-time L^2-admissibility follows already from a necessary condition on the resolvent, which is derived considering $\Phi_\infty u$ with $u(s) = e^{-\lambda s}u$, $u \in U$, $\operatorname{Re}\lambda > 0$, see Definition 3. By using the Laplace transform and Hölder's inequality, this yields that infinite-time L^p-admissibility, $p \in [1,\infty)$, implies that

$$\sup_{\operatorname{Re}\lambda>0} \|(p\operatorname{Re}\lambda)^{\frac{1}{p}} R(\lambda, A_{-1})B\|_{\mathcal{L}(U,X)} < \infty.$$

For $p = \infty$, this gives

$$\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A_{-1})B\|_{\mathcal{L}(U,X)} < \infty. \tag{11}$$

In analogy to $\mathfrak{b}_\infty(A)$, let us introduce the sets

$$\mathfrak{B}_{\infty,U}(A) := \{B \in \mathcal{L}(U, X_{-1}) \colon B \text{ is infinite-time } L^\infty\text{-admissible}\}.$$

This means that the Weiss conjecture for $p = \infty$ asks whether $\mathfrak{B}_{\infty,U}(A)$ equals $\{B \in \mathcal{L}(U, X_{-1})\colon \sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A_{-1})B\|_{\mathcal{L}(U,X)} < \infty\}$. Note that if the semigroup is bounded analytic, then (11) is satisfied for any $B = b \in \operatorname{ran} A_{-1}$. Thus, by Corollary 11 and Proposition 4, the Weiss conjecture for $p = \infty$ and finite-dimensional U has an affirmative answer for generators A on Hilbert spaces if the following conditions are met

- A generates a bounded analytic semigroup and has dense range,
- A has a bounded H^∞-calculus.

For a converse implication see Remark 17 below. However, for infinite-dimensional U this cannot be expected in general as the following result shows.

Proposition 14. *Let X be a Hilbert space with orthonormal basis $\{e_n\}$ and let $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}_-$ such that $(\operatorname{Re}\lambda_n)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence with $\lim_{n\to\infty} \operatorname{Re}\lambda_n = -\infty$ and $|\operatorname{Im}\lambda_n| \le k|\operatorname{Re}\lambda_n|$ for some $k > 0$ and all $n \in \mathbb{N}$. Let $A : D(A) \subset X \to X$ be given by*

$$Ae_n := \lambda_n e_n, \qquad D(A) := \{x = \sum_n x_n e_n \mid \sum_n |\lambda_n x_n|^2 < \infty\}.$$

Further, we define $U := X$ and $B \in L(U, X_{-1})$ by $B := A_{-1}$. Then we have:

(i) For every $u \in U$, the system $\Sigma(A, Bu)$ is L^∞-admissible.
(ii) $\Sigma(A, B)$ is not L^∞-admissible.
(iii) $\Sigma(A, B)$ satisfies the resolvent condition (11).

Proof. The operator A generates an exponentially stable analytic diagonal semigroup on X. Thus (i) follows from Corollary 8 and (iii) is clear by the resolvent identity and since $-A$ is sectorial. Thus it remains to prove (ii). We choose a subsequence $(\gamma_n)_{n\in\mathbb{N}}$ of $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$\operatorname{Re}\gamma_1 \le -1 \quad \text{and} \quad \operatorname{Re}\gamma_{n+1} < 2\operatorname{Re}\gamma_n, \qquad n \in \mathbb{N}.$$

Let $u \in L^\infty(0, 1; X)$ be given by $u(s) = \sum (u(s))_n e_n$ with

$$(u(s))_n := \begin{cases} 0, & \lambda_n \neq \gamma_m \text{ for all } m \in \mathbb{N}, \\ 1, & \lambda_n = \gamma_m \text{ for some } m \in \mathbb{N} \text{ and } s \in \left[-\frac{1}{2\operatorname{Re}\gamma_m}, -\frac{1}{\operatorname{Re}\gamma_m}\right]. \end{cases}$$

It is easy to see that $u \in L^\infty(0, 1; X)$ with $\|u\| = 1$. However,

$$\begin{aligned} \left\| \int_0^1 T_{-1}(s)Bu(s)\,ds \right\|^2 &= \sum_{n=1}^\infty \left| \int_0^1 e^{\lambda_n s} \lambda_n (u(s))_n\,ds \right|^2 \\ &= \sum_{m=1}^\infty \left(e^{-1/2} - e^{-1}\right)^2 \end{aligned}$$

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$$= \infty,$$

which shows that $\Sigma(A, B)$ is not L^∞-admissible. Note, however, that A has a bounded H^∞-calculus as the generated semigroup is a contraction semigroup. □

Proposition 14 also shows that Proposition 4 fails for infinite-dimensional spaces U and $Z = L^\infty$, and hence shows that the so-called *weak Weiss conjecture* is not true in this situation. Moreover, the example is in line with the following known characterization of when the L^∞-Weiss conjecture actually holds. For reflexive spaces X this follows already from the dual situation of L^1-admissible operators in [11, Sektion 2.3] and Bounit–Driouich–El-Mennaoui [2] by using [38, Thm. 6.9]. For completeness we provide a proof of the general case.

Theorem 15. *Let A generate a semigroup T on a Banach space X. Then the following assertions are equivalent.*

(i) A_{-1} is infinite-time L^∞-admissible,
(ii) T is bounded analytic and the L^∞-Weiss conjecture for A holds, i.e.

$$\mathfrak{B}_{\infty,U}(A) = \{B \in \mathcal{L}(U, X_{-1}) \colon \sup_{\lambda>0} \|R(\lambda, A_{-1})B\|_{\mathcal{L}(U,X)} < \infty\}$$

for all Banach spaces U.

If additionally $0 \in \rho(A)$, then (i) and (ii) are further equivalent to

(iii) T is bounded analytic and $\mathfrak{B}_{\infty,U}(A) = \mathcal{L}(U, X_{-1})$ for all Banach spaces U.

Proof. By the resolvent identity, it follows easily that $\|R(\lambda, A_{-1})A_{-1}\|$ is uniformly bounded for $\operatorname{Re}\lambda > 0$ if T is bounded analytic. Hence, *(ii)* implies *(i)* and if $0 \in \rho(A)$, then *(ii)* is equivalent to *(iii)*. Hence it remains to show *(i)* $\implies$ *(ii)*. We assume that A_{-1} is infinite-time L^∞-admissible. Then $\mathfrak{b}_\infty(A) = \operatorname{ran} A_{-1}$ and thus T is bounded analytic by Proposition 9. Let $B \in \mathcal{L}(U, X_{-1})$ for some Banach space U with $C := \sup_{\lambda>0} \|R(\lambda, A_{-1})B\|_{\mathcal{L}(U,X)} < \infty$. For $\lambda > 0$ we can write $B = (\lambda - A_{-1})B_\lambda$ for $B_\lambda = (\lambda - A_{-1})^{-1}B \in \mathcal{L}(U, X)$. Therefore, for $u \in L^\infty(0, t; U)$,

$$\int_0^t T_{-1}(s)Bu(s)\,ds = \lambda \int_0^t T(s)B_\lambda u(s)\,ds - \int_0^t T_{-1}(s)A_{-1}B_\lambda u(s)\,ds \in X$$

by the assumption that A_{-1} is L^∞-admissible and since $B_\lambda u(\cdot) \in L^\infty(0, t; X)$. Choosing $\lambda = \frac{1}{t}$,

$$\begin{aligned} \left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| &\le \left(\sup_{s>0} \|T(s)\| + \|\Phi_t^{A_{-1}}\| \right) \|B_{t^{-1}}\| \, \|u\|_{L^\infty(0,t;U)} \\ &\le \left(\sup_{s>0} \|T(s)\| + \|\Phi_t^{A^{-1}}\| \right) C \, \|u\|_{L^\infty(0,t;U)} \end{aligned}$$

where we used that $\|B_{t^{-1}}\| = \|R(t^{-1}, A_{-1})B\| \le C$. Since A_{-1} is infinite-time admissible, $\sup_{t>0} \|\Phi_t^{A_{-1}}\|_{\mathcal{L}(L^\infty(0,t;U),X)} < \infty$ and we conclude that B is infinite-time L^∞-admissible. □

We emphasize that A_{-1} being infinite-time L^∞-admissible is a very strong and restrictive condition; much stronger than A having a bounded H^∞-calculus as Proposition 14 shows. Moreover, it is an open question if the condition already implies that A is a bounded operator. Under an even (slightly) stronger condition, this can indeed be proved, as the following result shows. For that recall the following refinement of admissibility: a Z-admissible operator B is called zero-class Z-admissible if

$$\sup_{\|u\|_{Z(0,t;U)}\le 1} \left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| \to 0, \quad \text{as } t \searrow 0.$$

Proposition 16. *Let A generate a C_0-semigroup T and let Z be* C, *L^∞, L_1 or E_Φ. Then A_{-1} is a zero-class Z-admissible operator if and only if A is bounded.*

Proof. If A is bounded, then A_{-1} is easily seen to be zero-class Z-admissible for any of the considered choices of Z. To show the converse, note that it suffices to consider the case where Z refers to the continuous functions C as $\mathrm{C}(I)$ is embedded in $L^1(I)$ and $E_\Phi(I)$ for bounded intervals I. For $x \in X$, $\|x\| = 1$ choose the constant function $u(\cdot) = x$. Then

$$\|T(t)x - x\|_X = \left\| \int_0^t AT(s)u(s)\,ds \right\|_X \le \|\Psi_t\| \|u\|_{L^\infty(0,t;U)} = \|\Psi_t\|,$$

where Ψ_t is defined as in Definition 3. Since $\|\Psi_t\| \to 0^+$ by zero-class admissibility, T is uniformly continuous and hence A is bounded (see e.g. [6, Cor.II.1.5]). □

Proposition 16 shows that if we could find an L^∞-admissible A_{-1} such that A is unbounded, then A_{-1} is not zero-class L^∞-admissible and thus not E_Φ-admissible for any Young function Φ. We remark that in order to show the existence of such an A_{-1}, it suffices to find an unbounded generator A on a reflexive space such that A is an L^1-admissible observation operator. This follows from the duality results in [38] mentioned above. The difficulty is the reflexivity of the space — examples of unbounded L^1-admissible observation operators A for $X = \ell^1(\mathbb{N})$ can be easily constructed by diagonal operators.

Remark 17. Let A generate a bounded analytic semigroup T on a Hilbert space X and assume that A has dense range. Setting our results, in particular Theorem 10, in context with Le Merdy's characterization of when the L^2-Weiss conjecture holds true, [26], we arrive at the following list of equivalent conditions:

(i) The L^2-Weiss conjecture holds true for A and A^* and any space U.
(ii) T is similar to a contraction semigroup.
(iii) The L^∞-Weiss conjecture holds true for A and any finite-dimensional U.

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In particular this shows that the Weiss conjecture "does not extrapolate" in the sense that its validity for L^2 does not imply the validity for L^∞.

5. Applications to input-to-state stability

The results of Sections 2 and 3 have direct consequences for notions of input-to-state stability, because of their characterization via admissibility, see Theorem 19 below. For the sake of completeness, we include the definitions for which we need the following function classes commonly used in Lyapunov theory.

$$\begin{aligned}
\mathcal{K} &= \{\mu\colon [0,\infty) \to [0,\infty) \mid \mu(0)=0,\ \mu \text{ continuous, strictly increasing}\},\\
\mathcal{K}_\infty &= \{\theta \in \mathcal{K} \mid \lim_{x\to\infty} \theta(x) = \infty\},\\
\mathcal{L} &= \{\gamma\colon [0,\infty) \to [0,\infty) \mid \gamma \text{ continuous, strictly decreasing}, \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} &= \{\beta\colon [0,\infty)^2 \to [0,\infty) \mid \beta(\cdot,t) \in \mathcal{K}\ \forall t \ge 0 \text{ and } \beta(s,\cdot) \in \mathcal{L}\ \forall s > 0\}.
\end{aligned}$$

Definition 18 *(Input-to-state stability).* A system $\Sigma(A,B)$ of form (1) is called

- *input-to-state stable with respect to* Z (or Z-*ISS*), if there exist functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that for every $t \ge 0$, $x_0 \in X$ and $u \in Z(0,t;U)$,

$$x(t) \text{ lies in } X \text{ and } \|x(t)\| \le \beta(\|x_0\|,t) + \mu(\|u\|_{Z(0,t;U)}).$$

- *integral input-to-state stable (integral ISS)* if there exist functions $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for every $t \ge 0$, $x_0 \in X$ and $u \in L^\infty(0,t;U)$,

$$x(t) \text{ lies in } X \text{ and } \|x(t)\| \le \beta(\|x_0\|,t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)ds\right).$$

The following result shows that in the case of linear systems (integral) ISS can be characterized by exponential stability and admissibility in certain norms.

Theorem 19 *([13]). A system $\Sigma(A,B)$ is*

- *Z-ISS if and only if A generates an exponentially stable semigroup and B is Z-admissible.*
- *integral ISS if and only if A generates an exponentially stable semigroup and B is E_Φ-admissible for some Young function Φ.*

For a generalization of this theorem for non-exponentially stable semigroups, see [31]. Using Theorem 19, Proposition 4 has the following version.

Proposition 20. *Let U be a finite dimensional Banach space and let Z be either E_Φ, L^1 or L^∞. Then we have for any generator A and $B \in \mathcal{L}(U, X_{-1})$ that*

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(i) $\Sigma(A, B)$ *is Z-ISS if and only if* $\Sigma(A, Bf)$ *is Z-ISS for any* $f \in U$,
(ii) $\Sigma(A, B)$ *is integral ISS if and only if* $\Sigma(A, Bf)$ *is integral ISS for any* $f \in U$.

With this we can formulate direct consequences of Theorem 7 and Corollary 8.

Corollary 21. *Let A generate an exponentially stable analytic semigroup T on a Banach space X. If additionally either*

(i) *X is a Hilbert space and T is similar to a contraction semigroup, or*
(ii) $(-A)^{\frac{1}{2}}$ *and* $(-A_{-1})^{\frac{1}{2}}$ *are L^2-admissible (observation/control) operators,*

then $\Sigma(A, B)$ is integral ISS for any $B \in \mathcal{L}(U, X_{-1})$ with finite-dimensional U. In particular, the notions of integral ISS and ISS coincide for $\dim U < \infty$.

Remark 22. Corollary 21 generalizes Theorem 4.1 in [13] in the case of exponentially stable parabolic diagonal systems on Hilbert spaces, that is, operators A of the form $Ae_n = \lambda_n e_n$, $n \in \mathbb{N}$, where $(e_n)_{n\in\mathbb{N}}$ is a Riesz-basis of X and $(\lambda_n)_{n\in\mathbb{N}}$ lie in a suitable sector. In fact, the generated semigroup is similar to a contraction semigroup. Conversely, if $(\lambda_n)_{n\in\mathbb{N}}$ is an *interpolating sequence*, then the Riesz-property of the basis is implied by similarity to a contraction semigroup, see [26]. This shows that the assumption of the Riesz-property in [13] is not necessary in general.

Remark 23. Corollary 21 can be generalized to the weaker notions of 'strong input-to-state stability' and 'strong integral ISS' which are discussed in [31]. In these notions, exponential stability of the semigroup is replaced by strong stability and the result follows as above by using a generalization of Theorem 19 from [31].

6. Discussion and outlook

As described in the introduction, an open question is if integral ISS is always implied by ISS for linear systems (1). As mentioned before, this can be rephrased as an operator-theoretic problem.

Question 24. *Is $B \in \mathcal{B}(U, X_{-1})$ an E_Φ-admissible control operator for some Φ provided that B is an L^∞-admissible control operator?*

This in turn can be seen as the question whether for $B \in \mathcal{L}(U, X_{-1})$, the mapping $\Phi_t : L^\infty(0, t; U) \to X$, see Definition 3, can always be extended to some Orlicz space $E_\Phi(0, t; U)$. We can also formulate the dual problem.

Question 25. *Is $C \in \mathcal{B}(X_1, Y)$ an E_Φ-admissible observation operator for some Φ provided that C is an L^1-admissible observation operator?*

In contrast to Question 24, we can provide a negative answer to Question 25.

Example 26. Let $X = L^1(0, 1)$ and T be the left-shift semigroup on X. Then $C = \delta_0$ is an L^1-admissible observation operator, but C is not E_Φ-admissible for any Young function Φ.

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Proof. Since for any $f \in L^1(0,1)$ and $s \le 1$, we have that $CT(s)f = f(s)$, it follows that C is L^1-admissible and not E_Φ-admissible as $L^1(0,t) \supsetneq E_\Phi(0,t)$ for any Φ. $\square$

Note that Example 26 does not yield an answer for Question 24: in fact, if we had an example of a generator on a reflexive space together with an L^1-admissible C which is not E_Φ-admissible for any Φ, then the dual semigroup and $B = C^*$ would provide a (negative) answer for Question 24.

In this article we have shown that the answer to Question 24 is 'yes' if U is finite-dimensional, the semigroup is bounded analytic and strongly stable (which is equivalent to A having dense range here) on a Banach space and A satisfies ("two-sided") square function estimates of the form (7). The next step is of course to ask what happens without the latter assumption. In particular, we ask if the implication still holds true for any analytic semigroup on a Hilbert space. It is well-known that on every (separable) Hilbert space, a bounded analytic semigroup with generator A can be constructed such that A does not have a bounded H^∞-calculus, that is A does not satisfy square function estimates of the above type, [1,12,28]. The question remains whether for such operators the sets $\mathfrak{B}_{\infty,U}(A)$ and

$$\mathfrak{B}_{\mathrm{Orlicz},U}(A) = \{B \in \mathcal{L}(U, X_{-1}) \colon B \text{ is } E_\Phi\text{-admissible for some } \Phi\}$$

still coincide for U is finite-dimensional. By Proposition 4, it again suffices to consider $U = \mathbb{C}$.

Question 27. Let A generate an exponentially stable analytic semigroup on a Hilbert space. Suppose that the H^∞-calculus for A is not bounded. Does Question 24 have a positive answer, i.e. does the following implication hold

$$b \in X_{-1} \text{ is } L^\infty\text{-admissible} \quad \Longrightarrow \quad \exists \Phi : b \text{ is } E_\Phi\text{-admissible} \quad ?$$

By Theorem 10, it is clear that in the situation of Question 27, $\mathfrak{b}_\infty(A) \subsetneq X_{-1}$. Because of this strict inclusion, the space $\mathfrak{b}_\infty(A)$ is hard to characterize which makes it difficult to investigate the question. However, we have to emphasize that the condition of a bounded H^∞-calculus is not very restricting in practice when working with specific pde's as it is known to hold true for a large class of operators, including most differential operators, see e.g. [36, Sec. 3] for a survey. Therefore, Question 27 is rather of theoretic interest.

On the other hand, when leaving the Hilbert space setting, there is a well-known subtlety concerning the relation of a bounded H^∞-calculus and square function estimates. In general, classical square functions of the form (7) are only sufficient but not necessary for a bounded calculus, see e.g. [3]. This issue, however, can be overcome by using generalized square function estimates that were first introduced only for L^p spaces and later, in a seminal paper by Kalton and Weis [20], generalized to general Banach spaces. For $X = L^p(\Omega,\mu)$, $1 \le p < \infty$, the corresponding condition for $(-A)^{\frac{1}{2}}$ reads

$$\exists K > 0, \forall f \in L^p(\Omega,\mu) : \left\| \left(\int_0^\infty \left| ((-A)^{\frac{1}{2}} T(t) f)(\cdot) \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,\mu)} \le K \|f\|_{L^p(\Omega,\mu)},$$

which is consistent with L^2-admissibility in the case $p = 2$. It is not hard to see that this condition together with its dual version can be used to derive that any $b \in X_{-1}$ is L^∞-admissible along the same lines as in the proof of Theorem 7. Moreover, one even gets zero-class L^∞-admissibility. However, the difference is that it is not clear if an Orlicz-space norm can be recovered as in the case for classical L^2-admissibility.

Question 28. Let $X = L^p(\Omega, \mu)$ and let A generate an exponentially stable analytic semigroup such that A has a bounded H^∞-calculus. Does the implication

$$b \in X_{-1} \quad \Longrightarrow \quad \exists \Phi : b \text{ is } E_\Phi\text{-admissible}$$

hold true?

In the case that $L^p(\Omega, \mu) = \ell^p(\mathbb{N})$ this was proved in [13, Thm. 4.1] using a direct method without (explicitly) using the boundedness of the calculus. Also note that Fackler [7] recently provided an L^p-version of the Le Merdy–Grabowski–Callier result linking the boundedness of the H^∞-calculus to a corresponding positive, contractive semigroup, see also [35].

This article was mainly concerned with parabolic pde's, but it is an interesting question what happens for more general semigroups. In particular, it is important to investigate the case of contraction semigroups, where, in the Hilbert space case, boundedness of the functional calculus is guaranteed by von Neumann's inequality. However, at this point it is unclear how (and if at all) the latter property is useful when studying Question 24. This is subject to future work.

Finally, we want to study the consequences of the derived results for nonlinear (parabolic) pde's; starting with relatively simple semi-linear equations. Although the notion of admissibility is often perceived as restricted to linear problems, the relation to ISS, a nonlinear concept, seems promising to achieve this step. On the other hand, as mentioned in the introduction, there are recent results by Mironchenko and Wirth [30] for a class of (nonlinear) systems which the authors call *forward complete dynamical (control) systems*, where in particular — and crucially — trajectories are required to be continuous. Our results, e.g. Theorem 1, provide characterizations of this latter property which moreover implies that the considered parabolic systems then indeed fall into that class. This probably opens the way to use ISS features, such as Lyapunov functions, which becomes particularly interesting in the transition to above mentioned nonlinear parabolic problems, see e.g. [40] where a semilinear 1-D heat equation is studied.

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Appendix A. A technical lemma

Lemma 29. *Let $f \in L^1(0,\infty)$ and $f(\tau+\cdot) \in L^\infty(0,\infty)$ for some $\tau > 0$. Then there exists a Young function Φ such that $\Phi(|f(\cdot)|)$ is integrable. Thus, $f \in L_\Phi(0,\infty)$.*

Proof. By [24, Thm. 3.2.5] there exists a Young function Φ such that the assertion of the lemma holds in the case that f has bounded essential support (this only relies on the assumption that $f \in L^1(0,\infty)$). For the general case consider

$$\int_0^\infty \Phi(|f(s)|)\,ds = \int_0^\tau \Phi(|f(s)|)\,ds + \int_\tau^\infty \frac{\Phi(|f(s)|)}{|f(s)|}|f(s)|\,ds.$$

The first term on the right-hand-side is finite by the first case. For the second term, note that $\frac{\Phi(x)}{x}$ is bounded on any bounded subinterval of $[0,\infty)$, which follows from continuity of Φ and $\lim_{x\to 0^+}\frac{\Phi(x)}{x} = 0$. Hence, as $f \in L^\infty(\tau,\infty)$ and $L^1(0,\infty)$, the claim follows. □

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