On Fattorini’s paper: “Boundary Control Systems”

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Abstract: In 1968 Fattorini published in SIAM Journal on Control his well-known paper with the title Boundary Control Systems. In this paper he sets up the abstract theory for systems described by partial differential equations with a control at the boundary. The theory as presented is complete, meaning that it could be copied into textbooks and articles without major alternations. In his paper and in these textbooks one finds examples illustrating the use of this abstract concept. However, as we will show, these examples are much older, and are more or less standard within the theory of partial differential equations with an inhomogeneous boundary condition.

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1. INTRODUCTION

Partial differential equations (pde’s) as models for vibrations in plates and beams or for diffusion of heat and matter in a spatial domain are already standard for more than 200 years. Control of these systems is often restricted to the boundary of the spatial domain. This was already observed in the sixties of the last century, and so very soon after the birth of finite- and infinite-dimensional systems theory. In 1968 Fattorini publishes (Fattorini, 1968) in which he sets up an abstract general theory to deal with partial differential equations having control at the boundary. This paper is well-known and can be seen as the first paper treating partial differential equations with control at the boundary in an abstract functional analytic way, see also Emirsajlow and Townley (2000), (Lasiecka and Triggiani, 2000, Appendix 3B) or (Tucsnak and Weiss, 2009, Chapter 10). According to Google Scholar this paper has been cited 325 times (date: March 21, 2017).

In this paper we shall discuss Fattorini’s set-up and show that his theory was rather complete, i.e., the difference with for instance section 3.3 in (Curtain and Zwart, 1995) is only marginal. Furthermore, we show that the examples to which, and many authors after him, applied the theory were already known for a long time. In fact, already in the 1920’s they were so well-known that they were presented without a reference.

The organization of this paper is as follows. In the next section we give a short summary of Fattorini (1968). In Section 3 we discuss the class of boundary control systems as introduced by Fattorini and comment on the changes when comparing it with Section 3.3 of Curtain and Zwart (1995). In Section 4 we present some examples of boundary control systems and in the conclusion section we give historic remarks to earlier work.

2. SHORT SUMMARY OF FATTORINI (1968)

Although the title of Fattorini (1968) is “Boundary Control Systems”, this is not the main aim of his paper. The authors aim is to study controllability of boundary control systems. More precisely, he wants to relate the controllability property of the boundary control systems to that of the associated internally controlled systems, see also Section 3. The controllability property he wants to characterize is the ability to steer from zero to (almost) any other state in finite or infinite time, or as he states it in his introduction; The control is exerted on the system by means of the boundary conditions and (possibly) by means of parameters distributed all over I, and we seek conditions on the way the control is applied that assure that the system can be steered from an arbitrary initial state to (the vicinity of) an arbitrary final state. The author calls this null-controllability, but now we would name it approximate controllability, see e.g. (Curtain and Zwart, 1995, Section 4.1). He distinguished between first and second order abstract differential equation. Since the treatment of the first and second order case is very similar, we restrict ourselves to the first order case. The main difference is that the second derivative of the boundary input now becomes the input to the internally controlled systems, see also Section 4.
3. THE CLASS OF BOUNDARY CONTROL SYSTEMS

To define the class of boundary control system, we follow the notation \(^1\) of (Curtain and Zwart, 1995, Section 3.3). However, we reproduce here the original definition of Fattorini (1968).

Let \(X, U_s\) and \(U\) be Banach spaces. Let \(\mathfrak{A}\) be a closed, linear operator from \(D(\mathfrak{A}) \subseteq X\) to \(X\). Since it is closed its domain (with the graph norm) is again a Banach space. For \(B_s \in \mathcal{L}(U_s, U)\) and \(\mathfrak{P} : D(\mathfrak{P}) \supseteq D(\mathfrak{A}) \to U\) with \(\mathfrak{P}\) bounded with respect to the graph norm of \(\mathfrak{A}\), we define the following boundary control system

\[
\dot{x}(t) = \mathfrak{A}x(t), \quad x(0) = x_0, \quad \mathfrak{P}x(t) = B_s u_s(t). \tag{1}
\]

Two more assumptions are added (Fattorini, 1968, Assumption 2–4).

**Assumption 1.** For the system (1) we assume that:

1. The set of all \(x_0 \in D(\mathfrak{A})\) with \(\mathfrak{P}x_0 = 0\) is dense in \(X\), and for these initial conditions the solution of (1) is well-posed, meaning that there exists a unique solution continuous depending on the initial condition.
2. There exists a \(B \in \mathcal{L}(U_s, X)\) such that for all \(u_s \in U_s\), \(B u_s \in D(\mathfrak{A})\) and \(\mathfrak{P}(B u_s) = B u_s\), for all \(u_s \in U_s\). \tag{2}

Furthermore, there exists an \(m > 0\) such that

\[
\|B u_s\| \leq m \|B u_s\|, \quad \text{for all } u_s \in U_s. \tag{3}
\]

As Fattorini notices in his Remark 1.4, Assumption 1.1 is equivalent to the assumption that the operator \(A : D(A) \to X\) with \(D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{P})\) defined as

\[
A x = \mathfrak{A} x \quad \text{for } x \in D(A) \tag{4}
\]

is the infinitesimal generator of a \(C_0\)-semigroup on \(X\). This is the condition which we nowadays finds, see e.g. (Curtain and Zwart, 1995, Definition 3.3.2). In this book and in many papers, the condition as formulated in the second part of Assumption 1 is formulated differently. We return to this in Subsection 3.1.

Still in the first section of the paper, Fattorini states that under the above assumptions on the system any classical solution of (1) gives rise to a classical solution of

\[
\dot{v}(t) = Av(t) - B u_s(t) + \mathfrak{A} B u_s(t). \tag{5}
\]

with initial condition \(v(0) = x_0 - B u_s(0)\). Conversely, when \(v\) is a classical solution of (5), then \(x(t) = v(t) + B u(t)\) is a solution of (1). He does not include a proof of these statements. The equivalence between both statements is easy by substituting the solution of one equation into the other, and using the connection between \(v\) and \(x\). He notices that by the closed graph theorem \(\mathfrak{A}B\) is a bounded operator, and so equation (5) is a standard inhomogeneous (abstract) differential equation. From that he concludes that he has proved existence and uniqueness of (1) for smooth inputs and smooth initial conditions.

Although he does not formally define control systems as the abstract control differential equations satisfying Assumption 1, it is clear that this class has the right properties to study control properties. In Section 2 he extends the notion to second order systems, and in Section 3 and 4 he studies the relation between the controllability of (1) and (5). His examples can be found in Section 5 and 6. Before we discuss the examples, we discuss the second item in Assumption 1 in some more detail.

### 3.1 On the second condition in Assumption 1.

In Curtain and Zwart (1995) the abstract boundary control system is not given by (1), but by

\[
\dot{x}(t) = \mathfrak{A}x(t), \quad x(0) = x_0, \quad \mathfrak{P}x(t) = u(t). \tag{6}
\]

Thus the term \(B_s u_s(t)\) has been combined to \(u(t)\). Furthermore, on \(\mathfrak{P}\) the following conditions are made

**Assumption 2.** \(\mathfrak{P} : D(\mathfrak{P}) \subseteq X \to U_{CZ}, U_{CZ}\) a Banach space, satisfies \(D(\mathfrak{A}) \subseteq D(\mathfrak{P})\) and there exists a \(B_{CZ} \in \mathcal{L}(U_{CZ}, X)\) such that for all \(u \in U_{CZ}, B_{CZ} u \in D(\mathfrak{A})\), the operator \(\mathfrak{A}B_{CZ}\) is an element of \(\mathcal{L}(U_{CZ}, X)\) and

\[
\mathfrak{P} B_{CZ} u = u, \quad u \in U_{CZ}. \tag{7}
\]

So in contrast to the assumption in Fattorini, \(\mathfrak{P}\) needs not to be bounded from \(D(\mathfrak{A})\) to \(U\), and part 2 of Assumption 1 is differently formulated.

Since neither of the two formulations seems to incorporate the other, we have to discuss two implications. We begin with the formulation of Curtain and Zwart and show that it fits into the formulation of Fattorini.

Starting with the formulation of Curtain and Zwart, we see that by choosing \(U_s = U = U_{CZ}\) and \(B_s = I\), we obtain the formulation of Fattorini. However, we are still missing the boundedness of \(\mathfrak{P}\) as operator from \(D(\mathfrak{A})\) to \(U\). It is easy to show that \(\mathfrak{P}\) is a closed operator from \(D(\mathfrak{A})\) to \(U\), and since the domain of \(\mathfrak{P}\) contains the domain of \(\mathfrak{A}\) it is by the closed graph theorem a bounded operator from \(D(\mathfrak{A})\) to \(U\).

Now we show that the formulation of Fattorini can be written into (6). It is easy to see that without loss of generality we may assume that \(\ker B_s = \{0\}\).

As \(U_{CZ}\) we take the range of \(B_s\), with the following norm

\[
\|u\| = \|B_s u_s\|_s + \|u_s\|_s, \quad \text{where } u = B_s u_s. \tag{8}
\]

Since \(B_s\) is injective, this is well-defined. Furthermore, using the fact that \(B_s \in \mathcal{L}(U_s, U)\) and \(U, U_s\) are Banach spaces, it is not hard to show that \(U_{CZ}\) is a Banach space with respect to the norm (8).

For \(u \in U_{CZ}\) we define \(B_{CZ} v\) via

\[
B_{CZ} v = B_s u_s, \quad \text{where } u = B_s u_s. \tag{9}
\]

Now by (2) we find that

\[
\mathfrak{P} (B_{CZ} u) = \mathfrak{P} (B_s u_s) = B_s u_s = u,
\]

and so (7) is satisfied.

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\(^1\) Fattorini uses \(\sigma\) and \(\tau\) for \(\mathfrak{A}\) and \(\mathfrak{P}\), respectively. Since there is a real danger that these would be confused with “spectrum” and “time”, we decided to use the more recent notation.

\(^2\) Fattorini uses \(E, F_s\) and \(F\) for these spaces. Although these symbols are still standard in theory on Banach spaces, we changed them, since they could be confused with feedbacks.

\(^3\) In Curtain and Zwart (1995) all spaces are assumed to be Hilbert spaces, but it is easy to see that all the proofs in their Section 3.3 directly carry over to Banach spaces.
Using (3) and (8), we find that
\[\|BCZu\| = \|Bu\| \leq m\|Bz_u\| \leq m\|\|Bz_u\| + |u_s|\| = m\|u\|\]
and so \(BCZ\) is a bounded operator from \(UCZ\) to \(X\). Furthermore, its takes its values in the domain of \(\mathfrak{A}\). Hence using the closed graph theorem we conclude that \(AB_{CZ}\) is a bounded operator.

4. EXAMPLES

The section 5 and 6 of Fattorini (1968) contain the examples. Parabolic pde’s with control at the boundary of its one-dimensional domain are studied in section 5, whereas their hyperbolic counterpart are the topic of section 6. As mentioned before the emphasis of Fattorini’s paper is in studying controllability. Thus showing that section 6. As mentioned before the emphasis of Fattorini’s examples. Parabolic pde’s with control at the boundary. For his class, he shows that under certain conditions boundary control can be transformed into control within the spatial domain. However, no abstract set-up is given. Russell writes on that others have used the same trick, and thereby refers to recent (conference) papers of Ju.E. Egorov, P.K.C. Wang & Tung, and A.V. Balakrishnan. However, here he does not refer to the book Courant and Hilbert (1953), although volume II (Courant and Hilbert, 1989) from 1962 is on his reference list. On page 277 of Courant and Hilbert (1953) the following is written:

**Homogeneous differential equations with nonhomogeneous boundary conditions are essentially equivalent to nonhomogeneous differential equations with homogeneous boundary conditions.**

This is an exact translation of the statements in the original German edition (Courant and Hilbert, 1924, p. 222). However, in the later version of 1953, page 277, the following footnote is added: **However, one should note that the first step, transforming a problem with nonhomogeneous boundary conditions into one with homogeneous conditions, involves assumptions on continuity and differentiability not necessarily made for the original problem.** We see that the example of Fattorini as discussed here in the previous section shows some of these difficulties. How to calculate (explicitly) \(B\) can be a problem. The existence of \(B\) can only happen when the range of \(\mathfrak{A}\) equals that of \(B_\ast\), see also Tucsnak and Weiss (2009) or Emirsajlow and Townley (2000) for more information on the construction of \(B\).

Summarising, as already formulated in (Lasiecka and Triggiani, 2000, p. 420), we see that procedure as given by Fattorini (1968) is the abstract formulation of a very well-known technique from pde’s. Although it was not possible to trace back the first paper in which this technique was applied, it is very likely that it dates back to the nineteenth century. The article Fattorini (1968) fits precisely in the functional analytic formulation of pde’s, which has turned out to be very successful, see for a nice overview Brezis and Browder (1998).

5. CONCLUSION

As we have shown the set-up as presented in Fattorini (1968) became, except to some minor changes, the standard in formulating boundary control systems. Nowadays the term “boundary control systems” refers to an abstract system like equation (1) in which the operators satisfy conditions as formulated in Section 3, see Assumption 1. Since Fattorini (1968) was the first paper on this topic, it is strange that the author did not motivate his definition. However, even more surprising is that he did not link his abstract set-up to the standard way in which inhomogeneous boundary conditions in a partial differential equation were removed. In his introduction, referring to the technique as reproduced here in Section 3 he writes: **this type of argument has already been used in connection with controllability and optimal problems; see, for instance [14].** Here reference [14] is the paper by Russell, (Russell, 1966).

Indeed in his paper of 1966 D.L. Russell studies optimal control for hyperbolic systems on a one dimensional spatial domain with finite-dimensional control in the domain and at the boundary. For his class, he shows that under certain conditions boundary control can be transformed into control within the spatial domain. However, no abstract set-up is given. Russell writes on that others have used the same trick, and thereby refers to recent (conference) papers of

**REFERENCES**


\(^4\) Unfortunately, I was unable to retrieve these items

